Tachyon couplings on non-BPS D-branes and Dirac-Born-Infeld action

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ABSTRACT

By explicit evaluation of certain disk S-matrix elements in the presence of background B-flux, we find coupling of two open string tachyons to gauge field, graviton, dilaton or Kalb-Ramond antisymmetric tensor on the world-volume of a single non-BPS Dp-brane. We then propose an extension of the abelian Dirac-Born-Infeld action which naturally reproduces these couplings in field theory. This action includes non-linearly the dynamics of the tachyon field much like the other bosonic modes of the non-BPS Dp-brane. On the general grounds of gauge and T-duality transformations and the symmetrized trace prescription, we then extend the abelian action to non-abelian cases.

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1 Introduction

Recent years have seen dramatic progress in the understanding of non-perturbative aspects of string theory [1]. With these studies has come the realization that solitonic extended objects, other than just strings, play an essential role. An important object in these investigations has been Dirichlet branes [2]. D-branes are non-perturbative states on which open string can live, and to which various closed strings including Ramond-Ramond states can couple.

Type II string theories have two kind of D-branes, BPS Dp-branes for $p$ even(odd) [2] and non-BPS Dp-branes for $p$ odd(even) [3] in IIA(IIB) theory. The BPS branes are stable solitons which break half of the space-time supersymmetries and their dynamics are properly described in field theory by Dirac-Born-Infeld(DBI) action [4] (see also [5]) and Chern-Simons action [6]. The non-BPS Dp-branes on the other hand suffer from open string tachyonic mode whose mass causes the brane in a flat background to be unstable. However, there are other terms in the tachyon potential which makes it bounded from below. Consequently, the non-BPS branes decay to minimum of the tachyon potential. It has been conjectured that at the stationary point of the tachyon potential, the negative minimum energy of the tachyon potential plus the positive energy of the brane tension is exactly zero [7]. Hence, the unstable non-BPS branes in flat space-time vacuum should decay to the true vacuum of the theory in which there is no branes. This conjecture was studied in [8] by explicit calculation of the tachyon potential using the string field theory framework.

The dynamics of massless bosonic excitations of non-BPS Dp-branes are suitably described by the DBI action in field theory. This action has been generalized to the supersymmetric form to include the dynamics of massless fermionic modes of the branes as well [9]. The RR fields of the type II theory have no coupling to the non-BPS D-branes through the usual Chern-Simons action. However, there is a non-vanishing coupling between the RR field and tachyon on the world-volume of the branes [10]. The Chern-Simons action hence was modified in [11] to incorporate this coupling. In the present paper, on the other hand, we are interested in generalizing the DBI action to take into account the dynamics of the tachyon field. We study this by explicit evaluation of some nontrivial disk S-matrix elements in the first quantized string theory. From these matrix elements we conjecture an extension for the DBI action which includes the tachyon field as well.

An outline of the paper is as follows. We begin in section 2 by expressing our conjecture for extension of the DBI action which includes dynamics of the tachyon field. Then we expand this action around a background B-flux to produce various couplings involving two tachyons and one gauge field, graviton, dilaton or Kalb-Ramond antisymmetric tensor. In Section 2.1 we transform the above couplings between commutative fields to their non-commutative counterparts. We do this because the disk S-matrix elements in the presence of the background B-flux with which we are going to compare our conjectured action in the
subsequent section are corresponding to non-commutative open string fields [12]. In Section 3, we evaluate the S-matrix elements and check their consistency with the proposed field theory couplings. In Section 4 we extend our proposed action for describing dynamics of a single non-BPS Dp-brane to the case of the non-abelian theory of N coincident branes using the general grounds of the symmetrized trace, and non-abelian gauge and invariance under T-duality transformations. Appendix contains our conventions and some useful comments on conformal field theory propagators and vertex operators used in our calculations.

2 Abelian action

The world-volume theory of a single non-BPS D-brane in type II theory includes a massless U(1) vector \( A_a \), a set of massless scalars \( X^i \), describing the transverse oscillations of the brane, a tachyonic state \( T \) and their fermionic partners (see, e.g., [10]). The leading order low-energy action for the massless fields corresponds to a dimensional reduction of a ten dimensional U(1) Yang Mills theory. As usual in string theory, there are higher order \( \alpha' = \ell_s^2 \) corrections, where \( \ell_s \) is the string length scale. As long as derivatives of the field strengths (and second derivatives of the scalars) are small compared to \( \ell_s \), then the action takes a Dirac-Born-Infeld form [4]. To take into account the couplings of the massless open string states with closed strings, the DBI action may be extended naturally to include massless Neveu-Schwarz closed string fields, i.e., the metric, dilaton and Kalb-Ramond filed. In this case one arrives at the following world-volume action:

\[
S = -T_p \int d^{p+1} \sigma \ e^{-\Phi} \sqrt{-\det(P[G_{ab} + B_{ab}] + 2\pi \alpha' F_{ab})}.
\]

Here, \( F_{ab} \) is the abelian field strength of the world-volume ordinary gauge field, while the metric and antisymmetric tensors are the pull-backs of the bulk tensors to the D-brane world-volume, e.g.,

\[
P[G_{ab}] = G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}
= G_{ab} + 2G_{i(a} \partial_{b)} X^i + G_{ij} \partial_a X^i \partial_b X^j \quad (1)
\]

where in the second line above we have used that fact that we are employing static gauge throughout the paper, i.e., \( \sigma^a = X^a \) for world-volume and \( X^i(\sigma^a) \) for transverse coordinates.

