Large CP Violation, Large Mixings of Neutrinos and the $Z_3$ Symmetry

Takahiro Miura*, Eiichi Takasugi† and Masaki Yoshimura‡

Department of Physics, Osaka University
Toyonaka, Osaka 560-0043, Japan

Abstract
We present neutrino mass matrices which predict the atmospheric neutrino mixing to be almost maximal, $\sin^2 2\theta_{atm} > 0.999$, as well as the large solar neutrino mixing, $8/9 > \sin^2 2\theta_{sol} > 0.87$, and the large CP violation (the CP violation phase in the standard form is maximal $\delta = \pi/2$), based on the $Z_3$ symmetry.

*email address: miura@het.phys.sci.osaka-u.ac.jp
†email address: takasugi@het.phys.sci.osaka-u.ac.jp
‡email address: masaki@het.phys.sci.osaka-u.ac.jp
1 Introduction

The observation of the atmospheric neutrino by Super-Kamiokande[1] has shown the existence of the neutrino masses and the neutrino mixings. In particular, the data[1] show that the mixing between $\nu_\mu$ and $\nu_\tau$ is favored and

$$\sin^2 2\theta_{atm} \sim 1, \quad \Delta_{atm}^2 \equiv |m_3^2 - m_2^2| \sim 3.5 \times 10^{-3} \text{eV}^2.$$  (1)

The solar neutrino problem is now considered to be due to the $\nu_e$ and $\nu_\mu$ oscillation, but the information on masses and mixing angles is ambiguous. Now four solutions are possible[2]. The another crucial information is given by CHOOZ group[3] that gives

$$|(V_{MNS})_{13}| < 0.16,$$  (2)

where $V_{MNS}$ is the Maki-Nakagawa-Sakata (MNS)[4] neutrino mixing matrix.

If we interpret these information in the three generation mixing, we find

$$\sin^2 2\theta_{atm} \simeq \sin^2 2\theta_{23} s_{13}^4$$  (3)

and

$$|s_{13}| < 0.16,$$  (4)

where $\theta_{ij}$ is the mixing angle between the i-th and the j-th mass eigenstate neutrinos, $s_{ij} = \sin \theta_{ij}, c_{ij} = \cos \theta_{ij},$ and we used the standard parameterization of mixing matrix[5] given in Appendix A.

If we obtain the large $\sin^2 2\theta_{atm}$ with $|s_{13}| < 0.16$, we need $|s_{23}| \sim |c_{23}| \sim 1/\sqrt{2}$. Now we face the following questions:

(1) Why $|s_{23}| \sim |c_{23}| \sim 1/\sqrt{2}$?

(2) Why $s_{13}$ is so small?

(3) How large is the CP violation phase $\delta$?

In order to answer these questions, various mixing schemes have been proposed. Among them, the bi-maximal mixing matrix[6], $(V_{MNS})_{bi},$ and the democratic one[7],
demono, predict the large mixing for both the solar and the atmospheric neutrino mixings,

\[
(V_{\text{MNS}})_{bi} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} & -1 \\
\frac{1}{2} & \frac{1}{2} & -1
\end{pmatrix},
(V_{\text{MNS}})_{\text{demo}} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}.
\tag{5}
\]

The bi-maximal mixing matrix predicts \(\sin^2 2\theta_{\text{sol}} = 1\) and \(\sin^2 2\theta_{\text{atm}} = 1\), and the democratic mixing matrix predicts \(\sin^2 2\theta_{\text{sol}} = 1\) and \(\sin^2 2\theta_{\text{atm}} = 8/9\).

In our previous paper[8], we proposed a new type of the neutrino mixing matrix based on the democratic-type mass matrix which is derived from the \(Z_3\) invariant Lagrangian. This model predicts the most needed relation \(|s_{23}| = |c_{23}| = 1/\sqrt{2}\) and in addition \(\delta = \pi/2\), the largest CP violation phase in the standard parameterization. In this paper, we examine the democratic-type mass matrix further and explore the possibility of constructing more predictive models.

