We find the joint effect of non-zero temperature and finite conductivity onto the Casimir force between real metals. Configurations of two parallel plates and a sphere (lens) above a plate are considered. Perturbation theory in two parameters (the relative temperature and the relative penetration depth of zero point oscillations into the metal) is developed. Perturbative results are compared with computations. Recent evidence concerning possible existence of large temperature corrections at small separations between the real metals is not supported.

12.20.Ds, 11.10.Wx, 12.20.Fv

Currently the Casimir effect is attracting considerable interest. A large amount of information is available nowadays of both theoretical and experimental nature. Theoretical progress was made in elaborating different approximate methods [1–3] and in the problem of a dielectric sphere where, for instance, the structure of the ultraviolet divergencies had been clarified [4]. In [5] the additive method was successfully applied to a dilute dielectric ball, and in [6] a progress was made in obtaining analytical results. The Casimir force was demonstrated between metallic surfaces of a spherical lens above a disk using a torsion pendulum [7] and an atomic force microscope [8,9].

The increased accuracy of Casimir force measurements invites a further investigation of different theoretical corrections. In [10] the Casimir force for the configuration of a sphere above a plate was computed by taking into account surface roughness and finite conductivity corrections up to fourth order in respective small parameters. That result is in excellent agreement with the measured Casimir force. Except for contributions of surface roughness and finite conductivity, corrections due to non-zero temperature play a dominant role above some distance between the test bodies. The general expression for the temperature Casimir force between dielectric plates was firstly obtained in [11] (see also [12]). The temperature Casimir force between perfectly conducting plates was found in [13–15], including the limiting cases of large and small plate separations (high and low temperatures). These results were modified for the configuration of a spherical lens above a disk in [7]. The temperature corrections are found to be insignificant within the separations of experiments [8,9] (from a ≈0.1 μm till a=0.9 μm or 0.5 μm). As for experiment [7] they constitute up to 174% of the net force at room temperature T=300 K at the largest separation a=6 μm [16] (in spite of this, experimental data of [7] are not sufficiently accurate to demonstrate temperature corrections). In particular, it should be emphasized that the joint effect of non-zero temperature and finite conductivity of the boundary metal was not investigated up to the present.

The computations of the recent paper [17] have cast some doubt on the possibility to use the low temperature limit of the temperature corrections for perfect conductors [13–15] in order to describe the real metals. For the configuration of a sphere above a disk the difference of 4pN at a ~0.1 μm was found depending on whether one uses the expression for the temperature Casimir force for a real metal or the zero temperature one. This is in contradiction with the generally accepted behavior of the temperature correction between perfect conductors at small separations which is proportional to (kBTa/hc)^3 for a sphere above a disk, k_B being the Boltzmann constant [7,13–15]. The authors of [17] hypothesized that for a real conductor the temperature correction to the Casimir force at low temperature can behave as k_BTa/hc and be important.

Here we present a perturbative calculation of the joint influence of non-zero temperature and finite conductivity on the Casimir force. The obtained results are the generalization of [13–15] to the case of real metals. They can be used for the interpretation of precision experiments on Casimir force. No unexpected large temperature contributions arise at small separations. The erroneous assumption of [17] is explained below in detail.

We start with the configuration of two plane parallel plates with the dielectric permittivity ε separated by an empty gap of thickness a. At arbitrary temperature T the attractive force per unit area acting between plates is given by the Lifshitz formula [12]

\[ F_{pp}(a) = -\frac{k_BT}{\pi \epsilon_0^3} \sum_{n=0}^{\infty} \epsilon_n^3 \int_1^\infty p^2 dp \left( Q_1^{-1} + Q_2^{-1} \right), \]

where

\[ Q_1 = \frac{(s + \epsilon p)^2}{(s - \epsilon p)^2} e^{\frac{2\pi \epsilon a}{\epsilon}} - 1, \quad Q_2 = \frac{(s + p)^2}{(s - p)^2} e^{\frac{2\pi \epsilon a}{\epsilon}} - 1, \]
\[ s = \sqrt{\varepsilon - 1 + \frac{p^2}{c^2}}, \quad \varepsilon_n = \frac{2\pi k_B T}{h} n, \quad \varepsilon = \varepsilon(i\xi_n). \tag{2} \]

The prime on the sum indicates that the term with \( n = 0 \) is to be taken with the coefficient 1/2. Let us now introduce new variables \( x_n = 2a\xi_n/c \) and \( z = x_n p \). It is evident that \( x_n = \tau n \equiv 2\pi nT/T_{\text{eff}} \), where the effective temperature is defined by \( k_B T_{\text{eff}} = hc/(2a) \) [18]. In new variables Eq. (1) takes the form

\[ F_{pp}(a) = \frac{k_B T}{8\pi a^3} \sum_{n=0}^{\infty} \varphi_{pp}(x_n), \tag{3} \]

\[ \varphi_{pp}(x_n) = \int_{x_n}^{\infty} z^2 dz \left( Q_1^{-1} + Q_2^{-1} \right), \]

where \( Q_{1,2} \) in (2) are expressed now in terms of \( x_n, z \).