In order to extend this action to incorporate dynamics of the tachyonic mode as well, we shall evaluate some disk S-matrix elements in string theory and read from them various couplings involving the tachyon field. Our results indicate that the tachyon should appear in the following extension of DBI action:

\[
S = -T_p \int d^{p+1} \sigma \ e^{-\Phi} V(T) \sqrt{-\det(P[G_{ab} + B_{ab}] + 2\pi \alpha' F_{ab} + 2\pi \alpha' \partial_a T \partial_b T)}
\quad (2)
\]
where the tachyon potential is \( V(T) = 1 + 2\pi \alpha' m^2 T^2 / 2 + O(T^4) \) and the tachyon mass is \( m^2 = -1/2\alpha' \). In our conventions the tachyon field is dimensionless. The conjecture in [7] is that the tachyon potential is zero at the minimum of the potential, \( i.e., V(T_0) = 0 \). Hence, upon tachyon condensation at this point the abelian action (2) of the non-BPS brane becomes zero.

Note that appearance of the tachyon kinetic term and potential in (2) is similar in form to the kinetic term and potential of the transverse scalar fields in the non-abelian DBI action of \( N \) coincident BPS D-branes [13]. In this case though the kinetic term appears in the pull-back of the metric under the square root and the potential for scalar fields multiplies the square root in the DBI action (see eq. (22) for \( T = 0 \)).

We now continue backward, assuming the above action (2) and check its consistency with some S-matrix elements. To have nontrivial check, we shall evaluate disk amplitudes describing decay of two tachyons to dilaton, graviton or Kalb-Ramond antisymmetric tensor on the world-volume of a single non-BPS \( D_p \)-brane with background B-flux. The amplitude describing the world-volume coupling of two tachyons to gauge field will be evaluated as well. Therefore, we begin by expanding (2) for fluctuations around the background \( G_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = \mathcal{F}^{ab} \eta_{\mu\alpha} \eta_{\nu\beta}, \Phi = 0 \) and \( T = 0 \) to extract interactions expected from the proposed action (2). The fluctuations should be normalized as the conventional field theory modes which appear in the string vertex operators. As a first step, we recall that the graviton vertex operator corresponds to string frame metric. Hence, one should transform the Einstein frame metric \( G_{\mu\nu} \) to the string frame metric \( g_{\mu\nu} \) via \( G_{\mu\nu} = e^{\Phi/2} g_{\mu\nu} \). Now with our conventions for string vertex operators (see Appendix), the string mode fluctuations take the form

\[
\begin{align*}
g_{\mu\nu} &= \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \\
\Phi &= \sqrt{2\kappa}\phi \\
B_{\mu\nu} &= \mathcal{F}^{ab} \eta_{\mu\alpha} \eta_{\nu\beta} - 2\kappa h_{\mu\nu} \\
T &= \frac{1}{\sqrt{2\pi \alpha'} T_p} \\
A_a &= \frac{1}{2\pi \alpha' \sqrt{T_p}} a_a \\
X^i &= \frac{1}{\sqrt{T_p}} \lambda^i .
\end{align*}
\]

With these normalizations, the pull back of the Einstein frame metric becomes:

\[
P[G_{ab}] = \eta_{ab}(1 + \frac{\kappa}{\sqrt{2}} \phi) + 2\kappa P[h_{ab}] + \frac{1}{T_p} (1 + \frac{\kappa}{\sqrt{2}} \phi) \partial_a \lambda^i \partial_b \lambda_i + \cdots
\]

where the dots represents terms with two and more closed string fields.
Now it is straightforward, to expand eq. (2) using
\[
\sqrt{\det(M_0 + M)} = \sqrt{\det(M_0)}(1 + \frac{1}{2}\text{Tr}(M_0^{-1}M)) - \frac{1}{4}\text{Tr}(M_0^{-1}MM_0^{-1}M) + \frac{1}{8}(\text{Tr}(M_0^{-1}M))^2 + \ldots
\]
to produce a vast array of interactions. We are interested in the interactions quadratic in tachyon, and linear in massless open and closed string fluctuations. The appropriate Lagrangian are:

\[
\mathcal{L}_{2,0} = -c \left( \frac{1}{2}m^2 \tau^2 + \frac{1}{2}(V_S)^{ab} \partial_a \tau \partial_b \tau \right)
\]
\[
\mathcal{L}_{3,0} = -\frac{c}{2\sqrt{T_p}} \left( \frac{1}{2}m^2 (V_A)^{ab} f_{ba} \tau^2 + \frac{1}{2} V^{ab} f_{ba} V^{cd} \partial_c \tau \partial_d \tau - V^{ab} f_{ba} V^{cd} \partial_d \tau \partial_a \tau \right) = 0
\]
\[
\mathcal{L}_{2,1} = -\kappa c \left( (V^{ab}(h_{ba} - b_{ba}) + \frac{1}{2\sqrt{2}} (\text{Tr}(V) - 4)\phi) \left( \frac{1}{2} (V_S)^{ab} \partial_a \tau \partial_b \tau + \frac{1}{2} m^2 \tau^2 \right) - V^{ab}(h_{bc} - b_{bc} + \frac{1}{2\sqrt{2}} \phi \eta_{bc}) V^{cd} \partial_d \tau \partial_a \tau \right).
\]

where we have dropped some total derivative terms in \( \mathcal{L}_{3,0} \). In the above Lagrangian, \( f_{ab} = \partial_a a_b - \partial_b a_a \) and

\[
c \equiv \sqrt{-\det(\eta_{ab} + \mathcal{F}_{ab})} \quad , \quad V^{ab} \equiv \left( (\eta + \mathcal{F})^{-1} \right)^{ab},
\]

and \( V_S(V_A) \) is symmetric(antisymmetric) part of the V matrix above. It is important to note that the antisymmetric matrix \( V_A \) appears in total derivative terms in the Lagrangian \( \mathcal{L}_{3,0} \) which involves only open string fields.

In the case that the background B-flux is zero, the coupling of Kalb-Ramond field to tachyon in the second line of \( \mathcal{L}_{2,1} \) vanishes, and the graviton and dilaton couplings reduce to the natural coupling of these fields to the kinetic term of the tachyon field, \textit{i.e.}, \( e^{-\phi} G^{ab} \partial_a T \partial_b T \). In that way, there is no nontrivial coupling that confirms the conjectured action (2) is valid or not. So we continue our discussion for non-vanishing background B-flux.

