Double Scalar–Tensor Gravity Cosmologies

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Abstract

We investigate homogeneous and isotropic cosmological models in scalar–tensor theories of gravity where two scalar fields are nonminimally coupled to the geometry. Exact solutions are found, by Noether symmetries, depending on the form of couplings and self–interaction potentials. An interesting feature is that we deal with the Brans–Dicke field and the inflaton on the same ground since both are nonminimally coupled and not distinguished a priori as in earlier models. This fact allows to improve dynamics to get successful extended inflationary scenarios. Double inflationary solutions are also discussed.

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1 Introduction

Extended gravity theories have recently assumed a prominent role in theoretical physics investigations since any unification scheme, as supergravity or superstrings, in the weak energy limit, or viable early universe cosmological models, as extended inflation, seem to base on them.

Besides, any effective theory, where quantum fields are taken into account in a curved space–time, results in nonminimal couplings between geometry and matter (scalar) fields \[1\]. Furthermore, notwithstanding the fact that Einstein’s general relativity is experimentally tested with high degree of accuracy, from solar system tests to binary pulsar observational data, it has become a peremptory necessity to consider alternative theories of gravity. The issues is now: What kind of theory? The plethora of them is overwhelming from higher–order gravity theories, to Kaluza–Klein multidimensional theories, to induced–gravity theories, to gauge theories with torsion. Several of them seem to be consistent with some quantum gravity effect in the weak energy limit but, till now, no one can be universally considered the ”full” quantum gravity theory.

Despite of this shortcoming, most of them have remarkable physical implications from cosmology to particle physics. A particularly relevant role is played in inflationary cosmology where, from the early Starobinsky model \[2\] to the more recent hyperextended models, non–standard theories have been widely used. In fact, the goal of every inflationary model is to generate a brief period in which the scalar factor of the universe, \(a(t)\), increases superluminally, i.e. \(a(t) > t\). If \(a(t)\) grows by \(e^{60}\) or more during this period, the horizon, flatness and monopole problems can be resolved. In addition, inflation generates energy density fluctuations which may be seeds for large scale structure formation. This features can be obtained in several alternative theories of gravity.

However, designing a detailed microphysical model that accomplishes all of these goals has proven to be extremely difficult and several times the extended theories have to be adjusted to reach some partial goal.

As a general scheme, one needs a grand entrance into inflation which is a mechanism which drives the universe into a false vacuum phase. The large, positive vacuum density acts as an effective cosmological constant which triggers a period of a (quasi) de Sitter expansion. Then one needs a graceful exit: a mechanism capable of terminating the inflationary expansion, reheat the universe to a high temperature, and restore a Friedman expansion. This second issue can result highly problematic due to the fine–tuning of parameters which one needs to connect in the isotropy of the cosmic microwave background (e.g. the so called ”big bubble problem” \[3\]).

Cosmological models deduced from nonminimally coupled theories of gravity (e.g. Brans–Dicke or induced gravity) have provided schemes capable of escaping such difficulties. Extended inflationary models could accomplish several goals which earlier models failed and, in particular, they cure some shortcomings of ”old” inflation. Extended inflationary models also start when the universe is trapped in a false vacuum state by a large energy barrier during a first–order phase transition \[4\]. The failure of old inflation was
that the universe could never escape the false vacuum state since the rate of tunneling through the barrier remains small compared to the inflationary expansion rate [5]. Extended models, in the various versions, avoid the same failure by introducing mechanisms so that the tunneling rate exceeds the expansion rate and, hence, the transition to the true vacuum can be completed. For a comprehensive review, see [6].

The key ingredient on which extended inflation lies is the relation between the tunneling or bubble nucleation rate, $\lambda$, and the expansion rate (the Hubble parameter) $H$. The ratio is the dimensionless "bubble nucleation rate" $\epsilon = \lambda / H^4$. The false vacuum can be percolated by true vacuum bubbles only if $\epsilon$ exceeds a critical value, $\epsilon_{\text{crit}} \simeq 0.2$. In old inflation, $\epsilon$ is time-independent since $\lambda$ and $H$ (which depends on the false vacuum energy) do not vary during inflation [7].

Two situations are possible: i) $\epsilon < \epsilon_{\text{crit}}$, in which case the true vacuum bubbles never percolate and the universe inflates forever, or ii) $\epsilon > \epsilon_{\text{crit}}$, in which case the true vacuum percolates, but so quickly that there is insufficient inflation to solve cosmological problems. A way to bypass this shortcoming is to avoid bubble nucleation altogether. New inflation and chaotic inflation utilize this approach. For example, in new inflation, the energy barrier disappears altogether as the universe supercools and the universe evolves slowly but continuously from the false to true vacuum phase [9, 10]. However, the model has to be fine-tuned.

Extended inflation models employ an alternative approach asking for the time variation of $\epsilon$. Initially, $\epsilon$ is much less than $\epsilon_{\text{crit}}$ to achieve sufficient inflation, but then it grows during inflation to a value $\epsilon > \epsilon_{\text{crit}}$ so that the phase transition can be completed. Being $\lambda$ fixed by the form of the self interacting potential, the only quantity on which one can act is $H$. Cosmological solution with a time-varying $H$ can be easily obtained by using modified theories of gravity like scalar–tensor theories. A simple extended inflationary model [4] can be contructed by using the Brans–Dicke theory [11]. The gravitational action is

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ \phi R - \omega \left( \frac{\partial \phi \rho_m}{\phi} \right) + \mathcal{L}_m \right], \quad (1.1)$$

where $\phi$ is the Brans–Dicke scalar field, $\omega$ is a dimensionless parameter and $\mathcal{L}_m$ is the matter Lagrangian density including all the non-gravitational fields (from now on, we shall use natural units $\hbar = c = k_B = 8\pi G = 1$). The cosmological Friedman–Robertson–Walker (FRW) equations are

$$H^2 = \frac{\rho_m}{3\phi} - \frac{k}{a^2} + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - H \left( \frac{\dot{\phi}}{\phi} \right), \quad (1.2)$$

$$\ddot{\phi} + 3H \dot{\phi} = \frac{\rho_m - 3p_m}{3 + 2\omega}, \quad (1.3)$$

where $H = \dot{a}/a$, $\rho_m$ and $p_m$ are, respectively, the energy and pressure densities of matter, and finally, $k = 0, \pm 1$. For $p_m = -\rho_m$ and $k = 0$ we have

$$\phi(t) = \left( 1 + \frac{\chi t}{\alpha} \right)^2, \quad (1.4)$$
\[ a(t) = \left( 1 + \frac{\chi t}{\alpha} \right)^{\omega + 1/2}, \]  
where \( \chi \) is an integration constant connected to the energy of false vacuum, \( \alpha = (3 + 2\omega)(5 + 6\omega)/12 \).