The sum in (3) can be calculated with the help of the Abel-Plana formula [18]. The result is

\[ F_{pp}(a) = -\frac{k_B T}{8\pi a^3} \left[ \frac{1}{\tau} \int_{0}^{\infty} dx \int_{x}^{\infty} z^2 dz \left( Q_1^{-1} + Q_2^{-1} \right) \right] + i \int_{0}^{\infty} \frac{\varphi_{pp}(i\tau y) - \varphi_{pp}(-i\tau y)}{e^{z\tau y} - 1} dy. \tag{4} \]

The first term in the right-hand side of (4) is the Casimir force at zero temperature, the second one takes into account the temperature corrections. The zero temperature contribution was calculated in [19] numerically by the use of optical tabulated data for the complex refractive index (an alternative computation [20] contains some errors which are indicated in [19]). Independently, in [21] it was determined by perturbation theory up to the fourth order in the small parameter \( \delta_0/a \) (\( \delta_0 \) being the effective penetration depth of electromagnetic zero point oscillations into the metal). Thereby, the plasma model was used for the dielectric permittivity

\[ \varepsilon(i\xi) = 1 + \frac{\omega_p^2}{\xi^2} = 1 + \frac{\tilde{\omega}_p^2}{x^2}, \tag{5} \]

where \( \omega_p \) is the effective plasma frequency, and \( \tilde{\omega}_p = 2\omega_p/c \), so that \( \alpha \equiv 1/\tilde{\omega}_p = \delta_0/(2a) \). The results of [19] and [21] are in good agreement for space separations \( a \geq \lambda_p = 2\pi c/\omega_p \). It is well known that the plasma model does not take into account the contribution of relaxation processes which are taken into consideration by the Drude model (see below). However, the variation of the Casimir force obtained by both models remains smaller than 2\% [19].

Let us calculate the second term of (4) in the application range of plasma model and under the condition \( T \ll T_{\text{eff}} \). To do this we use the representation of (3)

\[ \varphi_{pp}(x) = \varphi_{pp}^{(1)}(x) + \varphi_{pp}^{(2)}(x) \equiv \int_{x}^{\infty} z^2 dz Q_1^{-1} + \int_{x}^{\infty} z^2 dz Q_2^{-1}. \tag{6} \]

Introducing \( x_0 < 1 \) one can write

\[ \varphi_{pp}^{(1)}(x) = \int_{x_0}^{x} z^2 dz Q_1^{-1} + \int_{x_0}^{\infty} z^2 dz Q_1^{-1}. \tag{7} \]

Considering firstly the case \( i = 2 \), we notice that in the plasma model (5) the second term from the right-hand side of (7) does not depend on \( x \). Expanding \( Q_2^{-1} \) from the first term of (7) into a series in powers of \( z \) and integrating one arrives at the result

\[ \varphi_{pp}^{(2)}(x) = C - \frac{1}{6} \frac{x^2}{1 + 4\alpha} + \frac{x^3}{2} - \frac{1 + 16\alpha + 96\alpha^2 + 264\alpha^3 + 288\alpha^4 x^4}{12(1 + 4\alpha)^3} + O(x^6), \tag{8} \]

where \( C = \text{const} \).

The even powers of \( x \) evidently do not contribute to the second term of (4). As a result there is only one temperature correction originating from \( Q_2 \) which is caused by the term \( x^3/6 \) and which does not depend on \( \tilde{\omega}_p \). Substituting (8) into the second term of (4) one obtains

\[ \Delta_T F_{pp}^{(2)}(a) = F_{pp}^{(0)}(a) \frac{1}{6} \left( \frac{T}{T_{\text{eff}}} \right)^4. \tag{9} \]

Here \( F_{pp}^{(0)}(a) = -\pi^2 hc/(240a^4) \). Note that we neglect the corrections \( O \left[(T/T_{\text{eff}})^5\right] \).

