Brane worlds: the gravity of escaping matter

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Abstract

In the framework of a five-dimensional model with one 3-brane and an infinite extra dimension, we discuss a process in which matter escapes from the brane and propagates into the bulk to arbitrarily large distances. An example is a decay of a particle of mass $2m$ residing on the brane into two particles of mass $m$ that leave the brane and accelerate away. We calculate, in the linearized theory, the metric induced by these particles on the brane. This metric does not obey the four-dimensional Einstein equations and corresponds to a spherical gravity wave propagating along the four-dimensional future light cone. The four-dimensional space-time left behind the spherical wave is flat, so the gravitational field induced in the brane world by matter escaping from the brane disappears in a causal way.
I. INTRODUCTION AND SUMMARY

It has been found recently [1] that four-dimensional gravity may emerge as a low energy effective theory in models with non-compact extra dimensions (see also Refs [2,3]). One ingredient of this scenario is a relatively old [4–6], and recently revived [7–9] idea that ordinary matter may reside on a 3-brane embedded in higher-dimensional space. Another key point is the existence in a class of models of a bound state of a multi-dimensional graviton which is localized near the brane [1]. Even though gravity at short distances is higher-dimensional, this bound state dominates gravitational interactions in the brane world at large distances and gives rise to four-dimensional behavior of the gravity force experienced by matter residing on the brane. Modifications of this scenario include models with a metastable graviton bound state [10,11] or more than one very light four-dimensional graviton [12]; these models predicts violations of the four-dimensional Newton’s law at ultralarge distances as well.

If the extra dimensions are non-compact, it is conceivable that energy may leak from the brane into the bulk. As an example, field theoretic models of 3-branes can be viewed as defects in higher dimensions (domain walls in (4 + 1)-, cosmic strings in (4 + 2)-dimensions etc.). In these models, highly energetic ordinary particles are able to leave the brane and propagate into the bulk. Another possibility is that besides ordinary matter, there exist particle species that are not trapped to the brane at all; pair creation of these particles would also lead to the transfer of energy from the brane to the bulk. The same role may be played by Kaluza-Klein gravitons whose interactions with the brane matter are weak but non-vanishing.

At first glance, the possibility that energy is carried away from the brane to arbitrarily large distances in the bulk seems, from the four-dimensional point of view, to be in conflict with locality, causality, and the four-dimensional Newton’s law (“Gauss’ law of general relativity“): changing mass in a finite region of three-dimensional space would seem to result in an instantaneous changes of the gravitational potential everywhere in this space. By this argument, an observer living on the brane would immediately realize that energy had been emitted from the brane to the bulk, no matter how far away this event occurred.

Off hand, one may think of several possible resolutions of this apparent paradox. Three of them are as follows:

(i) One may recall that in (1+1) dimensions, energy non-conservation is in fact consistent with locality, causality and the long-range character of forces analogous to gravity [13,14]. Even though the reasons for this property are peculiar to (1+1) dimensions, one may wonder whether a similar consistency may be possible in (3 + 1) dimensions.

(ii) The bulk geometry in the Randall-Sundrum (RS) model [1] and its analogs is anti-de Sitter, so that particles leaving the brane get accelerated away from the brane. Their energy increases as they move away, and one may wonder whether this effect could compensate for the decrease of the strength of gravitational interactions of these particles with matter on the brane. In that case, the mass measured through gravitational forces by an observer residing on the brane would remain constant, and the particles continuously accelerating in the bulk would behave, from the four-dimensional point of view, as dark matter particles of fixed mass which participate in gravitational interactions in a standard way.

(iii) Finally, gravity in the RS model is guaranteed to be effectively four-dimensional
only as far as interactions of matter residing on the brane are concerned. No argument
implies that the effective four-dimensional description is valid for gravity induced by bulk
matter. In other words, the gravitational field induced on the brane by particles moving in
the bulk need not obey the four-dimensional Einstein equations, so the above discussion of
the conflict between causality and Newton’s law may not apply.

In this paper we decide between these possibilities by calculating, in the linear theory
about the RS background, the gravitational field of a pair of particles escaping from the
brane and propagating in the bulk. We show that the most exotic option (i) has nothing
to do with the RS model. The possibility (ii) would be realized if the graviton bound state
were the only relevant mode, i.e., if the zero mode approximation were reliable. This may
be viewed as a consistency check: in the zero-mode approximation gravity is effectively
four-dimensional irrespective of the position of its source in extra dimensions, so the four-
dimensional gravitating mass must be conserved.