This paper is organized as follows: In Sec.2, we briefly review the democratic-type mass matrix and its predictions. We explain the origin of the mass matrix based on \(Z_3\) symmetry, by using the see-saw mechanism. In Sec.3, we consider the one Higgs doublet case and introduce the \(Z_3\) symmetry breaking terms. The predictions are discussed in detail. The summary is given in Sec.4.

2 The democratic-type mass matrix and the \(Z_3\) symmetry

In this section, we present another view of the democratic-type mass matrix and its origin. Throughout of this paper, we consider the neutrino mass matrix in the charged lepton mass eigenstate basis.

(a) The mixing matrix derived from the deformation from the tri-maximal mixing matrix

The tri-maximal mixing matrix was discussed extensively by many authors[9] and is
defined by

$$(V_{MNS})_{tri} \equiv V_T = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}. \quad (6)$$

This model predicts $|(V_{MNS})_{13}| = 1/\sqrt{3}$ which conflicts with the CHOOZ data[3], but it has an interesting property that it predicts the maximal value of the CP violation phase in the standard form, $\delta = \pi/2$, and the maximal CP violation, that is, the maximal value of the rephasing invariant Jarlskog parameter, $|J_{CP}|_{tri} = 1/6\sqrt{3}$.

In order to remedy the deficit of the model, it may be interesting to consider the deformation from the tri-maximal mixing matrix by an orthogonal matrix $O$,

$$V = V_T O. \quad (7)$$

The orthogonal matrix contains three angles so that naively we expect that the CP violation phase is expressed by three angles in a complicated expression. Contrarily to this expectation, we found[8]

$$|\sin \theta_{23}| = |\cos \theta_{23}| = \frac{1}{\sqrt{2}}, \quad \delta = \frac{\pi}{2} \quad (8)$$

and two Majorana phases are fixed uniquely independent of angle parameters in $O$. Explicitly, we found that the neutrino mixing matrix is

$$V_{MNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & -is_{13} \\ -\frac{s_{12}-ic_{12}s_{13}}{\sqrt{2}} & \frac{c_{12}+is_{12}s_{13}}{\sqrt{2}} & -\frac{c_{13}}{\sqrt{2}} \\ -\frac{s_{12}+ic_{12}s_{13}}{\sqrt{2}} & \frac{c_{12}-is_{12}s_{13}}{\sqrt{2}} & \frac{c_{13}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \quad (9)$$

The diagonal matrix $\text{diag}(1, 1, i)$ represents the Majorana phase matrix which is relevant to the purely lepton number violating processes such as the neutrinoless double beta decay[10]. One parameter out of three parameters in $O$ is converted to the phase which is absorbed by the redefinition of charged leptons. The brief derivation is given in Appendix A.
If we use the CHOOZ constraint, \(|(V_{MNS})_{13}| = |s_{13}| < 0.16\), the model predicts
\[
\sin^2 2\theta_{\text{atm}} = 4|(V_{MNS})_{23}|^2(1 - |(V_{MNS})_{23}|^2)
\]
\[
= 1 - s_{13}^4 > 0.999 ,
\]
\[
\frac{|J_{CP}|_{\text{our model}}}{|J_{CP}|_{\text{max}}} = \frac{3\sqrt{3}}{2} |\sin 2\theta_{12}s_{13}c_{13}^2| ,
\]
where \(\sin^2 2\theta_{12} \simeq \sin^2 2\theta_{\text{sol}}\) is determined by the solar neutrino data. Therefore, in our model, the size of the CP violation is determined by the solar neutrino mixing angle and \(|s_{13}|\) which has been bounded by the CHOOZ data. If we take the large mixing solution for the solar neutrino problem, \(\sin^2 2\theta_{\text{sol}} = 0.9\) and the largest allowed value for \(s_{13}\), \(|s_{13}| = 0.16\), our model predicts \(|J_{CP}|_{\text{our model}} = 0.38|J_{CP}|_{\text{max}}\).