Consider now \( i = 1 \) in (7). In this case both the first and the second terms in the right-hand side depend on \( x \). The second term, however, is an even function of \( x \) and for that reason it does not contribute to (4). Let us expand the quantity \( z^2Q_1^{-1} \) in powers of small parameters \( \alpha \) and \( z \). Integrating the obtained series between the limits \( z = x \) and \( z = x_0 < 1 \) we obtain

\[ \varphi_{pp}^{(1)}(x) = \frac{1}{6} \frac{x^3}{6} + 4x^2\alpha \ln x + \tilde{\varphi}_{pp}^{(1)}(x), \tag{10} \]

where the quantity \( \tilde{\varphi}_{pp}^{(1)}(x) \) contains terms which do not contribute to (4) or lead to contributions of order \( (T/T_{\text{eff}})^5 \) or higher. Substituting (10) into the second term of (4) we get

\[ \Delta_T F_{pp}^{(1)}(a) = F_{pp}^{(0)}(a) \left[ \frac{1}{6} \left( \frac{T}{T_{\text{eff}}} \right)^4 + \frac{30\zeta(3)}{\pi^3} \frac{\delta_0}{a} \left( \frac{T}{T_{\text{eff}}} \right)^3 \right], \tag{11} \]

where \( \zeta(3) \approx 1.202 \) is the Riemann zeta function.

Now, let us take together (9), (11) and the zero temperature contribution given by the first term of (4). In [21] the last one was calculated up to the fourth order. Here we add two more orders. The final result is

\[ F_{pp}(a) = F_{pp}^{(0)}(a) \left\{ 1 + \frac{1}{3} \left( \frac{T}{T_{\text{eff}}} \right)^4 - \frac{16 \delta_0}{3 a} \left[ 1 - \frac{5\zeta(3)}{8\pi^3} \left( \frac{T}{T_{\text{eff}}} \right)^3 + \sum_{i=2}^{6} \frac{c_i \delta_0^i}{a^i} \right] \right\}, \tag{12} \]
where \( c_2 = 24 \), and the other coefficients are
\[
\begin{align*}
    c_3 &= -640 \left( 1 - \frac{\pi^2}{210} \right), \\
    c_4 &= \frac{2800}{9} \left( 1 - \frac{163\pi^2}{7350} \right), \\
    c_5 &= -10752 \left( 1 - \frac{305\pi^2}{5292} + \frac{379\pi^4}{1693440} \right), \\
    c_6 &= \frac{37632}{13} \left( 1 - \frac{1135\pi^2}{9720} + \frac{2879\pi^4}{1358280} \right).
\end{align*}
\]

For \( \delta_0 = 0 \) (perfect conductor) Eq. (12) turns into the well known result [13–15]. It is significant that the first correction of mixing finite conductivity and finite temperature is of order \((T/T_{\text{eff}})^4\), and there are no temperature corrections up to \((T/T_{\text{eff}})^4\) in the higher conductivity corrections from the second up to the six order.

Analogous calculations can be performed for the configuration of a sphere (lens) of radius \( R \) above a plate starting from the force
\[
F_{pl}(a) = \frac{k_B T R}{c^2} \sum_{n=0}^\infty \frac{\gamma_n^2}{1} \int_0^\infty p dp \ln \left( \frac{Q_1 Q_2}{(Q_1 + 1)(Q_2 + 1)} \right). \tag{14}
\]

This formula is obtained from (1) using the proximity force theorem [22]. \( Q_{i,2} \) are defined in (2). After straightforward calculations, using [21] for the zero temperature contribution, the result is
\[
F_{pl}(a) = F_{pl}(0)(a) \left\{ 1 + \frac{45\zeta(3)}{\pi^3} \left( \frac{T}{T_{\text{eff}}} \right)^3 - \left( \frac{T}{T_{\text{eff}}} \right)^4 \right\} - \frac{\delta_0}{a} \left[ 1 - \frac{45\zeta(3)}{2\pi^3} \left( \frac{T}{T_{\text{eff}}} \right)^3 + \left( \frac{T}{T_{\text{eff}}} \right)^4 \right] + \sum_{i=2}^6 \frac{\gamma_i^2}{a^i}, \tag{15}
\]

where \( F_{pl}(0)(a) = -\pi^2\hbar c R/(360\alpha^3) \), \( \gamma_i = 3c_i/(3 + i) \), \( c_i \) are defined in (13). For the perfect conductor \( \delta_0 \to 0 \) the known asymptotic behavior [7] is reproduced.