Immediately we see that for \( \chi t < \alpha, \dot{\phi} \approx 0 \), we have a de Sitter solution. If, for example, \( \omega > 90 \), one obtains the 60 e–foldings necessary to solve cosmological problems of standard model. When \( \chi t > \alpha \), \( a(t) \) evolves as a power–law expansion and \( H \sim t^{-1} \). This feature allows the successful graceful exit since \( \epsilon > \epsilon_{\text{crit}} \). Of course, the main ingredients are the variation of Newton constant and the coupling of the geometry to the scalar field. In other words, the model succeeds because the effective Newton constant \( G_{\text{eff}} = \phi^{-1} \) is decreasing, then \( H \) is decreasing and \( \epsilon \) is increasing.

The main flaw of this model is related to the expected value of the parameter \( \omega \). In order to restore Einstein’s general relativity, we should have \( \omega \to \infty \) [12], then the value of \( \omega \) in constrained by the classical tests of general relativity: light deflection and time–delay experiments require \( \omega > 500 \) [13] while the bounds on the anisotropy of the microwave background radiation give \( \omega \leq 30 \) [14]. In conclusion, \( \omega \) must be a function of time in order to obtain viable models. A pure Brans–Dicke theory is not able to yield realistic models and we have to introduce, at least, a function \( \omega = \omega(\phi) \) in order to overcome the above difficulties. Several proposal have been done to improve the early extended inflationary model and, in some of them also \( \lambda \) is assumed to vary [6]. The strict condition which implies \( \lambda = \text{constant} \) is a feature connected to Brans–Dicke models as it is shown in [8].

In the so called hyperextended inflation [15], one can assume

\[ \omega(\phi) = \omega_0 + \omega_m \phi^m, \]  
(1.6)

to improve the Brans–Dicke model [22]. If \( m = 5 \), the microwave background bounds are satisfied [16]. Alternatively, assuming the coupling

\[ \phi = F(\varphi), \quad \omega(\phi) = \frac{F(\varphi)}{2(dF/d\varphi)^2}, \]  
(1.7)

one can get successful implementations without fine–tuning the initial conditions. In particular, if \( F(\varphi) \) is a sixth order polynomial, the big bubble problem is avoided since the model is independent of the bubble–size distribution [16].

Other approaches give interesting results. For example, it is possible to include a first- or second-order potential \( V(\phi) \) for the Brans–Dicke field \( \phi \) as in induced gravity theories [17]. In this case, the potential places constraints on the percolation time–scale in the graceful exit and, furthermore, it can give rise to multiple episodes of inflation which may reveal extremely useful for large scale structure formation (e.g. super-cluster, cluster and galaxies).

Another way to escape the \( \omega \)–parameter constraints is to consider a curvature–coupled inflation [18]. Also in this case, extended inflation results enhanced since, in some sense,
the roles of Brans-Dicke field and inflaton are mixed. This feature allows to satisfy the solar system constraints on $\omega$, to avoid the big-bubble problem, to construct models with double inflationary episodes.

A more sophisticated way to bypass the graceful exit problem can be obtained by coupling first-order phase transitions to curvature-squared inflation [19]. The mechanism (getaway inflation) is based on a nonminimally coupled higher-order gravity theory where terms like $\varphi^2 R^2$ appear in the usual gravity-inflaton action. Their role is to produce an inflationary phase of the background which has a classical end. At the same time, a stage of bubble production via semiclassical tunneling occurs allowing useful spectra for large scale structure formation.

A final remark concerns the role which the Brans-Dicke scalar could have for dark matter in extended inflation. Its oscillations in the various models could account for the discrepancy between the dynamical estimate of the density of matter in the universe, $\Omega \simeq 0.2$, and the prediction of inflation, $\Omega = 1$ [20] which seems to be confirmed by the BOOMERANG experiment [21].

All these arguments and several more make extremely interesting to search for cosmological solutions useful for extended inflation. A first investigation in this sense is in [22] where general scalar-tensor theories of gravitation were studied in order to ”model” useful extended inflationary behaviours. Intermediate inflationary universes with expansion scale factor of the form

$$a(t) = a_0 \exp t^p, \quad 0 < p < 1,$$

were found. These models allow to succeed in realizing phase transition and graceful exit.

More recently, Modak and Kamilya [23] derived exact cosmological solutions by the so called Noether Symmetry Approach [24] in scalar-tensor gravity theories discussing the role of the coupling function $\omega(\varphi)$ connected to the Noether symmetry. They improved the approach in [25], where symmetries and solutions were found for theories with a scalar field nonminimally coupled to gravity, by introducing a second scalar field (the inflaton) as in extended inflationary models. Exponentially expanding solutions, in asymptotic region, were found and this feature does not allow to solve the graceful exit problem also if general relativity was asymptotically recovered.

In this paper we want to discuss, by Noether Symmetry Approach, a further generalization taking into account two nonminimally coupled scalar fields and their self interaction potentials. In this way, the roles of the Brans-Dicke field and the inflaton are mixed and both fields are taken on the same ground. This fact could be coheren with the stochastic approach for the fundamental laws of nature since the role of the fields is not attributed a priori [26].