(b) The possible origin of the mixing matrix in the form of \(V = V_T O\)

Let us consider what kind of mass matrix leads to \(V = V_T O\). In order to clarify the structure, we change the flavor eigenstate basis to the one which is obtained by transforming by \(V_T\) (hereafter we call it as the \(\psi\) basis),
\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix}
= V_T^\dagger
\begin{pmatrix}
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix}
= \frac{1}{\sqrt{3}}
\begin{pmatrix}
1 & \omega^2 & \omega \\
1 & \omega & \omega^2 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix}.
\]
(11)
The relation \(V = V_T O\) implies that when we look at the mass matrix in the \(\psi\) basis, the neutrino mass matrix should be a real symmetric matrix. Thus, we may parameterize
\[
\begin{pmatrix}
\tilde{m}_1 \\
\tilde{m}_2 \\
\tilde{m}_3
\end{pmatrix}
= V_T^\dagger m_\nu V_T
= \begin{pmatrix}
m_1^0 + \tilde{m}_1 & \tilde{m}_3 & \tilde{m}_2 \\
\tilde{m}_3 & m_2^0 + \tilde{m}_2 & \tilde{m}_1 \\
\tilde{m}_2 & \tilde{m}_1 & m_3^0 + \tilde{m}_3
\end{pmatrix},
\]
with the real parameters, \(m_i^0\) and \(\tilde{m}_i\). Here, \(\tilde{m}_\nu\) is the mass matrix in the \(\psi\) basis and \(m_\nu\) is the one in the flavor eigenstate basis. If \(\tilde{m}_\nu\) is a real symmetric matrix, then it is diagonalized by the orthogonal matrix \(O\) and thus the mixing matrix becomes \(V_T O\).

By inverting, we obtain the neutrino mass matrix \(m_\nu\) as
\[
m_\nu
= \frac{m_1^0}{3}
\begin{pmatrix}
1 & \omega^2 & \omega \\
\omega & \omega & \omega \\
\omega^2 & \omega & \omega
\end{pmatrix}
+ \frac{m_2^0}{3}
\begin{pmatrix}
1 & \omega & \omega^2 \\
\omega & \omega^2 & 1 \\
\omega^2 & 1 & \omega
\end{pmatrix}
+ \frac{m_3^0}{3}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\]
4
\[
+\tilde{m}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} + \tilde{m}_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} + \tilde{m}_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\] (13)

which we called the democratic-type mass matrix\[8\].

(c) The \(Z_3\) symmetric dimension five Lagrangian

We analyze the Lagrangian which gives the democratic-type neutrino mass matrix. From the transformation in Eq.(11), we define

\[
\Psi_1 = \frac{1}{\sqrt{3}}(\ell_e + \omega^2\ell_\mu + \omega\ell_\tau),
\]
\[
\Psi_2 = \frac{1}{\sqrt{3}}(\ell_e + \omega\ell_\mu + \omega^2\ell_\tau),
\]
\[
\Psi_3 = \frac{1}{\sqrt{3}}(\ell_e + \ell_\mu + \ell_\tau),
\] (14)

where \(\ell_i\) is the left-handed lepton doublet defined by, say, \(\ell_e = (\nu_{eL}, e_L)^T\). With the definition, \(\Psi_i = (\psi_i, e_i)^T\), the above relation is interpreted as the transformation from the flavor eigenstate basis to the \(\psi\) basis.