Now we consider space separations \( a \) for which \( T \sim T_{\text{eff}} \) or even larger. In this case perturbation theory in \( T/T_{\text{eff}} \) does not work. Let us compute the values of temperature force (14) numerically in dependence on \( a \) for \( T \) surfaces used in experiments [8,9] with \( \omega_p = 1.92 \times 10^{16} \text{rad/s} \) [23], \( T = 300 \text{K} \), and \( R = 100 \mu m \). The numerical results are shown in Fig. 1 by the solid curve. In the same figure the asymptotic behavior (15) is presented by the pointed line. The dashed line shows the Casimir force at zero temperature (but with account of finite conductivity). Here, the force was computed by Eq. (14) in which the sum has been changed into the integral [15]. It is seen that perturbation theory works well within the range \( 0.1 \mu m \leq a \leq 3.5 \mu m \) (note that all six perturbation orders are essential near the left verge of this interval). Starting from \( a = 6 \mu m \) the solid line represents the asymptotics at large separations (temperatures)
\[
F_{pl}(a) = -\frac{\gamma(3)}{4a^2} R k_B T \left( 1 - 2\frac{\delta_0}{a} \right). \tag{16}
\]

This result follows from the term of (14) with \( n = 0 \) (the other terms being exponentially small in \( T/T_{\text{eff}} \)). For \( \delta_0 = 0 \) one obtains from (16) the known expression for perfect conductors [7,13–15]. Finite conductivity corrections of higher orders do not contribute at large separations.

By way of example, consider the contribution of temperature correction for the \( AI \) sphere and the plate of experiments [8,9] at smallest separations (\( a \sim 0.1 \mu m \)) as calculated in [17]. According to the numerical results obtained by the plasma model \( \Delta_T F_{pl} = F_{pl}(T) - F_{pl}(T = 0) \approx 0.03 \text{pN} \), where \( F_{pl}(T) \) is computed by Eq. (14). Almost the same result is obtained by the perturbative result of Eq. (15). These values fall far short of the computational result \( |\Delta_T F_{pl}| \approx 4 \text{pN} \) presented in [17]. Although the computational procedure is not described explicitly in [17] we have been able to reproduce the value obtained there as follows.

In [17], instead of the plasma model, the Drude model was used at small frequencies for which the dielectric permittivity on the imaginary axis is
\[
\varepsilon(i\xi) = 1 + \frac{\omega_p^2}{\xi(\xi + \gamma)}.
\]

Here, \( \gamma \) is the relaxation frequency. Substituting the dielectric permittivity (17) into (14) and performing computations the incorrect values \( |\Delta_T F_{pl}| \approx (2.5 - 8.5) \text{pN} \) are obtained when separations decrease from \( 0.1 \mu m \) till \( 0.09 \mu m \) (compare with \( \approx 4 \text{pN} \) in [17]). It arises for the following reasons.

When performing computations, proper allowance must be made for a critical issue resolved by J. Schwinger, L. L. DeRaad, Jr., and K. A. Milton [15]. In line with [15] the prescription should be adopted that we take the limit \( \varepsilon \to \infty \) before setting \( \xi = 0 \) in order that the Casimir force between perfect conductors be obtained from Eqs. (1), (14). Otherwise the \( n = 0 \) terms of (1), (14), containing \( Q_2 \), would not contribute, which would imply incorrect limits both at low and high temperatures [15]. For a real metal the prescription of [15] is satisfied automatically when the plasma model (5) is used. In the case of Drude model (17), however, the contribution of \( Q_2 \) in the terms of (1), (14) with \( n = 0 \) is identical zero. Because of this, it is impossible to follow the prescription of [15]. It is necessary to stress that if the prescription of [15] is not carried out for the calculation of the zeroth term of (14) one would obtain \( -\gamma(3) R k_B T/(8a^2) \) instead of (16) (remind that the term containing \( Q_2 \) does not contribute in this case). The last expression is evidently incorrect. It is two times smaller than the main contribution to (16), which is valid for perfect conductor, and is independent of the actual value of conductivity of the real metal under consideration. Now it has been evident that the extra contribution of \( \approx 4 \text{pN} \) discussed above originates from the missing contribution of \( Q_2 \), when the Drude model is used, and is equal to it by the modulus. Because of this, Drude model can not be
used to provide a correct extraction of the case of metals from the Lifshits formula for dielectrics.

In conclusion it may be said that the joint effect of non-zero temperature and finite conductivity on the Casimir force was examined. It turned out to be in agreement with the previous knowledge for the real metals at zero temperature from one side and for the perfect conductors at non-zero temperature from the other. (Note that some of the above results related to the plasma model only were obtained independently in the recent preprint [24].) The obtained results are the topical ones for the interpretation of precision measurements of the Casimir force.

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FIG. 1. The Casimir force in pN as a function of the surface separation in configuration of a sphere above a disk. The solid line represents the computational results obtained by Eqs. (5), (14). The dotted line is calculated by the perturbative Eq. (15) up to sixth order in relative penetration depth and fourth order in relative temperature. The dashed line is the zero temperature result.
\[ \log \frac{F_{\text{TPN}}}{F_{\text{TPN}}} \]