The paper is organized as follows. In Sect. 2, we discuss the double scalar-tensor action, derive the equations of motion, the point-like FRW Lagrangian and the cosmological equations. Sect. 3 is devoted to the Noether Symmetry Approach which has to be improved for the double field case since the configuration space results enlarged. The summary of found symmetries is given considering also the subcases where one and not
two nonminimal couplings are present. The cosmological solutions are given in Sect. 4
while the graceful exit problem is discussed in Sect. 5. Conclusions are drawn in Sect. 6.

## 2 Double Scalar-Tensor Action and Equations of Motion

The most general action in four dimensions, where gravity is nonminimally coupled to
two scalar fields noninteracting between them, is

\[
A = \int d^4x \sqrt{-g} \left[ F(\varphi)R + G(\psi)R + \frac{1}{2} \varphi_\mu \varphi^\mu - V(\varphi) + \frac{1}{2} \psi_\mu \psi^\mu - W(\psi) \right],
\]

(2.1)

where we have not specified the four functions \(F(\varphi), V(\varphi), G(\psi),\) and \(W(\psi)\). This action
generalizes those used till now to construct extended inflationary models.\(^1\) The Brans–
Dicke action (1.1) can be immediately recovered by using the transformations (1.7). In
our units, the standard Newton coupling is recovered in the limit \(F(\varphi) + G(\psi) \rightarrow -1/2\).

The field equations can be derived by varying with respect to \(g_{\mu\nu}\)

\[
[F(\varphi) + G(\psi)] \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = T^{(\varphi)}_{\mu\nu} + T^{(\psi)}_{\mu\nu}.
\]

(2.2)

In the right hand side of (2.2) there is the effective stress–energy tensor containing the
nonminimal coupling contributions, the kinetic terms and the potentials of the scalar
fields \(\varphi\) and \(\psi\), that is

\[
T^{(\varphi)}_{\mu\nu} = -\frac{1}{2} \varphi_{;\mu} \varphi_{;\nu} + \frac{1}{4} g_{\mu\nu} \varphi_{;\alpha} \varphi^{;\alpha} - \frac{1}{2} g_{\mu\nu} V(\varphi) - g_{\mu\nu} \Box F(\varphi) + F(\varphi);_{\mu\nu}
\]

(2.3)

and analogously,

\[
T^{(\psi)}_{\mu\nu} = -\frac{1}{2} \psi_{;\mu} \psi_{;\nu} + \frac{1}{4} g_{\mu\nu} \psi_{;\alpha} \psi^{;\alpha} - \frac{1}{2} g_{\mu\nu} W(\psi) - g_{\mu\nu} \Box G(\psi) + G(\psi);_{\mu\nu}.
\]

(2.4)

\(\Box\) is the d’Alambertian operator. The variation with respect to \(\varphi\) and \(\psi\) gives the Klein–
Gordon equations

\[
\Box \varphi - R \left( \frac{dF}{d\varphi} \right) + \frac{dV}{d\varphi} = 0,
\]

(2.5)

and

\[
\Box \psi - R \left( \frac{dG}{d\psi} \right) + \frac{dW}{d\psi} = 0.
\]

(2.6)

\(^1\)We point out that the more general action is

\[
A = \int d^4x \sqrt{-g} \left[ F(\varphi, \psi)R - V(\varphi, \psi) + A(\varphi, \psi)(\nabla \varphi)^2 + B(\varphi, \psi)(\nabla \psi)^2 \right],
\]

but for the purpose of the paper, we will confine ourselves to the action (2.1)
Their sum is equivalent to the contracted Bianchi identities [25]. Let us now take into account a FRW metric of the form

$$ds^2 = dt^2 - a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$$

and substitute it into the action (2.1). Integrating by parts and eliminating the boundary terms, we get the point-like Lagrangian

$$\mathcal{L} = 6 \frac{dF}{d\varphi} a^2 \dot{\varphi} + 6 F a \ddot{a} - 6 \frac{a^3 \dot{\varphi}^2}{2} - a^3 V(\varphi) + 6 \frac{dG}{d\psi} a^2 \dot{\psi} + 6 G a \ddot{a} - 6 \frac{a^3 \dot{\psi}^2}{2} - a^3 W(\psi).$$

The Euler–Lagrange equations, corresponding to the cosmological Einstein equations are

$$[F + G] \left[ 2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] + 2 \left[ \dot{\varphi} \frac{dF}{d\varphi} + \dot{\psi} \frac{dG}{d\psi} \right] \left( \frac{\dot{a}}{a} \right) +$$

$$+ \left[ \ddot{\varphi} \frac{d^2 F}{d\varphi^2} + \ddot{\psi} \frac{d^2 G}{d\psi^2} - \frac{1}{2} \left( \ddot{\varphi}^2 + \ddot{\psi}^2 \right) - (V + W) \right] = 0,$$

$$6[F + G] \left( \frac{\ddot{a}}{a} \right)^2 + 6 \left[ \dot{\varphi} \frac{dF}{d\varphi} + \dot{\psi} \frac{dG}{d\psi} \right] \left( \frac{\dot{a}}{a} \right) + 6k \left[ F + G \right] + \frac{1}{2} \left( \ddot{\varphi}^2 + \ddot{\psi}^2 \right) + V + W = 0,$$ (2.10)

$$\ddot{\varphi} + 3 \left( \frac{\ddot{a}}{a} \right) \dot{\varphi} + 6 \left[ \ddot{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \left( \frac{dF}{d\varphi} \right) + \frac{dV}{d\varphi} = 0,$$ (2.11)

$$\ddot{\psi} + 3 \left( \frac{\ddot{a}}{a} \right) \dot{\psi} + 6 \left[ \ddot{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \left( \frac{dG}{d\psi} \right) + \frac{dW}{d\psi} = 0,$$ (2.12)

where Eq. (2.10) is the energy constraint corresponding to the (0,0)–Einstein equation.