The open string fields appearing in the DBI action (2) or (3) are ordinary commutative fields. Whereas, open string vertex operators in string theory with background B-flux correspond to non-commutative fields [12]. Hence, in order to compare the couplings in (3) with corresponding S-matrix elements, one has to transform the commutative fields in (3) to their non-commutative variables.
2.1 Change of variables

In [12] differential equation for transforming commutative gauge field to its non-commutative counterpart was found to be

$$\delta \hat{A}_a(\theta) = \frac{1}{4} \delta \theta^{cd} \left( \hat{A}_c * \hat{F}_{ad} + \hat{F}_{ad} * A_c - \hat{A}_c * \partial_d \hat{A}_a - \partial_d \hat{A}_a * A_c \right)$$

$$\delta \hat{F}_{ab}(\theta) = \frac{1}{4} \delta \theta^{cd} \left( 2 \hat{F}_{ac} * \hat{F}_{bd} + 2 \hat{F}_{bd} * \hat{F}_{ac} - \hat{A}_c * (\hat{D}_d \hat{F}_{ab} + \partial_d \hat{F}_{ab}) - (\hat{D}_d \hat{F}_{ab} + \partial_d \hat{F}_{ab}) * \hat{A}_c \right)$$

where the gauge field strength and product were defined to be

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i \hat{A}_a * \hat{A}_b + i \hat{A}_b * \hat{A}_a$$

$$\hat{f}(x) * \hat{g}(x) = e^{\frac{i}{2} \theta_{ab} \partial_y \partial_z} \hat{f}(y) \hat{g}(z) \big|_{y=z=x} .$$

These differential equations can be integrated perturbatively to find a relation between ordinary fields appearing in (3) and non-commutative fields corresponding to open string vertex operators. The result for abelian case that we are interested in is [14]:

$$A_a = \hat{A}_a - \frac{1}{2} \theta^{cd} \left( \hat{A}_c *' \hat{F}_{ad} - \hat{A}_c *' \partial_d \hat{A}_a \right) + O(\hat{A}^3)$$

$$F_{ab} = \hat{F}_{ab} - \theta^{cd} \left( \hat{F}_{ac} *' \hat{F}_{bd} - \hat{A}_c *' \partial_d \hat{F}_{ab} \right) + O(\hat{A}^3)$$

where the commutative multiplication *' operates as

$$\hat{f}(x) *' \hat{g}(x) = \frac{\sin(\frac{1}{2} \theta_{ab} \partial_y \partial_z)}{\frac{1}{2} \theta_{ab} \partial_y \partial_z} \hat{f}(y) \hat{g}(z) \big|_{y=z=x} .$$

In [14] we verified by explicit calculation of S-matrix elements of one massless closed and two open string states that the transformation (6) is exactly reproduced by perturbative string theory. Appropriate transformations for scalar fields such as the tachyon can be read from (6), i.e.,

$$T = \hat{T} + \theta^{cd} \hat{A}_c *' \partial_d \hat{T} + \cdots$$

$$\partial_a T = \partial_a \hat{T} - i [\hat{A}_a, \hat{T}]_M + \theta^{cd} \left( \hat{F}_{ca} *' \partial_b \hat{T} + \hat{A}_c *' \partial_a \partial_d \hat{T} \right) + \cdots$$

where dots represent terms which involve more than two open string fields. They produce couplings between more than three fields upon replacing them into (3) in which we are not interested.

The differential equation (5) expresses infinitesimal variation of linear field, e.g., $\delta \hat{A}_a$, in terms of infinitesimal variation of the non-commutative parameter, i.e., $\delta \theta^{cd}$. Upon integration (6), this transforms the linear commutative gauge field in terms of nonlinear combination of non-commutative fields. However, we are interested in transforming quadratic
combinations of commutative fields appearing in (3) in terms of non-commutative fields. Such a transformation, in principle, might be found from a differential equation alike (5) that expresses infinitesimal variation of the quadratic fields in terms of infinitesimal variation of non-commutative parameter. Upon integration, that would produced the desired transformation. In that way, one would find that not only the fields transform as in (7) but the multiplication rule between fields also undergo appropriate transformation. We are not going to find such a differential equation here. Instead, we simply note that the transformation for multiplication rule between two open string fields can be conjectured from the right hand side of eq. (6) to be

\[ fg|_{\theta=0} \rightarrow f \ast' g|_{\theta \neq 0} \]  

(8)

for \( f \) and \( g \) being any arbitrary open string fields. This transformation rule was confirmed in [14] by explicit evaluation of S-matrix elements of one massless closed and two open string states.

Now with the help of equation (7) and (8), one can transform the commutative Lagrangian (3) to non-commutative counterparts. In doing so, one should first using (8) replace ordinary multiplication of two tachyons by the \( \ast' \) multiplication. Then, using (7) the ordinary tachyon fields should be transformed to their non-commutative counterparts. The results are

\[ \hat{\mathcal{L}}_{2,0} = -c \left( \frac{1}{2} m^2 \hat{\tau} \hat{\tau} + \frac{1}{2} (V_{S}^{ab}) \partial_a \hat{\tau} \partial_{b} \hat{\tau} \right) + \frac{i c}{4 \pi \sqrt{\theta}} (V_{S}^{ab}) \partial_a \hat{\tau} [a_b, \hat{\tau}]_M + \cdots \]  