However, the zero-mode approximation is in fact not adequate to the problem in question,
and the Kaluza–Klein (KK) tower of gravitons plays an important role. When KK states are
included in the analysis, the outcome is option (iii). The final picture is that the particles
moving away from the brane produce a spherical gravity wave on the brane, which expands
in three-dimensional space with the speed of light (or almost the speed of light). The
four-dimensional space-time left behind this spherical wave is flat, whereas in front of this
wave the four-dimensional metric is of the usual Schwarzschild asymptotic form, in accord
with causality. The spherical gravitational wave itself does not obey the four-dimensional
Einstein equations, so the gravitational effects of particles moving in the bulk are intrinsically
higher-dimensional, even if these effects are measured by an observer residing on the brane.

II. THE PHYSICAL SET UP

The Randall–Sundrum model contains a 3-brane with tension \( \sigma > 0 \) embedded in five-
dimensional space-time. The bulk cosmological constant between the branes, \( \Lambda \), is negative
and tuned in such a way that the intrinsic geometry on the brane is that of flat four-
dimensional Minkowski spacetime. The solution to the five-dimensional Einstein equations
is

\[
ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu - dz^2
\]

where

\[
a(z) = e^{-k|z|}
\]

Here \( \mu, \nu = 0, 1, 2, 3 \) and \( \eta_{\mu\nu} \) is the Minkowski metric. The constant \( k \) is related to \( \sigma \) and
\( \Lambda \) as follows: \( \sigma = \frac{3k}{4\pi G_5} \), \( \Lambda = -\sigma k \), where \( G_5 \) is the five-dimensional Newton constant. The
four-dimensional hypersurfaces \( z = \text{const.} \) are flat, the five-dimensional space-time to the
right of the brane is a part of anti-de Sitter (adS) space.

It is sometimes convenient to introduce another coordinate,

\[
\zeta = \frac{1}{k}e^{kz}
\]
in terms of which the background metric is conformally flat,
\[ ds^2 = \frac{1}{k^2 \zeta^2} \left( \eta_{\mu \nu} dx^\mu dx^\nu - d\zeta^2 \right) \] (4)

Consider now a process in which some matter is emitted from the brane, and then freely moves in the bulk adS background. Let us assume for simplicity that this matter is symmetric with respect to \( z \to -z \); this will enable us to take symmetric metric perturbations and effectively consider only the space to the right of the brane. An example which we discuss throughout this paper is two particles of mass \( m \) which are emitted from the brane at time \( t = 0 \) along the line \( x = 0 \) in opposite directions in \( z \) with zero initial velocities. The coordinates of these particles obey the geodesic equations in adS. It is straightforward to see that the world line of a particle right to the brane is described as follows,
\[ x_c(t) = 0, \quad z_c(t) = \frac{1}{2k} \ln(1 + k^2 t^2) \] (5)

In terms of the coordinate \( \zeta \), this means
\[ \zeta_c(t) = \sqrt{k^{-2} + t^2} \] (6)

The particle accelerates towards \( z \to \infty \), and at \( t \gg k^{-1} \) its world line approaches the light cone, \( \zeta = t \). Similar observations have been made independently in Ref. [15].

The energy-momentum tensor of this particle,
\[ \hat{T}^{ab} = \frac{m}{\sqrt{-g}} \frac{dx^a}{ds} \frac{dx^b}{dt} \delta(x - x_c(t))\delta(z - z_c(t)) \] (7)

has the following non-vanishing components,
\[ T_{zz} = \frac{m}{a^3} \frac{v^2}{\sqrt{1 - v^2}} \delta(z - z_c(t))\delta(x) \] (8a)
\[ T_{z0} = -\frac{m}{a^2} \frac{v}{\sqrt{1 - v^2}} \delta(z - z_c(t))\delta(x) \] (8b)
\[ T_{00} = \frac{m}{a} \frac{1}{\sqrt{1 - v^2}} \delta(z - z_c(t))\delta(x) \] (8c)

where
\[ v = \frac{z_c}{a(z_c)} = \frac{kt}{\sqrt{1 + k^2 t^2}} \] (9)

Hereafter we consider tensors with all lower indices as basic ones; quantities with and without hats have upper indices raised by full adS and Minkowski metrics respectively. As an example, \( T^\mu_\nu = \eta^{\mu \lambda} T_{\lambda \nu}, T^\mu_\nu = a^{-2} \eta^{\mu \lambda} T_{\lambda \nu} \).