Let us now go to solve the system (2.9)–(2.12) by using the Noether Symmetry Approach. The solutions strictly depend on the form of the functions $F, G, V$ and $W$. By the Noether symmetries it is possible to select these functions so that the system (2.10)–(2.12) can be reduced and then integrated.

### 3 Selecting Couplings and Potentials by the Noether Symmetries

Given an undefined extended gravity theory, the existence of a Noether symmetry can select the form of the coupling and the scalar field potential [25] or the form of the higher–order Lagrangian density, e.g. $f(R, \Box R)$ [27].

At the same time, as we will show in the next section, the symmetry allows to reduce the dynamical system by a cyclic variable making it easier to solve. Taking into account the Lagrangian (2.8), its configuration space is three–dimensional, $Q = \{a, \varphi, \psi\}$. In the language of quantum cosmology, it can be identified with a minisuperspace [25]. The
tangent space on which the Lagrangian (2.8) is defined is $TQ = \{a, \dot{a}, \varphi, \dot{\varphi}, \psi, \dot{\psi}\}$ so that the lift vector $X$, the infinitesimal generator of symmetry, is
\begin{equation}
X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \varphi} + \gamma \frac{\partial}{\partial \psi} + \frac{d\alpha}{dt} \frac{\partial}{\partial \dot{a}} + \frac{d\beta}{dt} \frac{\partial}{\partial \dot{\varphi}} + \frac{d\gamma}{dt} \frac{\partial}{\partial \dot{\psi}}, \tag{3.1}
\end{equation}
where $\alpha, \beta, \gamma$ are functions of $a, \varphi, \psi$. A Noether symmetry exists if the condition
\begin{equation}
L_X \mathcal{L} = 0, \tag{3.2}
\end{equation}
is realized. $L_X$ is the Lie derivative with respect to $X$. Properly speaking, Eq. (3.2) corresponds to the contraction of the vector $X$ with the Lagrangian (2.8). The constant of motion connected to the Noether symmetry is nothing else but
\begin{equation}
\Sigma_0 = i_X \vartheta_{\mathcal{L}} \tag{3.3}
\end{equation}
where
\begin{equation}
\vartheta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}} da + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} d\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} d\psi \tag{3.4}
\end{equation}
is the Cartan one–form given by a Lagrangian $\mathcal{L}$ and $i_X$ is the contraction with respect to $X$. The relation between Eqs. (3.2) and (3.3) can be easily seen if the vector is generally expressed as
\begin{equation}
X = \alpha^i \frac{\partial}{\partial q^i} + \frac{d\alpha^i}{dt} \frac{\partial}{\partial \dot{q}^i}. \tag{3.5}
\end{equation}
Using the Euler–Lagrange equations, it can be shown that [25]
\begin{equation}
\frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}. \tag{3.6}
\end{equation}
If the Noether symmetry exists, Eq. (3.6) gives (3.2). In the Hamiltonian formalism, by a Legendre transformation, we get
\begin{equation}
\mathcal{H} = \dot{a} \pi_a + \dot{\varphi} \pi_{\varphi} + \dot{\psi} \pi_{\psi} - \mathcal{L}, \tag{3.7}
\end{equation}
where $\pi_q \equiv \partial \mathcal{L}/\partial \dot{q}$, $q = \{a, \varphi, \psi\}$ are the conjugate momenta. The phase–space vector for the symmetry is now
\begin{equation}
\Gamma = \dot{a} \frac{\partial}{\partial a} + \dot{\varphi} \frac{\partial}{\partial \varphi} + \dot{\psi} \frac{\partial}{\partial \psi} + \ddot{a} \frac{\partial}{\partial \dot{a}} + \ddot{\varphi} \frac{\partial}{\partial \dot{\varphi}} + \ddot{\psi} \frac{\partial}{\partial \dot{\psi}}, \tag{3.8}
\end{equation}
and a Noether symmetry exists if
\begin{equation}
L_\Gamma \mathcal{H} = 0. \tag{3.9}
\end{equation}
The conserved quantity (3.3) and the Hamiltonian (3.7) gives the Poisson brackets
\begin{equation}
\{\Sigma_0, \mathcal{H}\} = 0. \tag{3.10}
\end{equation}
Our issue is now to determine the functions $F, G, V, W$ by this Noether symmetry technique. We shall adopt the Lagrangian formalism. The condition (3.2) gives the system of partial differential equations

\[ F\left(\alpha + 2a\frac{\partial \alpha}{\partial a}\right) + G\left(\alpha + 2a\frac{\partial \alpha}{\partial a}\right) + a\left(\frac{dF}{d\varphi}\right)\left(\beta + a\frac{\partial \beta}{\partial a}\right) + a\left(\frac{dG}{d\psi}\right)\left(\gamma + a\frac{\partial \gamma}{\partial a}\right) = 0, \]

\[ 3\alpha + 12\left(\frac{dF}{d\varphi}\right)\frac{\partial \alpha}{\partial \varphi} + 2a\frac{\partial \beta}{\partial \varphi} = 0, \tag{3.11} \]

\[ 3\alpha + 12\left(\frac{dG}{d\psi}\right)\frac{\partial \alpha}{\partial \psi} + 2a\frac{\partial \gamma}{\partial \psi} = 0, \tag{3.12} \]

\[ a\beta\frac{d^2 F}{d\varphi^2} + \left(2\alpha + a\frac{\partial \alpha}{\partial a} + a\frac{\partial \beta}{\partial \varphi}\right)\left(\frac{dF}{d\varphi}\right) + 2\frac{\partial \alpha}{\partial \varphi}\frac{dF}{d\varphi} + a\frac{\partial \beta}{\partial \varphi} + a\frac{\partial \gamma}{\partial \psi}\left(\frac{dG}{d\psi}\right) + 2\frac{\partial \alpha}{\partial \varphi}G = 0, \tag{3.13} \]