(9)

\[ \hat{\mathcal{L}}_{2,1} = -\kappa c \left( (V_{ab}^{ab})(h_{ba} - b_{ba}) + \frac{1}{2 \sqrt{2}} (\text{Tr}(V) - 4) \phi \left( \frac{1}{2} (V_{S}^{ab}) \partial_a \hat{\tau} \ast' \partial_{b} \hat{\tau} + \frac{1}{2} m^2 \hat{\tau} \ast' \hat{\tau} \right) \right. \]

\[ \left. -V_{ab}^{ab}(h_{bc} - b_{bc}) + \frac{\phi m_{bc}}{2 \sqrt{2}} V_{cd}^{cd} \partial_d \hat{\tau} \ast' \partial_{a} \hat{\tau} \right) + \cdots \]  

(10)

where ellipsis represent terms which have more than three fields. Here we have dropped some total derivative terms which appeared in \( \hat{\mathcal{L}}_{2,0} \) and also replaced \( \ast' \) between two non-commutative tachyons in (9) with ordinary multiplication rule because the difference is some total derivative terms. In the eq. (10) on the other hand, the difference between \( \ast' \) and ordinary multiplication rules is not just a total derivative terms. Note that upon inserting the transformation (7) into (3), the antisymmetric matrix \( (V_{A}^{ab}) \) appears in eq. (9) only in the \( \ast \) product terms\(^1\).

It is interesting to note that the symmetric part of the \((\eta + F)^{-1}\) matrix, \( i.e., V_{S} \), appears as the metric in (9) and the antisymmetric part appears in the definition of \( \ast \) product in the Moyal bracket. This is consistent with the conclusion reached in [12]. In our discussion, however, the symmetric part \( V_{S} \) appears naturally as a result of expanding the ordinary DBI action around the background B-flux, and the antisymmetric part \( V_{A} \) appears as a

\(^1\)Note that our conventions set \( \theta^{ab} = 4\pi(V_{A}^{ab}) \).
result of transforming commutative fields to non-commutative variables. In the Lagrangian (10), on the other hand, which involves open and closed string fields, both symmetric and antisymmetric matrices appear in its different coupling terms.

3 Scattering Calculations

The couplings in (9) and (10) should be reproduced by disk S-matrix elements of string theory if the proposed action (2) is going to be valid. In this section, we work at the string theory side and evaluate these couplings using the conformal field theory technique. We begin with the evaluation of the coupling of two tachyons to gauge or scalar fields.

3.1 Open-Open-Open couplings

In the world-sheet conformal field theory framework, the coupling of two tachyons to a gauge or scalar field is described properly by the correlation of their corresponding vertex operators inserted at the boundary of the disk world-sheet, that is

\[
A \sim (\zeta_3 \cdot \mathcal{G})_\mu \int dx_1 dx_2 dx_3 \langle : V_{-1}(2k_1 \cdot V^T, x_1) : \cdot V_{-1}(2k_2 \cdot V^T, x_2) \cdot : V_0^\mu(2k_3 \cdot V^T, x_3) : \rangle
\]

where the vertex operators are given in the Appendix. Using the world-sheet conformal field theory technique, it is not difficult to perform the correlators above and show that the integrand is invariant under \( SL(2, \mathbb{R}) \). Gauging this symmetry by fixing the positions of the vertices at arbitrary points, one finds \( A(\tau_1, \tau_2, \lambda_3) = 0 \) and

\[
A(\tau_1, \tau_2, a_3) = \frac{c \sin(\pi l)}{\pi \sqrt{T_p}} (k_1 \cdot V_S \cdot \zeta_3)
\]

where we have defined \( l \equiv -2k_1 \cdot V^T \cdot F \cdot V \cdot k_2 = 2k_1 \cdot V_A \cdot k_2 \). We have also normalized the amplitude by the appropriate coupling factor \(-c/2\pi \sqrt{T_p}\). The \( \sin(\pi l) \) factor above arises basically from two different phase factors corresponding to two distinct cyclic orderings of the vertex operators. Each phase factor stems from the second term of the world-sheet propagator (29). Using the fact that our conventions set \( \theta^{ab} = 4\pi V_A^{ab} \), it is not difficult to verify that the S-matrix element (11) is exactly reproduced by the second term in (9). At the same time, vanishing of \( A(\tau_1, \tau_2, \lambda_3) \) is consistent with (9).
3.2 Closed-Open-Open amplitudes

The amplitudes describing decay of two open string tachyons to one massless closed string NSNS mode is given by the following correlation:

\[ A \sim (\varepsilon_3 \cdot D)_{\mu\nu} \int dx_1 dx_2 d^2 z \langle V_0(2k_1 \cdot V^T, x_1) :: V_0(2k_2 \cdot V^T, x_2) :: V_{\mu 1}^\nu (p_3, z_3) :: V_{\nu 1}^\mu (p_3 \cdot D, z_3) : \rangle \]

where the closed string vertex operator inserted at the middle and open string vertex operators at the boundary of the disk world-sheet. Explicit form of the vertex operators in terms of world-sheet fields are given in the Appendix. Here again using appropriate world-sheet propagators, one can evaluate the correlations above and show that the integrand is \( SL(2, R) \) invariant. We refer the reader to Refs. [15, 16, 17] for the details of the calculations.

Gauging the \( SL(2, R) \) symmetry by fixing \( z_3 = i \) and \( x_1 = 1 \), one arrives at

\[
A \sim 2^{-s-2} \int dx \left( (2s + 1) \text{Tr}(\varepsilon_3 \cdot D) - \frac{8i k_2 \cdot V^T \cdot \varepsilon_3 \cdot D \cdot V \cdot k_1}{x - i} + \frac{8i k_1 \cdot V^T \cdot \varepsilon_3 \cdot D \cdot V \cdot k_2}{x + i} \right) \\
\times (x - i)^{s-l}(x + i)^{s+l}
\]

where the integral is taken from \(-\infty\) to \(+\infty\), and \( s = -p_3 \cdot V_S \cdot p_3 = -1/2 - 2k_1 \cdot V_S \cdot k_2 \). This integral is doable and the result is

\[
A = -\frac{i\kappa c}{2} \left( a_1(s + l) - a_2(s - l) \right) \frac{\Gamma(-2s)}{\Gamma(1 - s - l)\Gamma(1 - s + l)}
\]