Equations (7) and (8) are valid at \( t > 0 \); before that the energy-momentum tensor is concentrated on the brane. We will assume for definiteness that at \( t < 0 \), the source on the brane is a point-like particle with mass \( 2m \). The physical picture is that this brane particle decays at \( t = 0 \) into two particles of equal mass, one of which moves according to eq. (5) and another which is its mirror image.
The linearized five-dimensional Einstein equations in RS background have been considered in a number of papers, see, e.g., Refs. [1,10,16–18]. In particular, the advantages and disadvantages of the Gaussian Normal (GN) gauge have been emphasized [17,18,10]. As we are interested in the gravitational effects on the brane of a particle moving outside the brane, we find it convenient to work entirely in GN coordinates, however, this will mean that there will not be a global coordinate system which is GN once a matter source has been introduced. In a GN frame one has

\[ g_{zz} = -1, \quad g_{z\mu} = 0 \] (10)

and the linearized theory is described by the metric

\[ ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu + h_{\mu\nu}(x,z)dx^\mu dx^\nu - dz^2 \] (11)

The linearized Einstein equations have the following form,

\[ \delta R_{zz} = 8\pi G_5 \theta_{zz} \] (12a)
\[ \delta R_{z\mu} = 8\pi G_5 \theta_{z\mu} \] (12b)
\[ \delta R_{\mu\nu} - 4k^2 h_{\mu\nu} = 8\pi G_5 \theta_{\mu\nu} \] (12c)

where

\[ \theta_{zz} = \left( \frac{2}{3} T_{zz} + \frac{1}{3a^2} T^\lambda_\lambda \right) \] (13a)
\[ \theta_{z\mu} = T_{z\mu} \] (13b)
\[ \theta_{\mu\nu} = \left( T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} T^\lambda_\lambda + \frac{a^2}{3} \eta_{\mu\nu} T_{zz} \right) \] (13c)

and

\[ \delta R_{zz} = -\left( \frac{h'}{2a^2} \right)' \] (14a)
\[ \delta R_{z\mu} = \left[ \frac{1}{2a^2} (h'_{\mu\nu} - h_{\mu}) \right]' \] (14b)
\[ \delta R_{\mu\nu} = \frac{1}{2} h''_{\mu\nu} + 2k^2 h_{\mu\nu} - \left( k^2 h + \frac{k}{2} h' \right) \eta_{\mu\nu} \]
\[ + \frac{1}{2a^2} (2h^\lambda_{(\mu,\nu)\lambda} - h_{\mu\nu,\lambda} - h_{\mu\nu}) \] (14c)

Hereafter \( h = h^\mu_{\mu} \equiv \eta^{\mu\nu} h_{\mu\nu} \).

We will explicitly consider times \( t > 0 \) when the source is in the bulk. Since there is no matter on the brane, the metric should obey the Israel junction condition on the brane, which, due to the \( Z_2 \) symmetry reads

\[ h'_{\mu\nu} + 2kh_{\mu\nu} = 0 \] (15)

Let us first solve eq. (12a). Its general solution is...
\[ h(z, x) = -\frac{8\pi G_5}{k} \left[ a^2(z) \int_z^\infty dz' \theta_{zz}(z') - \int_z^\infty dz' \ a^2(z') \theta_{zz}(z') \right] + a^2(z) C(x) + D(x) \] (16)

From the junction condition, \( h' + 2kh = 0 \), we find

\[ D(x) = -\frac{8\pi G_5}{k} \int_0^\infty dz' \ a^2(z') \theta_{zz}(z') \] (17)

Consider now eq. (12c). Let us introduce

\[ \xi_\mu = h_\lambda^{\mu \lambda} - \frac{1}{2} h_\mu \] (18)

This function can be found from eq. (12b), but its explicit form will be irrelevant. With the definition (18), eq. (12c) becomes

\[ \frac{1}{2} h''_{\mu \nu} - 2k^2 h_{\mu \nu} - \frac{1}{2a^2} \Box h_{\mu \nu} = 8\pi G_5 \theta_{\mu \nu} + \left( k^2 h + \frac{k}{2} h' \right) \eta_{\mu \nu} - \frac{1}{2a^2} \left( \xi_{\mu \nu} + \xi_{\nu, \mu} \right) \] (19)

Hence, one can write

\[ h_{\mu \nu} = \bar{h}_{\mu \nu} (u_{\mu, \nu} + u_{\nu, \mu}) \] (20)

where \( u_\mu \) accounts for \( \xi_\mu \) in eq. (19), and \( \bar{h}_{\mu \nu} \) obeys the following equation,

\[ \frac{1}{2} \bar{h}''_{\mu \nu} - 2k^2 \bar{h}_{\mu \nu} - \frac{1}{2a^2} \Box \bar{h}_{\mu \nu} = 8\pi G_5 \theta_{\mu \nu}^{eff} \] (21)

with

\[ 8\pi G_5 \theta_{\mu \nu}^{eff} = 8\pi G_5 \theta_{\mu \nu} + \left( k^2 h + \frac{k}{2} h' \right) \eta_{\mu \nu} \] (22)