The fields \(\Psi_i\) behave as irreducible representations of \(Z_3\) symmetry under the permutation of \(\ell_e, \ell_\mu\) and \(\ell_\tau\),

\[
\Psi_1 \rightarrow \omega \Psi_1, \quad \Psi_2 \rightarrow \omega^2 \Psi_2, \quad \Psi_3 \rightarrow \Psi_3.
\] (15)

If we introduce two Higgs doublets that behave as

\[
H_1 \rightarrow \omega^2 H_1, \quad H_2 \rightarrow \omega H_2,
\] (16)

then we can construct the \(Z_3\) invariant dimension five effective Lagrangian as

\[
\mathcal{L}_y = -\left( (m_0^0 + \tilde{m}_1) (\Psi_1)^\dagger \Psi_1 \frac{H_1 H_1}{u_1^2} + (m_0^0 + \tilde{m}_2) (\Psi_2)^\dagger \Psi_2 \frac{H_2 H_2}{u_2^2} \right.
\]

\[
+ (m_0^0 + \tilde{m}_3) (\Psi_3)^\dagger \Psi_3 \frac{H_1 H_2}{u_1 u_2} \Bigg)
\]
\[
-2 \left( \tilde{m}_1 (\Psi_2)^\dagger \Psi_3 \frac{H_1 H_1}{u_1^2} + \tilde{m}_2 (\Psi_1)^\dagger \Psi_3 \frac{H_2 H_2}{u_2^2} + \tilde{m}_3 (\Psi_1)^\dagger \Psi_2 \frac{H_1 H_2}{u_1 u_2} \right),
\] (17)
where $u_i$ is the vacuum expectation value of the neutral component of $H_i$. After the Higgs fields acquire the vacuum expectation values, the neutrino mass matrix in the $\psi$ basis defined by Eq.(12) is obtained.

(d) The $Z_3$ symmetric Lagrangian and the see-saw mechanism

In addition to the $\Psi_i$ and two Higgs doublets, $H_1$ and $H_2$, we introduce the right-handed neutrinos, $\nu_{eR}$, $\nu_{\mu R}$ and $\nu_{\tau R}$ and one more Higgs $H_3$ which behave under the $Z_3$ as

$$\nu_{eR} \rightarrow \nu_{eR} \, , \, \nu_{\mu R} \rightarrow \omega \nu_{\mu R} \, , \, \nu_{\tau R} \rightarrow \omega^2 \nu_{\tau R} \, ,$$

$$H_3 \rightarrow H_3 \, .$$

Now the $Z_3$ invariant Yukawa interaction and the Majorana mass term for the right-handed neutrinos are given by

$$\mathcal{L}_D = -(a \overline{\nu_{eR}} H_1 \frac{u_1}{u_2} + b \overline{\nu_{\tau R}} H_2 \frac{u_2}{u_3} + d' \overline{\nu_{\mu R}} H_3 \frac{u_3}{u_3}) \Psi_1$$

$$- (c \overline{\nu_{\mu R}} H_1 \frac{u_1}{u_1} + d \overline{\nu_{eR}} H_2 \frac{u_2}{u_2} + f' \overline{\nu_{\tau R}} H_3 \frac{u_3}{u_3}) \Psi_2$$

$$- (e \overline{\nu_{\tau R}} H_1 \frac{u_1}{u_1} + f \overline{\nu_{\mu R}} H_2 \frac{u_2}{u_2} + b' \overline{\nu_{eR}} H_3 \frac{u_3}{u_3}) \Psi_3 + \text{h.c.} \, ,$$

where $u_i$ is the vacuum expectation value of $H_i$ and

$$\mathcal{L}_R = -M (\overline{\nu_{eR}})^C \nu_{eR} - M' (\overline{\nu_{\mu R}})^C \nu_{\mu R} + (\overline{\nu_{\tau R}})^C \nu_{\tau R} \, .$$

After the Higgs fields acquire the vacuum expectation values, the Dirac mass term, $m_D$ in the $\psi$ basis and Majorana mass term $M_R$ for $\nu_R$ are given by

$$m_D = \begin{pmatrix} a & d & b' \\ d' & c & f \\ b & f' & e \end{pmatrix} \, , \quad M_R = \begin{pmatrix} M & 0 & 0 \\ 0 & 0 & M' \\ 0 & M' & 0 \end{pmatrix} \, .$$