(12)

where \( a_1 \) and \( a_2 \) are two kinematic factors depending only on the space time momenta and the closed string polarization tensor

\[
a_1 = -4k_2 \cdot V^T \cdot \varepsilon_3 \cdot D \cdot V \cdot k_1 \\
a_2 = (s + l) \text{Tr}(\varepsilon_3 \cdot D) + 4k_1 \cdot V^T \cdot \varepsilon_3 \cdot D \cdot V \cdot k_2.
\]

We have also normalized the amplitude (12) at this point by the coupling factor \(-i\kappa c/2\pi\). As a check of our calculations, we have inserted the dilaton polarization (31) into (12) and found that the result is independent of the auxiliary vector \( \ell^\mu \). The amplitude (12) has the pole structure at \( m_{\text{open}}^2 = n/\alpha'^2 \). This does not have tachyon pole which is consistent with the fact that coupling of three tachyons is zero. In fact due to the world-sheet fermions in the tachyon vertex operator, coupling of any odd number of tachyons is zero. Hence, the world-volume of the non-BPS Dp-branes has a \( Z_2 \) symmetry under which the tachyon changes sign.

3.2.1 Massless poles

Given the general form of the string amplitude in eq. (12), one can expand this amplitude as an infinite sum of terms reflecting the infinite tower of open string states that propagate on
the world-Volume of D-brane. In the domain where $s \to 0$, the first term of the expansion representing the exchange of massless string states dominate. In this case the scattering amplitude (12) reduces to

$$A = \frac{i \kappa c \sin(\pi l)}{4\pi s} (a_1 + a_2) + \cdots$$

(13)

where dots represent contact terms and the infinite series of massive poles. Making the appropriate explicit choices of polarizations, we find

$$A_s(\tau_1, \tau_2, \phi_3) = \frac{i \kappa c \sin(\pi l)}{8\pi \sqrt{2} s} \left( \frac{l}{2} (\text{Tr}(D) + 2) - 4k_1 \cdot V \cdot V \cdot k_2 \right) + 1 \leftrightarrow 2$$

(14)

$$A_s(\tau_1, \tau_2, h_3) = \frac{i \kappa c \sin(\pi l)}{4\pi s} \left( \frac{l}{2} \text{Tr}(\varepsilon_3 \cdot D) - 4k_1 \cdot V \cdot \varepsilon^T_3 \cdot V \cdot k_2 \right) + 1 \leftrightarrow 2.$$

Here $h_3$ stands for both graviton and Kalb-Ramond antisymmetric tensors. In writing explicitly the above massless poles, one finds some terms which are proportional to $s$ as well. We will add these terms which have no contribution to the massless poles of field theory to the contact terms in (16). The amplitudes (14) should be reproduced in $s$-channel of field theory. They can be evaluated in field theory as

$$A'_s(\tau_1, \tau_2, \phi_3) = (\tilde{V}_{\phi_3})^a (\tilde{G}_a)_{ab} (\tilde{V}_{\tau_1 \tau_2})^b$$

$$A'_s(\tau_1, \tau_2, h_3) = (\tilde{V}_{h_3})^a (\tilde{G}_a)_{ab} (\tilde{V}_{\tau_1 \tau_2})^b$$

(15)

where the propagator and the vertices can be read from expansion of (2) in terms of non-commutative fields. They are

$$\tilde{G}_a = \frac{i (V^{-1})^{ab}}{c s}$$

$$\tilde{V}_{\phi_3} = \frac{\sqrt{T_p} \kappa c}{2\sqrt{2}} \left( \frac{1}{2} (\text{Tr}(D) + 2)p_3 \cdot V^a - p_3 \cdot V \cdot V^a + V^a \cdot V \cdot p_3 \right)$$

$$\tilde{V}_{h_3} = \frac{\sqrt{T_p} \kappa c}{2\sqrt{2}} \left( \frac{1}{2} \text{Tr}(\varepsilon_3 \cdot D)p_3 \cdot V^a - p_3 \cdot V \cdot \varepsilon^T_3 \cdot V^a + V^a \cdot \varepsilon_3 \cdot V \cdot p_3 \right)$$

$$\tilde{V}_{\tau_1 \tau_2} = \frac{c \sin(\pi l)}{2\pi \sqrt{T_p}} k_1 \cdot V_S^a + 1 \leftrightarrow 2.$$

In writing the above propagator, we have used the covariant gauge $V^{ab}_s \partial_b \dot{A}_b = 0$. Replacing above propagator and vertices into (15), one finds exactly the string massless poles (14).