The last term in this equation is known explicitly from eqs. (16) and (17):

\[ \left( k^2 h + \frac{k}{2} h' \right) = -8\pi G_5 k \int_0^z dz' \ a^2(z') \theta_{zz}(z') \] (23)

We are interested in the induced metric on the brane,

\[ h_{\mu \nu} (z = 0) = \bar{h}_{\mu \nu} (z = 0) + (u_{\mu, \nu} + u_{\nu, \mu})(z = 0) \] (24)

The last term in this equation is a pure gauge in the four-dimensional brane world and has no effect on the motion of matter confined to the brane, so we may omit it. In effect, we have to solve eq. (21) and then find \( \bar{h}_{\mu \nu} \) on the brane, i.e., at \( z = 0 \).

It is worth looking at the additional term (23) in the case of the point-like particle moving in the bulk. We have

\[ \left( k^2 h + \frac{k}{2} h' \right) = -8\pi G_5 k \theta(z - z_c(t)) a^2(z_c(t)) \Phi_{zz}(x) \] (25)

where \( \Phi_{ab} \) is defined by
\[ \theta_{ab}(z, x) = \delta(z - z_c(t)) \Phi_{ab}(x) \]  

(26)

Explicitly

\[ \Phi_{zz}(x) = \frac{m}{a^3(z_c) \sqrt{1-v^2}} \left( \frac{2}{3} v^2 + \frac{1}{3} \right) \delta(x) \]  

(27a)

\[ \Phi_{00}(x) = \frac{m}{a(z_c) \sqrt{1-v^2}} \left( \frac{1}{3} v^2 + \frac{2}{3} \right) \delta(x) \]  

(27b)

\[ \Phi_{ij}(x) = \frac{m}{a(z_c) \sqrt{1-v^2}} \left( -\frac{1}{3} v^2 + \frac{1}{3} \right) \delta(x) \delta_{ij} \]  

(27c)

A special feature of the additional term (25) is that this is a non-local expression with a “string” extending from \( z = z_c \) to \( z = \infty \). This string is of course a gauge artifact due to the breakdown of the brane GN gauge as we pass the particle in the \( z \)-direction, and represents a caustic of the normal geodesic congruence used to define the affine parameter \( z \). In fact, a similar string is also characteristic to the asymptotic four-dimensional Schwarzschild solution transformed into a gauge which is GN with respect to an arbitrarily chosen 2-plane. This artifact is easily removable (as noted in [18]) via a construction reminiscent of the Wu-Yang gauge patching for a Dirac monopole [19]. One introduces an additional GN gauge patch to the right of the accelerating particle, valid for \( z > z_c(t) - \epsilon \tan^{-1} |x| \), with the brane GN patch being valid for \( z < z_c(t) + \epsilon \tan^{-1} |x| \). The gauge transformation on the overlap is readily seen to be a bulk analog of the Garriga-Tanaka transformation:

\[ \epsilon_{\mu} = -\frac{\epsilon_z(x)}{2k} ; \quad \partial^2 \epsilon_z = 8\pi G a^2 (z_c(t)) \Phi_{zz}(x) \]  

(28)

One could always choose a harmonic bulk gauge \( (\nabla^a h_{ab} - \frac{1}{2} h, b = 0) \), however, the computation of the Green’s function in this gauge is cumbersome and not particularly illuminating, therefore we simply use a GN gauge. Indeed, since we are primarily interested in the metric induced on the brane, we stick with the brane GN system, the string singularity outside the brane not leading to any inconvenience.

III. SOLUTION OF THE LINEARIZED PROBLEM

A. First trial: zero mode approximation

We are going to treat the parameter \( k \) of the RS model as microscopic, and are interested in the induced metric on the brane, \( \bar{h}_{\mu\nu}(x, z = 0) \), at distance scales large compared to \( k^{-1} \). In particular, we consider the region of the four-dimensional space-time such that \( r \equiv |x| \gg k^{-1}, t \gg k^{-1} \). The solution to eq. (21) involves the retarded Green’s function of the operator

\[ \frac{1}{2} \frac{d^2}{dz^2} - 2k^2 - \frac{1}{2a^2} \Box \]  