\[ a\gamma\frac{d^2 G}{d\psi^2} + \left(2\alpha + a\frac{\partial \alpha}{\partial a} + a\frac{\partial \gamma}{\partial \psi}\right)\left(\frac{dG}{d\psi}\right) + 2\frac{\partial \alpha}{\partial \psi}G + a^2\frac{\partial \gamma}{\partial \psi}\left(\frac{dF}{d\varphi}\right) + 2\frac{\partial \alpha}{\partial \psi}F = 0, \tag{3.14} \]

\[ 6\frac{\partial \alpha}{\partial \varphi}\left(\frac{dG}{d\psi}\right) + 6\frac{\partial \alpha}{\partial \psi}\left(\frac{dF}{d\varphi}\right) + 6\frac{\partial \beta}{\partial \varphi}a + 6\frac{\partial \gamma}{\partial \psi}a = 0, \tag{3.15} \]

\[ 6k\left[\alpha F + \beta \left(\frac{dF}{d\varphi}\right) a + \alpha G + \gamma \left(\frac{dG}{d\psi}\right) a\right] + a^2 \left[3\alpha V + \beta a \left(\frac{dV}{d\varphi}\right) + 3\alpha W + \gamma a \left(\frac{dW}{d\psi}\right)\right] = 0, \tag{3.16} \]

obtained by equating to zero the second degree coefficients in $\dot{a}$, $\dot{\varphi}$, $\dot{\psi}$. The number of these equations is $1 + n(n + 1)/2$, where $n$ is the dimension of the configuration space $Q$.

The system (3.11)–(3.17) is the straightforward generalization of the system (5.30)–(5.34) in [25] for the case of two scalar fields.

The integration of (3.11)–(3.17) gives as a result the functions $\alpha(a, \varphi, \psi), \beta(a, \varphi, \psi), \gamma(a, \varphi, \psi), F(\varphi), V(\varphi), G(\psi), W(\psi)$. Solutions are not unique and the various cases which we have found are summarized in Table I.

In Table II, being $G(\psi) = 0$, the cases where only one nonminimal coupling is present are summarized. The value of the spatial curvature constant $k$ is also given. The quantities $F_0, G_0, F_0', \Lambda_{1,2}$ are constants. Using these results, the dynamical system (2.9)–(2.12) can be reduced since, as we shall show below, a change of variables can be found where a cyclic coordinate is present. This feature allows to integrate more simply the dynamics.

4 The Cosmological Solutions

The existence of a Noether symmetry gives, in any case, a cyclic variable so that the transformation

\[ \mathcal{L}(a, \dot{a}, \varphi, \dot{\varphi}, \psi, \dot{\psi}) \longleftrightarrow \mathcal{L}(w, \dot{w}, u, \dot{u}, z) \]  

(4.1)
is always possible. If more than one symmetry exists, more than one cyclic variable can be present [25]. In a geometric language, it is always possible to choose a new set \(q_i = q_i(Q_k)\), \(i, k = 1, 2, 3\), adapted to the foliation given by \(X\)

\[
i_X dQ_3 = 1, \quad i_X dQ_j = 0, \quad j = 1, 2,
\]

where \(i_X\), as before, is the contraction given by \(X\) and \(dQ_j = (\partial Q_j/\partial q_i) dq_i\). Explicitly, in our case, Eqs. (4.2) become

\[
\frac{\partial w}{\partial a} + \frac{\partial w}{\partial \varphi} + \gamma \frac{\partial w}{\partial \psi} = 0, \tag{4.3}
\]

\[
\frac{\partial u}{\partial a} + \frac{\partial u}{\partial \varphi} + \gamma \frac{\partial u}{\partial \psi} = 0, \tag{4.4}
\]

\[
\frac{\partial z}{\partial a} + \frac{\partial z}{\partial \varphi} + \gamma \frac{\partial z}{\partial \psi} = 1, \tag{4.5}
\]

where \(z\) is the cyclic variable. However, the transformation (4.3)–(4.5) are specified as soon as the functions \(\alpha, \beta, \gamma\) are given. As an example, let us take into account Case 1 in Table I. Cases 2 and 3 of Table I and 1, 2, 3 of Table II can be deduced from it. The system (4.3)–(4.5) is solved by the choice of the new variables

\[
w = a^3 \psi^2, \quad u = a^3 \varphi^2, \quad z = \ln a. \tag{4.6}
\]

For the scalar–field functions, as we said, we choose Case 1 in Table I. Lagrangian (2.8) becomes

\[
\mathcal{L} = 6F_0 \dot{z}(\ddot{w} - 2 \dot{z} w) + \frac{1}{8w}(\dot{w} - 3 \dot{z} w)^2 - V_0 w + 6G_0 \dot{z}(\ddot{u} - 2 \dot{z} u) + \frac{1}{8w}(\ddot{u} - 3 \dot{z} u)^2 - W_0 u \tag{4.7}
\]

while the equations of motion are

\[
6(8G_0 - 1) \ddot{z} + 3(32G_0 - 3) \dot{z}^2 + 2 \left( \frac{\ddot{u}}{u} - \frac{\dot{u}^2}{2u^2} \right) + 8W_0 = 0, \tag{4.8}
\]

\[
6(8F_0 - 1) \ddot{z} + 3(32F_0 - 3) \dot{z}^2 + 2 \left( \frac{\ddot{w}}{w} - \frac{\dot{w}^2}{2w^2} \right) + 8V_0 = 0, \tag{4.9}
\]

\[
(8F_0 - 1) \ddot{w} - (32F_0 - 3) \dot{z} w + (8G_0 - 1) \ddot{u} - (32G_0 - 3) \dot{z} u = \Sigma_0 \tag{4.10}
\]

\[
6(8F_0 - 1) \ddot{w} - 3(32F_0 - 3) \dot{z}^2 w + \frac{\dot{w}^2}{w} + 8V_0 w + 6(8G_0 - 1) \ddot{u} - 3(32G_0 - 3) \dot{z}^2 u + \frac{\dot{u}^2}{u} + 8W_0 u = 0, \tag{4.11}
\]