3.2.2 Contact terms

Having examined in detail the massless poles of string amplitude, we now extract the low energy contact terms of the string amplitude (12). Expanding the gamma function
appearing in this amplitude for $s \to 0$, one will find
\[
A = \frac{ikc}{2} \left( \frac{(a_1 + a_2)}{2\pi} \frac{\sin(\pi l)}{s} + \frac{(a_1 - a_2)}{2\pi l} \right)
+ (a_1 + a_2) \frac{\sin(\pi l)}{\pi l} \sum_{n=1}^{\infty} \zeta(2n+1) l^{(2n+1)} + k^2 O(s, l) \right).
\]
The factor $\sin(\pi l)/(\pi l)$ appears for all the contact terms which is consistent with the transformation of multiplication rule in (8). The second term of the first line above is the contact term with minimum number of momentum in which we are interested, that is,
\[
A_c = \frac{ikc \sin(\pi l)}{4\pi l} (a_1 - a_2)
\]
Inserting appropriate polarization (see Appendix) and adding the residue contact terms of the massless poles (13), one finds
\[
A_c(\tau_1, \tau_2, \phi_3) = \frac{ikc \sin(\pi l)}{8\pi \sqrt{2} l} \left( \frac{s}{2} (\text{Tr}(D) + 2) - 4k_1 \cdot V \cdot k_2 \right) + 1 \leftrightarrow 2
\]
\[
A_c(\tau_1, \tau_2, h_3) = \frac{ikc \sin(\pi l)}{4\pi l} \left( \frac{s}{2} \text{Tr}(\varepsilon_3 \cdot D) - 4k_1 \cdot V \cdot \varepsilon_3^T \cdot V \cdot k_2 \right) + 1 \leftrightarrow 2.
\]
where again $h_3$ stands for both graviton and Kalb-Ramond antisymmetric tensors. These contact terms are reproduced exactly by the Lagrangian in (10). The first terms in (17) by the terms in the first line of (10) and the second terms in (17) by the terms in the second line of (10). Note that, while the first terms in (17) can be reproduced in field theory by an action in which the tachyon kinetic term appears linearly like the one proposed in [9], the second terms in (17) can be reproduced only if the tachyon kinetic term appears nonlinearly in the determinant under the square root in the BDI action, i.e., eq. (2). This ends our illustration of consistency between disk S-matrix elements and the proposed action (2).

4 Non-abelian action

In this section we extend the proposed action (2) for a non-BPS Dp-brane to the case of N coincident non-BPS Dp-brane where the world-volume theory involves a U(N) gauge theory. Our guiding principle in constructing such a non-abelian action is that the action should be consistent with the familiar rules of T-duality. This guideline has been recently employed by Myers [13] to construct non-abelian DBI action. In this way, one should start with non-abelian action for D9-branes and then use some sort of T-duality transformations to convert the D9-brane action to non-abelian action for Dp-branes. Therefore, we begin by extending the abelian action (2) to non-abelian action for non-BPS D9-branes. In this case there is no scalar field corresponding to the transverse direction of D9-branes. Hence,
the non-abelian action may be constructed from the corresponding abelian case by simply extending the derivative of open string fields to its non-abelian covariant derivative \[18\], and a trace over the U(N) representations \[19\] (see also \[5\]), that is,

\[
S = -T_9 \int d^{10} \sigma \, \text{Tr} \left( e^{-\Phi} V(T) \sqrt{-\text{det}(G_{\mu\nu} + B_{\mu\nu} + 2\pi \alpha' F_{\mu\nu} + 2\pi \alpha' D_\mu T D_\nu T)} \right)
\]

(18)

where \(G_{\mu\nu}, B_{\mu\nu}\) and \(F_{\mu\nu}\) are the metric, antisymmetric tensor and the non-abelian gauge field strength, respectively, and

\[
D_\mu T = \frac{\partial T}{\partial \sigma^\mu} - i [A_\mu, T].
\]

This action is still incomplete without a precise prescription for how the gauge trace should be implemented. We expect that a prescription similar to that given for Born-Infeld action \[19\] should also be given here. That is, the gauge trace should be completely symmetric between all non-abelian expression of the form \(F_{\mu\nu}, D_\mu T\) and individual \(T\) appearing in the tachyon potential \(V(T)\).

Now we generalize (18) to the action appropriate for non-BPS Dp-branes for any \(p\). To this end, we apply familiar T-duality transformations rules to the non-abelian D9-brane action (18). T-duality transformations in \(i = p + 1, \ldots, 9\) directions of the D9-brane world-volume converts the D9-brane to Dp-brane, the gauge fields in those direction to \(\tilde{A}_i = X^i/2\pi \alpha'\) and leaves unchanged the tachyon, \(i.e., \tilde{T} = T\). The new scalar fields \(X^i\) are now transverse coordinates of the new Dp-brane. Under this transformation, the covariant derivative of tachyon becomes

\[
\tilde{D}_i T = \frac{\partial T}{\partial \sigma^i} - \frac{i}{2\pi \alpha'} [X^i, T].
\]

Using the fact that we are always working in the static gauge, the first term on the right hand side becomes zero because of the assumption implicit in the T-duality transformations that all fields must be independent of the coordinates \(\sigma^i\). Now adding this transformation to the T-duality transformation of massless fields (see e.g., \[13\]), one has complete list of T-duality transformations for the fields appearing in (18);

\[
\begin{align*}
\tilde{E}_{ab} &= E_{ab} - E_{ai} E^{ij} E_{jb}, & \tilde{E}_{ai} &= E_{aj} E^{ji}, \\
\tilde{E}_{ij} &= E^{ij}, & E_{ia} &= -E^{ij} E_{ja}, \\
\tilde{E}^2 &= e^{2\phi} \det(E^{ij}), & \tilde{D}_i \tilde{T} &= -\frac{i}{2\pi \alpha'} [X^i, T] \\
\tilde{F}_{ab} &= F_{ab}, & \tilde{F}_{ai} &= \frac{1}{2\pi \alpha'} D_a X^i, \\
\tilde{F}_{ij} &= -\frac{i}{(2\pi \alpha')^2} [X^i, X^j], & \tilde{F}_{ia} &= -\frac{1}{2\pi \alpha'} D_a X^i
\end{align*}
\]

(19)
where we have defined $E_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}$. Here $E^{ij}$ denotes the inverse of $E_{ij}$, i.e., $E^{ik}E_{kj} = \delta^i_j$. Under above T-duality transformations the determinant in (18) becomes

$$
\tilde{D} = \det \begin{pmatrix}
\tilde{E}_{ab} + 2\pi\alpha'F_{ab} + 2\pi\alpha'D_aTD_bT & \tilde{E}_{aj} + D_aX^j - iD_aT[X^j, T] \\
\tilde{E}_{ib} - D_bX^i - i[X^i, T]D_bT & \tilde{E}_{ij} - i\frac{1}{2\pi\alpha'}[X^i, X^j] - \frac{1}{2\pi\alpha'}[X^i, T][X^j, T]
\end{pmatrix}
$$