(29)

with boundary conditions enforcing the junction property (15). This Green’s function is expressed in terms of the eigenfunctions of the corresponding one-dimensional problem [1,17],
\[ G_R(x - x', z, z') = 4ka^2(z)a^2(z')D_0(x - x') + 2\int_0^\infty dm u_m(z)u_m(z')D_m(x - x') \quad (30) \]

where \( D_0 \) and \( D_m \) are retarded Green’s functions of massless and massive scalar fields in four flat dimensions, and

\[ u_m(z) = \sqrt{\frac{mJ_1(m/k)Y_2(m\zeta) - Y_1(m/k)J_2(m\zeta)}{\sqrt{J_1(m/k)^2 + Y_1(m/k)^2}}} \quad (31) \]

Here \( \zeta(z) \) is defined by eq. (3), \( J_n \) and \( Y_n \) are the Bessel functions. The first term in eq. (30) is the contribution of the bound state of the five-dimensional graviton (the zero mode), whereas the second term comes from the continuous KK spectrum. Note that our expression for the Green’s function, eq. (30), differs from that of Refs. [1,17] by an overall factor of 4. One factor of 2 has to do with our form of the operator (29), and the other is due to symmetry \( z \to -z \) and effectively accounts for the two particles moving left and right from the brane.

If the source in eq. (21) were on the brane, the long-distance behaviour of the induced metric would be governed by the zero mode contribution,

\[ G_R^{zm}(x - x', z, z') = 4ka^2(z)a^2(z')D_0(x - x') \quad (32) \]

As our first trial, let us boldly use the zero mode approximation (32) in the problem at hand. As mentioned in the Introduction, this approximation is not adequate in our case, but rather provides a non-trivial consistency check.

In the zero mode approximation, the metric induced on the brane is effectively determined by the following equation,

\[ -\Box h_{\mu\nu}(x) = 8\pi G_N \tau_{\mu\nu}(x) \quad (33) \]

where \( G_N = kG_5 \) is the four-dimensional Newton constant and

\[ \tau_{\mu\nu}(x) = 4\int_0^\infty dz a^2(z)\theta_{\mu\nu}^{eff}(z) \quad (34) \]

In the case of a point particle, the right hand side of this equation is straightforward to evaluate,

\[ \tau_{00} = 2m \frac{a(z_c(t))}{\sqrt{1 - \nu^2}} \delta(x) \quad (35a) \]
\[ \tau_{ij} = 2m \frac{a(z_c(t))}{\sqrt{1 - \nu^2}} \delta(x)\delta_{ij} \quad (35b) \]

Making use of eq. (5) one finds

\[ \frac{a(z_c(t))}{\sqrt{1 - \nu^2}} = 1 \quad (36) \]

Therefore, eq. (33) coincides with the linearized Einstein equation in four dimensions with a static source of mass \( 2m \). As discussed in the Introduction, this is consistent with the general property that gravity in the zero mode approximation is always effectively four-dimensional, irrespective of the position of its source in the fifth dimension.
B. Full treatment

To obtain the correct expression for the induced metric on the brane, we have to consider the complete Green’s function (30). It is convenient to work in the coordinate representation, and make use of the explicit form of the retarded scalar Green’s function in four flat dimensions [20],

\[ D_m(x) = -\frac{1}{2\pi} \theta(t) \delta(\lambda^2) + \frac{m}{4\pi \lambda} \theta(t - |x|) J_1(m\lambda), \quad \lambda = \sqrt{t^2 - |x|^2} \]  

(37)

At \( m = 0 \) (i.e., in the zero mode case), only the first term in this expression survives. The first term in eq. (37) is independent of \( m \). Now, one recalls that the functions \( u_m(z) \), with the zero mode included, form a complete set, so that

\[ 2ka^2(z)a^2(z') + \int u_m(z)u_m(z')dm = a^2(z)\delta(z - z') \]  

(38)

We see that the contribution of the zero mode to the five-dimensional Green’s function (30) is cancelled out at \( z = z' \) by KK states. Since we are interested in the metric on the brane induced by the particle outside the brane, we set \( z = z' \) and obtain

\[ G_R(x; z, z') = \int_0^\infty dm \frac{m}{2\pi \lambda} \theta(t - |x|) J_1(m\lambda) u_m(z)u_m(z'), \quad z \neq z' \]  

(39)