where, clearly, \(z\) is the cyclic variable and \(\Sigma_0\) is the constant of motion connected to \(z\). With respect to the system (2.9)–(2.12), system (4.8)–(4.11) is reduced and it is highly symmetric due to the functions \(F(\varphi), G(\psi), V(\varphi), W(\psi)\) selected by the Noether symmetry.
Since the role of the two fields is completely symmetric, we can suppose that, depending on the value of the parameters, one can select two regimes of physical interest. For example, in the first case, dynamics is \( \varphi (u) \)-dominated, in the second case, it is \( \psi (w) \)-dominated. The relation among the initial data is given by the energy condition (4.11). We get in the \( \varphi (u) \)-dominated regime

\[
 u(t) = c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_1 t],
\]

\[
 z(t) = z_1 \arctan \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) + z_2 \ln |c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_1 t]| + z_0,
\]

for \( c_1 c_2 > 0 \), and

\[
 z(t) = -z_1 \arctanh \left( \sqrt{\frac{c_1}{c_2}} \exp[-\lambda_1 t] \right) + z_2 \ln |c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_1 t]| + z_0,
\]

for \( c_1 c_2 < 0 \), while in the \( \psi (w) \)-dominated regime

\[
 w(t) = c_3 \exp[\lambda_2 t] + c_4 \exp[-\lambda_2 t],
\]

\[
 z(t) = z_3 \arctan \left( \sqrt{\frac{c_3}{c_4}} \exp[\lambda_2 t] \right) + z_4 \ln |c_3 \exp[\lambda_2 t] + c_4 \exp[-\lambda_2 t]| + z_0,
\]

for \( c_3 c_4 > 0 \), and

\[
 z(t) = -z_3 \arctanh \left( \sqrt{\frac{c_3}{c_4}} \exp[-\lambda_2 t] \right) + z_4 \ln |c_3 \exp[\lambda_2 t] + c_4 \exp[-\lambda_2 t]| + z_0,
\]

for \( c_3 c_4 < 0 \). The constants \( z_0, c_i, z_j, i, j = 1, \ldots, 4 \) are integration constants while

\[
 \lambda_1 = \frac{V_0 (32 F_0 - 3)}{4 F_0 (12 F_0 - 1)}, \quad \lambda_2 = \frac{W_0 (32 G_0 - 3)}{4 G_0 (12 G_0 - 1)},
\]

assume the role of cosmological constants depending on the couplings and the potentials. Inverting the relations (4.6), we get

\[
 a(t) = a_0 \left( c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_1 t] \right) \exp \left[ z_1 \arctan \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) \right],
\]

\[
 \varphi(t) = \varphi_0 \left( c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_1 t] \right)^{-2} \left( \exp \left[ z_1 \arctan \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) \right] \right)^{-3},
\]

for \( c_1 c_2 > 0 \), and

\[
 a(t) = a_0 \left( c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_2 t] \right) \exp \left[ z_1 \arctanh \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) \right],
\]

\[
 \varphi(t) = \varphi_0 \left( c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_2 t] \right)^{-2} \left( \exp \left[ z_1 \arctanh \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) \right] \right)^{-3},
\]

for \( c_1 c_2 > 0 \), and

\[
 a(t) = a_0 \left( c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_2 t] \right) \exp \left[ z_1 \arctan \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) \right],
\]

\[
 \varphi(t) = \varphi_0 \left( c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_2 t] \right)^{-2} \left( \exp \left[ z_1 \arctan \left( \sqrt{\frac{c_1}{c_2}} \exp[\lambda_1 t] \right) \right] \right)^{-3},
\]

for \( c_1 c_2 < 0 \).
\[ \varphi(t) = \varphi_0 (c_1 \exp[\lambda_1 t] + c_2 \exp[-\lambda_2 t])^{-2} \left( \exp \left[ z_1 \text{arctanh} \left( \frac{c_1}{\sqrt{c_2}} \exp[\lambda_1 t] \right) \right] \right)^{-3}, \] (4.22)

for \( c_1, c_2 < 0 \) in the \( \varphi \)-dominated regime.

The situation is analogous in the \( \psi \)-dominate regime, but the constants \( \lambda_1, z_1, c_{1,2} \) and \( \varphi_0 \) have to be substituted with \( \lambda_2, z_2, c_{3,4} \) and \( \psi_0 \).

It is easy to see that we have two inflationary eras. Their durations are ruled by the parameter \( \lambda_1, \lambda_2 \) which strictly depends on the strength of the couplings. Another interesting particular solution is

\[ a(t) = a_0 e^{z_0 t}, \quad \varphi(t) = \varphi_0 \exp \left\{ \frac{4F_0 z_0}{8F_0 - 1} t \right\}, \quad \psi(t) = \psi_0 \exp \left\{ \frac{4G_0 z_0}{8G_0 - 1} t \right\}, \] (4.23)

where

\[ V_0 = -\frac{z_0^2}{8(8F_0 - 1)} F_0 \left( F_0 - \frac{1}{12} \right) \left( F_0 - \frac{1}{10.6} \right) \] (4.24)

\[ W_0 = -\frac{z_0^2}{8(8G_0 - 1)} G_0 \left( G_0 - \frac{1}{12} \right) \left( G_0 - \frac{1}{10.6} \right). \] (4.25)

Also here the inflationary behaviour is clear.