Manipulating the matrix inside the determinant, one finds

$$
\tilde{D} = \det \left( P[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij}E_{jb}] + 2\pi\alpha'F_{ab} + T_{ab} \right) \det(Q^i_j)\det(E^{ij}) \quad (20)
$$

where now the definition of the pull-back above is the extension of (1) in which ordinary derivative is replaced by its non-abelian covariant derivative. Here the matrices $Q^i_j$ and $T_{ab}$ are defined to be

$$
Q_{ij} = \delta_{ij} - \frac{i}{2\pi\alpha'}[X^i, X^k]E_{kj} - \frac{1}{2\pi\alpha'}[X^i, T][X^k, T]E_{kj}
$$

$$
T_{ab} = 2\pi\alpha'D_aTD_bT - D_aT[X^i, T](Q^{-1})_{ij}[X^j, T]D_bT - iE_{ai}(Q^{-1})_{ij}[X^j, T]D_bT - iD_aT[X^i, T](Q^{-1})_{ij}E_{jb}
$$

$$
- iD_aX^i(Q^{-1})_{ij}[X^j, T]D_bT - iD_aT[X^i, T](Q^{-1})_{ij}D_bX^j
$$

In equations (20) and (21), indices are raised and lowered by $E^{ij}$ and $E_{ij}$, respectively. Now replacing (20) into T-dual of (18) and using the transformation for dilaton field (19), one finds the final T-dual action

$$
\tilde{S} = -T_p \int d^{p+1}\sigma \times \text{Tr} \left( e^{-\Phi V(T)}V'(T, X^i)\sqrt{-\det(P[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij}E_{jb}] + 2\pi\alpha'F_{ab} + T_{ab})} \right) \quad (22)
$$

where we have defined $V'(T, X^i) = \sqrt{\det(Q^i_j)}$. This potential term is one for abelian case. If the tachyon field is set to zero, this action would get to the result of non-abelian action for $N$ coincident BPS Dp-branes [13]. In this case, the prescription for the gauge trace is studied in [13]. The trace is completely symmetric between $F_{ab}$, $D_aX^i$, $i[X^i, X^j]$ and individual $X^i$. The latter non-abelian field stems from non-abelian Taylor expansion of the closed string fields that appear in the DBI action [20]. Natural extension of this prescription for the trace in the action (22) is that the trace should be completely symmetric between all non-abelian expressions of the form $F_{ab}$, $D_aX^i$, $i[X^i, X^j]$, $X^i$, $D_aT$, $i[X^i, T]$ and individual $T$ of the tachyon potential.

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A  Perturbative string theory with background field

In perturbative superstring theories, to study scattering amplitude of some external string states in conformal field theory frame, one usually evaluate correlation function of their corresponding vertex operators with use of some standard conformal field theory propagators [21]. In trivial flat background one uses an appropriate linear $\sigma$-model to derive the propagators and define the vertex operators. In nontrivial D-brane background the vertex operator remain unchanged while the standard propagators need some modification. Alternatively, one may use a doubling trick to convert the propagators to standard form and give the modification to the vertex operators [15]. In this appendix we would like to consider a D-brane with constant gauge field strength / or antisymmetric Kalb-Ramond field in all directions of the D-brane. The modifications arising from the appropriate linear $\sigma$-model appear in the following boundary conditions [22]:

$$\partial_y X^a - i \mathcal{F}^a_{\ b} \partial_x X^b = 0 \quad \text{for} \quad a, b = 0, 1, \ldots p$$
$$X^i = 0 \quad \text{for} \quad i = p + 1, \ldots 9$$

(23)

where $\mathcal{F}_{ab}$ are the constant background fields, and these equations are imposed at $y = 0$. The world-volume (orthogonal subspace) indices are raised and lowered by $\eta^{ab}(N_{ij})$ and $\eta_{ab}(N_{ij})$, respectively. Now we have to understand the modification of the conformal field theory propagators arising from these mixed boundary conditions. To this end consider the following general expression for propagator of $X^\mu(z, \bar{z})$ fields:

$$< X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) > = -\eta^{\mu\nu} \log(z - w) - \eta^{\mu\nu} \log(\bar{z} - \bar{w})$$
$$- D^{\mu\nu} \log(z - \bar{w}) - D^{\mu\nu}(\bar{z} - w)$$

(24)

where $D^{\mu\nu}$ is a constant matrix. To find this matrix, we impose the boundary condition (23) on the propagator (24), which yields

$$\eta^{ab} - D^{ba} = \mathcal{F}^{ab} - \mathcal{F}^{ac} D^{bc} = 0$$

(25)

for the world-volume directions, $D^{ij} = -N^{ij}$ for the orthogonal directions, and $D^{ia} = 0$ otherwise. Now equation (25) can be solved for $D^{ab}$; that is

$$D_{ab} = 2(\eta - \mathcal{F})_{ab}^{(-1)} - \eta_{ab}$$
$$= 2V_{ba} - \eta_{ab}$$

(26)

(27)

\[8\]Our notation and conventions follow those established in [15]. So we are working on the upper-half plane with boundary at $y = 0$ which means $\partial_y$ is normal derivative and $\partial_x$ is tangent derivative. And our index conventions are that lowercase Greek indices take values in the entire ten-dimensional space-time, e.g., $\mu, \nu = 0, 1, \ldots, 9$; early Latin indices take values in the world-volume, e.g., $a, b, c = 0, 1, \ldots, p$; and middle Latin indices take values in the transverse space, e.g., $i, j = p + 1, \ldots, 8, 9$. Finally, our conventions set $\ell_s^2 = \alpha' = 2$. 

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where matrix $V$ is the dual metric that appears in the expansion of DBI action (4). Note that the $D^{\mu\nu}$ is orthogonal matrix, i.e., $D^{\mu}_\alpha D^{\nu\alpha} = \eta^{\mu\nu}$.