To analyze the long-distance physics, we consider the regime \( t, |x|, \zeta' \gg k^{-1} \) (recall that \( \zeta \sim t \) at the position of the source, see eq. (6)). Then the main contribution to the Green’s function comes from modes with small \( m \), that is \( m \ll k \). Thus, we retain the terms in eq. (31) which are leading order in \( m/k \) (not assuming that \( m\zeta' \) is small) and obtain for the Green’s function with one argument on the brane and another off the brane

\[ G_R(x; 0, z') = -\frac{\theta(t - |x|)}{2\pi k \lambda} \int_0^\infty dm \frac{m^2 J_1(m\lambda)J_2(m\zeta')}{k^2} \]  

(40)

The integration here is performed by making use of the following relations,

\[ \int_0^\infty dx \ x^2 J_1(\alpha x)J_2(\beta x) = \left( \frac{\partial^2}{\partial \beta^2} - \frac{1}{\beta} \frac{\partial}{\partial \beta} \right) \int_0^\infty dx J_1(\alpha x)J_0(\beta x) \]  

\[ = \left( \frac{\partial^2}{\partial \beta^2} - \frac{1}{\beta} \frac{\partial}{\partial \beta} \right) \frac{1}{\alpha} \theta(\alpha - \beta) \]  

(41)

In this way we find

\[ G_R(x; 0, z') = -\frac{\theta(t - |x|)}{2\pi k \lambda^2} \left( \delta'(\lambda - \zeta') + \frac{1}{\zeta'} \delta(\lambda - \zeta') \right) \]  

(42)

The Green’s function \( G_R(x - x'; 0, z') \) is concentrated on the five-dimensional light cone, \( \lambda = \zeta' \), i.e.,

\[ (t - t')^2 - (x - x')^2 - \zeta'^2 = 0 \]  

(43)
This feature may (or may not) be an artifact of our coarse-graining: if we were able to resolve short distances, the Green’s function might spread over the region of size $k^{-1}$ inside the light cone.

To obtain the metric induced on the brane, we have to integrate the Green’s function (42) with $\theta^\text{eff}_{\mu\nu}$,

$$\bar{h}_{\mu\nu}(x) \equiv \bar{h}_{\mu\nu}(x, z = 0) = 8\pi G_5 \int d^4x' \int_0^{\infty} d\zeta' G_R(x - x'; 0, \zeta') \theta^\text{eff}_{\mu\nu}(x', \zeta')$$

$$= 8\pi G_5 \int d^4x' \int_{1/k\zeta}^{\infty} \frac{d\zeta'}{k\zeta} G_R(x - x'; 0, \zeta') \theta^\text{eff}_{\mu\nu}(x', \zeta')$$

(44)

The source here is given by eqs. (22), (25), (26),

$$\theta^\text{eff}_{\mu\nu}(x, \zeta) = k\zeta \delta(\zeta - \zeta_c(t)) \Phi_{\mu\nu}(x) - \frac{1}{k\zeta^2_c(t)} \eta_{\mu\nu} \theta(\zeta - \zeta_c(t)) \Phi_{zz}(x)$$

(45)

Even though this source contains the non-local “string”, the integrand in eq. (44) is in effect local. This is due to the particular structure of the Green’s function (42),

$$G_R(x; 0, \zeta') = \frac{\theta(t - |x|)}{2\pi k \lambda^2} \zeta' \frac{\partial}{\partial \zeta'} \left( \frac{1}{\zeta'} \delta(\lambda - \zeta') \right)$$

(46)

Upon integrating by parts in eq. (44), the second term in eq. (45) becomes a delta-function of $(\zeta - \zeta_c(t))$, i.e., only the region $\zeta \approx \zeta_c(t)$ in fact contributes to this integral. This of course had to be the case, as the “string” is a gauge artifact.

In the regime considered, $t \gg k^{-1}$, one has $\zeta_c(t) = t$, and the leading terms in $\Phi_{ab}$ are (see eqs. (27a,27b,27c))

$$\Phi_{zz} = m(kt)^4 \delta(x)$$

(47a)

$$\Phi_{00} = m(kt)^2 \delta(x)$$

(47b)

$$\Phi_{ij} = m \frac{1}{3} \delta_{ij} \delta(x)$$

(47c)

Substituting these expressions into eq. (45) we get

$$\theta^\text{eff}_{00}(x, \zeta) = m[(kt)^3 \delta(\zeta - t) - k^2 t^2 \theta(\zeta - t)] \delta(x)$$

(48a)

$$\theta^\text{eff}_{ij}(x, \zeta) = mk^2 t^2 \theta(\zeta - t) \delta_{ij} \delta(x)$$

(48b)

The integration in eq. (44) is now straightforward. We end up with

$$\bar{h}_{00}(x) = -4G_N m \frac{2t^2 - r^2}{t^3},$$

$$\bar{h}_{ij}(x) = -4G_N m \frac{1}{t} \delta_{ij}, \quad t - r > 0$$

(49)