By using similar arguments, we can analyze Case 2 in Tab.I. Here the potential terms cancel each other in the Lagrangian (4.7). We get the power-law solution

\[ a(t) = a_0 t^n, \quad \varphi(t) = \varphi_0 t^{-n}, \quad \psi(t) = \psi_0 t^{-n}, \] (4.26)

which is particularly useful for extended inflation being \( n \) an arbitrary constant depending on the initial conditions. A similar situation holds in Case 4, which is a minimally coupled case with two fields and two cosmological constants. For \( k = 0 \) and \( \Lambda_1 = -|\Lambda|_2 \), one finds

\[ a(t) = a_0 t^n, \quad \varphi(t) = \frac{\Sigma_0}{a_0^2(1 - 3n)} t^{1 - 3n} + \varphi_0, \quad \psi(t) = \frac{\Sigma_1}{a_0^2(1 - 3n)} t^{1 - 3n} + \psi_1, \] (4.27)

for \( n \neq 1/3 \), and

\[ a(t) = a_0 t^{1/3}, \quad \varphi(t) = \frac{\Sigma_0}{a_0^2} \ln t + \varphi_0, \quad \psi(t) = \frac{\Sigma_1}{a_0^2} \ln t + \psi_0, \] (4.28)

when \( n = 1/3 \). In this case, two Noether symmetries are present and they assign the value of gravitational constant being

\[ F_0 + G_0 = -\frac{3(\Sigma_0^2 + \Sigma_1^2)}{4a_0^6}. \] (4.29)

Let us now analyse, in detail, Case 5. Without loosing of generality, we can assume \( F'_0 = F_0 = 0, \gamma_0 = 1 \) and studying the couplings \( F(\varphi) = \varphi^2/12 \) and \( G(\psi) = G_0 \). Lagrangian (2.8) becomes

\[ \mathcal{L} = \left( \frac{\varphi^2}{2} + 6G_0 \right) a^2 \dot{a} + a^2 \ddot{a} \varphi \dot{\varphi} + a^3 \left( \frac{\dot{\varphi}^2}{2} - \Lambda \right) + \frac{a^3 \dot{\psi}^2}{2} \] (4.30)
where $\Lambda = \Lambda_1 + \Lambda_2$. Clearly $\psi$ is the cyclic variable. Eqs. (4.3)–(4.5) are satisfied by
\[ w = a, \quad u = a\varphi - \psi, \quad z = a\varphi. \tag{4.31} \]

With the further change
\[ \chi = z - u, \tag{4.32} \]

Lagrangian (4.30) reads
\[ \mathcal{L} = \left( 6G_0\dot{\psi}^2 + \frac{\dot{z}^2}{2} \right) w + \left( \frac{\chi^2}{2} - \Lambda \right) w^3, \tag{4.33} \]

where two cyclic variables appear. The dynamical system is
\[ 6G_0(2\ddot{\psi}w + \dot{\psi}^2) = \frac{\dot{z}^2}{2} + 3w^2 \left( \frac{\chi^2}{2} - \Lambda \right), \tag{4.34} \]
\[ \dot{\chi}w^3 = \Sigma_1, \tag{4.35} \]
\[ \dot{\psi}w = \Sigma_2, \tag{4.36} \]
\[ \dot{z}^2 + 12G_0w^2 + \chi^2 w^2 + 2\Lambda w^2 = 0, \tag{4.37} \]

whose general solution is given by the elliptic integral
\[ \int \frac{w^2 dw}{\sqrt{A_1w^6 + A_2w^2 + A_3}} = \pm t, \tag{4.38} \]

where
\[ A_1 = -\frac{A_1}{6G_0}, \quad A_2 = -\frac{\Sigma_2}{6G_0}, \quad A_3 = -\frac{\Sigma_1}{12G_0}. \tag{4.39} \]

In the particular case where $A_2 = 0$, we get the explicit solution
\[ a(t) = a_0^3 \sqrt{\frac{\Sigma_1^2}{2\Lambda}} \sinh(\pm 3\sqrt{A_1} t), \tag{4.40} \]
\[ \varphi(t) = \sqrt{\frac{\Sigma_2^2}{2\Lambda}} \sinh(\pm 3\sqrt{A_1} t), \tag{4.41} \]
\[ \psi(t) = -\frac{\cosh(\pm 3\sqrt{A_1}) t}{3\sqrt{A_1} \sinh(\pm 3\sqrt{A_1} t)} - \frac{1}{6\sqrt{A_1}} \ln \tanh \left( \frac{\pm 3\sqrt{A_1} t}{2} \right) + \psi_0, \tag{4.42} \]

which asymptotically gives a de Sitter behaviour. A last interesting case is 8 in Tab.I, which can be assigned by the functions
\[ F = F_0, \quad V(\varphi) = \Lambda, \quad k = -1, \]
\[ G(\psi) = G_0 + \frac{\psi^2}{16} + \left( \frac{\varphi_0^2}{12\psi_0^2} \right) \psi^4, \quad W(\psi) = \frac{\psi^4}{4\psi_0^4} - \Lambda, \tag{4.43} \]

being $G(\psi)$ and $W(\psi)$ free for the Noether symmetry. The model is relevant for hyper-extended inflation (see e.g. [15]). Power law solutions like in [15, 16] are easily found.

The cases in Tab.II are essentially subcases of those discussed above.
5 Inflation and graceful exit

As we discussed in Introduction, the goal to get a sufficient inflationary period and then to exit from it, without imposing any particular fine tuning, can be achieved by assuming the variation of the bubble nucleation rate $\epsilon$. However, we are taking into account a first order phase transition after which we recover a Friedman stage [6]. In principle, being $\epsilon = \lambda/H^4$, we can expect the variation of both $\lambda$ and $H$. The form of $\lambda$ strictly depends on the form of the theory and, as it is discussed in [8], it is time independent toward the late times, if we are dealing with a Brans–Dicke theory. In our cases, by using (1.7), we get that most of the couplings selected by the existence of Noether symmetry can be recast in a Brans–Dicke–like form.

In spite of the variation of the effective gravitational coupling, it is reasonable to assume $\lambda$ to be approximatively constant [6], so that the mechanism of the graceful exit can be essentially connected to the variation of $H$.