Using two dimensional equation of motion, one can write the world-sheet fields in terms of right- and left-moving components. In terms of these chiral fields, closed NSNS and open NS vertex operators are

$$V^{\text{NSNS}} = : V^\text{NS}_n(X(z), \psi(z), \phi(z), p) :: V^\text{NS}_m(\tilde{X}(\bar{z}), \tilde{\psi}(\bar{z}), \tilde{\phi}(\bar{z}), p) :$$

$$V^{\text{NS}} = : V^\text{NS}_n(X(x) + \tilde{X}(x), \psi(x) + \tilde{\psi}(x), \phi(x) + \tilde{\phi}(x), k) :$$

where $\psi^\mu$ is super partner of world-sheet field $X^\mu$ and $\phi$ is world-sheet superghost field. The indices $n, m$ refer to the superghost charge of vertex operators, and $p$ and $k$ are closed and open string momentum, respectively. In order to work with only right-moving fields, we use the following doubling trick:

$$\tilde{X}(\bar{z}) \longrightarrow D^\mu_\nu X^\nu(\bar{z}) \quad \tilde{\psi}^\nu(\bar{z}) \longrightarrow D^\mu_\nu \psi^\nu(\bar{z}) \quad \tilde{\phi}(\bar{z}) \longrightarrow \phi(\bar{z}) \ . \quad (28)$$

These replacements in effect extend the right-moving fields to the entire complex plane and shift modification arising from mixed boundary condition from propagators to vertex operators. Under these replacement, world-sheet propagator between all right-moving fields take the standard form [23] except the following boundary propagator:

$$< X^\mu(x_1) X^\nu(x_2) > = -\eta^{\mu\nu} \log(x_1 - x_2) + \frac{i\pi}{2} F^{\mu\nu} \Theta(x_1 - x_2) \quad (29)$$

where $\Theta(x_1 - x_2) = 1(-1)$ if $x_1 > x_2 (x_1 < x_2)$. Note that the orthogonal property of the $D$ matrix is an important ingredient for writing the propagators in the standard form. The vertex operators under transformation (28) becomes

$$V^{\text{NSNS}} = : V^\text{NS}_n(X(z), \psi(z), \phi(z), p) :: V^\text{NS}_m(D \cdot X(z), D \cdot \psi(z), \phi(z), p) :$$

$$V^{\text{NS}} = : V^\text{NS}_n(X(x) + D \cdot X(x), \psi(x) + D \cdot \psi(x), 2\phi(x), k) :$$

The vertex operator for massless NSNS and NS states and open string tachyon are

$$V^{\text{NSNS}} = (\varepsilon \cdot D)_{\mu\nu} : V^\text{NS}_n(p, z) :: V^\text{NS}_m(p \cdot D, \bar{z}) :$$

$$V^{\text{NS}} = (\zeta \cdot G)_{\mu} : V^\text{NS}_n(2k \cdot V^T, x) :$$

$$V^\tau = : V^\text{NS}_n(2k \cdot V^T, x) : \quad (30)$$

where $G^{ab} = (\eta^{ab} + D^{ab})/2 = \bar{V}^{ba}$ for gauge field, $G^{ij} = (\eta^{ij} - D^{ij})/2 = N^{ij}$ for scalar field and $G^{ai} = 0$ otherwise. The open string vertex operators in (0) and (-1) pictures are

$$V^\text{0}_0(k, x) = (\partial X^\mu(x) + ik \cdot \psi(x) \psi^\mu(x)) e^{ik \cdot X(x)}$$

$$V^\text{1}_1(k, x) = e^{-\phi(x) \psi^\mu(x)} e^{ik \cdot X(x)}$$

$$V^\text{0}_0(k, x) = ik \cdot \psi(x) e^{ik \cdot X(x)}$$

$$V^\text{-1}_1(k, x) = e^{-\phi(x)} e^{ik \cdot X(x)} \ .$$
The physical conditions for the massless open string and tachyon are

massless: \( k \cdot V_S \cdot k = 0 \), \( k \cdot V_S \cdot \zeta = 0 \)
tachyon: \( k \cdot V_S \cdot k = \frac{1}{4} \)

and for massless closed string are \( p^2 = 0 \) and \( p_\mu \varepsilon^{\mu \nu} = 0 \) where \( \varepsilon \) is the closed string polarization which is traceless and symmetric(antisymmetric) for graviton(Kalb-Ramond) and

\[
\varepsilon^{\mu \nu} = \frac{1}{\sqrt{8}} (\gamma^{\mu \nu} - \ell^\mu p^\nu - \ell^\nu p^\mu) \quad \ell \cdot p = 1
\]  

(31)

for the dilaton. Using the fact that \( D^{\mu \nu} \) is orthogonal matrix, one finds the following identities:

\[
\mathcal{G} \cdot \mathcal{G}^T = \mathcal{G}^S \quad (D \cdot \mathcal{G}^T)^{ab} = \mathcal{G}^{ab} \quad (D \cdot \mathcal{G}^T)^{ij} = - N^{ij}
\]

where the \( \mathcal{G}^S \) is symmetric part of the \( \mathcal{G} \) matrix.
References

    C. Vafa, “Lectures on Strings and Dualities,” hep-th/9702201;
    J. Polchinski, Rev. Mod. Phys. 68 (1996) 1245 [hep-th/9607050];


    A. Sen, JHEP 9809 (1998) 023 [hep-th/9808141];
    A. Sen, JHEP 9810 (1998) 021 [hep-th/9809111];


    W. Taylor, “D-brane effective field theory from string field theory,” hep-th/0001201;
    N. Moeller and W. Taylor, “Level truncation and the tachyon in open bosonic string
theory,” hep-th/0002237;
N. Berkovits, “The Tachyon Potential in Open Neveu-Schwarz String Field Theory,” hep-th/0001084;