Here we stress the fact that these expressions are valid inside the four-dimensional future light cone, $t - r > 0$, as is clear from the explicit $\theta$-function in eq. (39). Outside this light cone, the induced metric is determined by the source existing at $t < 0$. With our model for this source (particle of mass $2m$ on the brane at rest at $x = 0$), the induced metric
outside the light cone is the asymptotic Schwarzschild solution (see, e.g. Ref. [17] for explicit derivation in RS model), which in an appropriate gauge reads

\[
\begin{align*}
\bar{h}_{00}(x) &= -4G_Nm \frac{1}{r}, \\
\bar{h}_{ij}(x) &= -4G_Nm \frac{1}{r} \delta_{ij}, \quad t - r < 0
\end{align*}
\]  

(50)

We see that the induced metric (49), (50) is continuous on the four-dimensional light cone, but its derivatives are not. Because of the latter property, the induced metric does not obey the (linearized) four-dimensional Einstein equations. It describes a spherical gravity wave propagating in four dimensions with the speed of light. Again, within our approximation we do not resolve distances of order \( k^{-1} \), so the wave may actually spread over the region of this size.

The four-dimensional space-time on the brane left behind the spherical wave is in fact flat. Indeed, a coordinate transformation on the brane

\[
\begin{align*}
\delta t &= -4G_Nm \left( \frac{r^2}{4t^2} + \ln t \right), \quad \delta x^i = -4G_Nm \frac{x^i}{2t}
\end{align*}
\]  

(51)

reduces the four-dimensional metric perturbation (49) to \( \bar{h}_{\mu\nu} = 0 \) inside the light cone \( t - r > 0 \). The gravitational field induced on the brane by matter escaping into the bulk finally disappears, this process occurring in a causal way.

**IV. DISCUSSION**

We have shown that within linearized perturbation theory, the metric on the brane does indeed react to the ‘loss’ of the sources in the bulk in an intrinsically five-dimensional fashion, a spherical shock wave expanding outwards from the moment of emission leaving behind flat space. It is tempting to ask if something similar could perhaps be obtained by an appropriate use of the far field Schwarzschild-adS solution

\[
ds^2 = \left( 1 + k^2 \rho^2 - \frac{\mu}{k^2 \rho^2} \right) d\tau^2 - \left( 1 + k^2 \rho^2 - \frac{\mu}{k^2 \rho^2} \right)^{-1} d\rho^2 - \rho^2 d\Omega^2_{11}
\]  

(52)

In the absence of the mass term, \( \mu \), the transformation between the brane coordinates and the spherical coordinates is

\[
\begin{align*}
\rho &= \frac{1}{2k\varsigma} \left[ (\varsigma^2 - k^{-2})^2 + k^2(x^2 - t^2)^2 + 2k^2\varsigma^2(x^2 - t^2) + 2(t^2 + x^2) \right]^{1/2} \\
\tan k\tau &= \frac{2t}{k} \left[ k^{-2} + \varsigma^2 + x^2 - t^2 \right]^{-1} \\
\tan \chi &= \frac{2|x|}{k} \left[ k^{-2} + x^2 - t^2 - k^{-2} \right]^{-1}
\end{align*}
\]  

(53a,b,c)

where the parametrisation of \( S^3 \) in the spherical system is the direct generalisation of the \( S^2 \) one, rather than Euler angles, and the angular \( \theta, \phi \) variables coincide with the angular
variables on the brane-world. Notice the periodic relation of the $\tau$-variable to the brane coordinates, this signifies that the spherical coordinates represent in fact the universal covering space of adS, whereas the brane exists in a single patch. In a sense, there are an infinite family of branes existing in the maximally extended spacetime, as discussed in Chamblin, Hawking and Reall [21]. These branes start off 'planar', become parabolic, then return to planar again, oscillating indefinitely in the maximally extended spherical coordinates. From the perspective of the brane spacetime, $\rho = 0$ corresponds to the geodesic trajectory (5). It is tempting therefore to ask whether we can, by modifying our spacetime to Schwarzschild-adS, come up with a metric corresponding to the particle accelerating away from the brane, at least at large spherical $\rho$-coordinate. From the perspective of the metric (52), this would correspond to placing a brane at some angular coordinate $\chi = \chi(t, \rho)$.