However, this argument does not work for more general classes of theories, as hyper-extended inflation [15], where $\omega(\phi)$ is not a constant. Among the cases in Tabs. I and II, we have also couplings of the form

$$\omega(\phi) = \frac{F(\varphi)}{2(dF/d\varphi)^2} = \frac{1}{2} \varphi^2 + F_0' \varphi + F_0$$

(5.1)

see e.g. the cases 5 in Tab.I and 4 in Tab.II. This situation deserves more attention since we can distinguish a regime where we match a sort of hyperextended inflation ($\varphi \to 0$) and a regime where the extended inflationary scheme is recovered ($\varphi \to \infty$). In any case, the microwave background bounds have to be satisfied, as discussed in [16]. Furthermore, taking into account double–field models, the contributions to $\lambda$ come from $\varphi$ and $\psi$. Being both fields nonminimally coupled, and from the forms of couplings selected by the Noether symmetry, we are dealing with a double Brans–Dicke–like theory where the extended inflationary mechanism is improved. Looking at the solutions of previous section, we can have double inflationary stages ruled by the parameters of couplings and self–interaction potentials (see e.g. (4.19)–(4.22) or (4.26)). This situation is extremely interesting since ”very” large scale structure and large scale structure can be selected by these inflationary phases. In fact, we can have ”two” first–order phase transitions and then ”two” bubble nucleations where the size of bubbles is given by the coupling parameters. In other words, we can expect two graceful exits given by the superposition of two extended inflationary phases. To be more precise, at a given time $t > t_0$ after nucleation, the ”comoving” bubble radius is

$$r(t, t_0) = \int_{t_0}^{t} dt (a(t))^{-1},$$

(5.2)

while the ”physical” size of the bubble is

$$R(t, t_0) = a(t)r(t, t_0).$$

(5.3)
When $t \to \infty$, the form of $a(t)$ selects the size of the bubble. In the cases (4.19), (4.22) and (4.23), this size is finite since the asymptotic behaviour is $a(t) \sim \exp H_0 t$. For power law behaviours, the growth of the bubble size is linear.

Besides, we have a variation of the Hubble parameter in most of the cases we dealt with: in (4.19), (4.21), and (4.40), it converges to a constant for $t \to \infty$, in (4.23), it is exactly a constant, in the other cases, it is $H \sim t^{-1}$.

Graceful exit is achieved if, being $\lambda$ a constant, $\epsilon$ is less than $\epsilon_{\text{crit}}$ during inflation and, after bubble nucleation, $\epsilon > \epsilon_{\text{crit}}$. In our cases, $H$ is the key parameter which governs the behaviour of $\epsilon$. For power–law solutions, as for standard extended inflationary models, the graceful exit is easily recovered (see Eqs(4.26) and (4.27)). In the asymptotically exponential cases, the parameter $\epsilon$ goes to a constant for $t \to \infty$ and the graceful exit problem has no solution. In fact, the function $H$, calculated from (4.19) and (4.21), is a sort of step function with two different constant values at $t \to \pm \infty$. The related $H^{-4}$ has a singularity in the origin which does not allow a graceful exit from inflation. The situation for the solution (4.40) is similar.

6 Conclusions

In this paper, we derived exact cosmological solutions in double scalar–tensor gravity theories by the general approach of searching for Noether symmetries. This work generalizes those in [23, 24, 25]. The couplings and the potentials of both scalar fields are connected with the existence of the symmetries, and the solutions of dynamics furnish power law or de Sitter evolutions. As a consequence, in all the above cases it is easy to calculate the bubble nucleation rate $\epsilon = \lambda/H^4$ to test if one succeeds in graceful exit. Depending on the value of intervening parameters, this can be accomplished in several cases.

Furthermore, being in principle, both fields nonminimally coupled and self–interacting with a potential, their role is mixed and it is not possible to distinguish, a priori, a Brans–Dicke field and an inflaton field as in other extended inflationary models. This distinction seems, in our opinion, rather artificial in view of a stochastic approach to the fundamental interactions where the effective role of the various fields is distinguishable only in the low energy limit (see [26] and reference therein) and there is no reason why a field should interact with the gravitational field and the other one not.

Another remark deserves double inflation which is ruled by the parameters of the theory and, then, by the Noether symmetry. As it is well known this feature is of extreme interest in perturbation theory since it can furnish the seeds for the formation of structures at large and at very large scales. As we have seen, it is very common in our approach and it could contribute to the enhancement of a successful extended inflation.

Finally, we want to stress the fact that the standard Newtonian coupling can be recovered in several of the above models, being

$$G_{\text{eff}} = -\frac{1}{2[F(\varphi) + G(\psi)]},$$  

(6.1)
so that as soon as $F(\varphi) \to F_0$ and/or $G(\psi) \to G_0$, general relativity is restored (in our units $F_0 + G_0 \to -1/2$) and both fields can contribute to its recovering. This means that in an accurate setting of the models, one could succeed both in graceful exit and in recovering of standard gravity.

Table I – Symmetries in double nonminimally coupled models.

<table>
<thead>
<tr>
<th>N.</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$F(\varphi)$</th>
<th>$G(\psi)$</th>
<th>$V(\varphi)$</th>
<th>$W(\psi)$</th>
<th>$k$</th>
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<td>$-3\psi/2$</td>
<td>$F_0\varphi^2$</td>
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<td>$-3\psi/2$</td>
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<td>$-\Lambda$</td>
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<td>$-3\psi/2$</td>
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<td>$W_0\psi^2$</td>
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</tr>
<tr>
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<td>$F_0$</td>
<td>$G_0$</td>
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<td>$\Lambda_2$</td>
<td>$\forall k$</td>
</tr>
<tr>
<td>5</td>
<td>$1/a$</td>
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<td>$\frac{1}{12}\varphi^2 + \frac{1}{2}\varphi + F_0\varphi + F_0$</td>
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Table II – Symmetries in single nonminimally coupled models.

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<th>$V(\varphi)$</th>
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<td>$-3\psi/2$</td>
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<td>0</td>
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<td>$a$</td>
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<td>$F_0\varphi^2$</td>
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<tr>
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<td>$\gamma_0$</td>
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