In Ref. [21], it was shown that it was not possible to find a static trajectory, $\chi(\rho)$, which might correspond to a particle sitting on the brane, however, it was suspected that this was due to the non-accelerating nature of the black hole – a more appropriate exact metric, such as some sort of C-metric would be a better candidate. These expectations were partly backed up by the lower-dimensional calculation of [22]. In our case however, the accelerating particle in the bulk translates into a non-accelerating $\rho = 0$ geodesic in the spherical spacetime (52) so we might hope that a time dependent brane trajectory will work. Unfortunately, as we show in the appendix, it is not possible even to find a time dependent trajectory corresponding to a particle accelerating in the bulk. The reason why this approach fails (as well as that in [21]) becomes apparent once we think of the difference in the causal structure of the adS and Schwarzschild-adS spacetimes. In the former, we have an infinite family of oscillating branes, whereas the latter has simply one copy of the brane. The trajectory of the brane therefore cannot be a simple perturbation of the pure adS trajectory, which oscillates periodically in $\tau$. Indeed, if one were to try this approach, one would be trying to consider the cumulative effect of the attractive central potential generated by the Schwarzschild source over an infinite number of oscillations! It becomes clear that (52) is an inappropriate metric to use in this context when one tries to use it as a far-field approximation, i.e. assuming that the matter source is extended in such a fashion as to avoid an event horizon. A brane-world observer sitting outside the domain of influence of this accelerating particle (the past light cone of $t = -k^{-1}$ for example) should see no effect, however, the metric (52) still induces a nonzero perturbation in this region, which is of course due to the infinitely extended nature of the maximal adS spacetime. This behaviour should be contrasted with the causal behaviour of the metric perturbation derived above.

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APPENDIX

In this appendix, we show that it is not possible to use the Schwarzschild-adS solution (52) to construct a spacetime corresponding to a brane with an accelerating particle in the bulk. To do this, we need only two of the four separate Israel conditions appropriate to the hypersurface defined by the brane:

$$X^a = (\tau, \rho, \chi(\tau, \rho), \theta, \phi)$$  \hspace{1cm} (54)

with normal

$$n_a = (\dot{\chi}, \dot{\chi}', -1, 0, 0)/n$$  \hspace{1cm} (55)

where $$n^2 = 1/\rho^2 + A^2\chi'^2 - \dot{\chi}^2A^{-2}$$, writing $$A^2 = g_{tt}$$ for convenience. The $\theta$ and $\phi$ Israel conditions for the brane are identical, and give

$$\frac{(\cos \chi - \rho A^2 \chi' \sin \chi)}{n\rho^2 \sin \chi} = k$$  \hspace{1cm} (56)

In order to get the remaining Israel conditions, rather than working with the fundamental forms of the brane hypersurface, it is easier to generalise the technique used in Ipser and Sikivie, [23], and use the normal jump in the parallel derivatives of the unit vectors corresponding to the remaining time and space-like directions on the brane:

$$u^a = \dot{X}^a/|\dot{X}|, \quad v^a = X'^a/|X'|$$  \hspace{1cm} (57)

which satisfy

$$n_a \nabla_a u^a = -k, \quad n_a \nabla_a v^a = k, \quad n_a \nabla_a u^a = n_a \nabla_a v^a = -ku^a v_a$$  \hspace{1cm} (58)

This last relation giving

$$-\dot{\chi}' + \dot{\chi} \left( \frac{A'}{A} - A^2 \rho \chi'^2 - \frac{1}{\rho} \right) = k\rho^2 \dot{\chi}\chi' n$$  \hspace{1cm} (59)

Combining (56) and (59) requires

$$\dot{\chi} \sin \chi = \frac{A}{\rho} f(k\tau) \Rightarrow \cos \chi = -A f(k\tau) + C(\rho)$$  \hspace{1cm} (60)

where $F = f d\tau f$. But then (56) implies

$$f^2 + F^2 - 2\frac{AF}{k\rho} (C + \rho C') \left(1 + \frac{2\mu}{k^4 \rho^4} \right) + A^2 (C + \rho C')^2 + \frac{\mu}{k^4 \rho^4} \left(3F^2 - C^2 - \rho^2 A^2 C'^2 \right) = 1$$  \hspace{1cm} (61)

It is not difficult to see that since $f, F$ are functions of $\tau$ this cannot be satisfied once $\mu \neq 0$. For $\mu = 0$ the solution is of course the adS brane trajectory: $f = -\sin k\tau, C = -1/k\rho$.

We should also note that this approach is distinct from that of Kraus and Ida [24], who use the Schwarzschild-adS spacetime to obtain a cosmological brane world, i.e. a homogeneous universe with an evolving scale factor, which we would not expect to obtain from a localised particle accelerating in the bulk.
REFERENCES