By the see-saw mechanism, the neutrino mass matrix for the left-handed neutrinos in the $\psi$ basis is given by

$$\tilde{m}_\nu = -m_D^T M_R^{-1} m_D = - \begin{pmatrix} \frac{a^2}{M} + \frac{b d e}{M'} & \frac{a d'}{M} + \frac{b c e}{M'} + \frac{d' f'}{M'} & \frac{a b'}{M} + \frac{b f}{M} + \frac{d e}{M'} \\ \frac{a d'}{M} + \frac{b c e}{M'} + \frac{d' f'}{M'} & \frac{d^2}{M} + \frac{2 d f'}{M'} & \frac{a b'}{M} + \frac{b f}{M} + \frac{d e}{M'} \\ \frac{a b'}{M} + \frac{b f}{M} + \frac{d e}{M'} & \frac{a b'}{M} + \frac{b f}{M} + \frac{d e}{M'} & \frac{b^2}{M} + \frac{2 f e}{M'} \end{pmatrix} \, .$$
If we parameterize

\[
\begin{align*}
\tilde{m}_1 &= \frac{a^2}{M} + 2 \frac{bd'}{M'} \\
\tilde{m}_2 &= \frac{d^2}{M} + 2 \frac{cf'}{M'} \\
\tilde{m}_3 &= \frac{b^2}{M} + 2 \frac{ef}{M'}
\end{align*}
\]

we obtain the neutrino mass matrix \( \tilde{m}_\nu \) in the \( \psi \) basis given in Eq.\((12)\). The ansatz is that all parameters \( a, b, c, d, e, f, b', d', f' \), \( M \) and \( M' \) are real. The Lagrangian in Eqs.\((20)\) and \((21)\) are the most general one to derive the democratic mass matrix, although it contains redundant parameters.

The minimal model is those which contain only two Higgs doublets, say, \( H_1 \) and \( H_2 \), where \( \tilde{m}_\nu \) contains six independent parameters to specify \( m^0_i \) and \( \tilde{m}_i \).

### 3 A restricted model -One Higgs case-

In order to construct the more predictive mass matrix, we try to reduce the number of parameters in the democratic-type mass matrix, \( m^0_i \) and \( \tilde{m}_i \). Here, we consider the possibility of one Higgs model.

If we keep only \( H_1 = H \) with \( \langle H \rangle = u \) in the Lagrangian \((20)\), then

\[
\mathcal{L}_D = \frac{-1}{u} \left( a \overline{\nu}_{eR} H \Psi_1 + c \overline{\nu}_{\mu R} H \Psi_2 + e \overline{\nu}_{\tau R} H \Psi_3 \right) + \text{h.c.}.
\]

Then, the Dirac mass term in the \( \psi \) basis is \( m_D = \text{diag}(a, c, e) \). After the see-saw mechanism, the left-handed neutrino mass matrix in the \( \psi \) basis is

\[
\tilde{m}_\nu = -m_D^T M_R^{-1} m_D = -\begin{pmatrix}
m^0_1 + \tilde{m}_1 & 0 & 0 \\
0 & 0 & \tilde{m}_1 \\
0 & \tilde{m}_1 & 0
\end{pmatrix}.
\]

In the flavor eigenstate basis

\[
m_\nu = V_T^* \tilde{m}_\nu V_T^T = \frac{m^0_1}{3} \begin{pmatrix}
1 & \omega^2 & \omega \\
\omega^2 & \omega & 1 \\
\omega & 1 & \omega^2
\end{pmatrix} + \tilde{m}_1 \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}.
\]
From Eq.(26), we see that the neutrino mass matrix $m_\nu$ is diagonalized by

$$V_1 = V_T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{28}$$

This mixing scheme is interesting, although this model is unrealistic because it predicts the degenerate mass, $m_2 = -m_3$.

(a) The model with the $Z_3$ symmetry breaking terms

We introduce the $Z_3$ symmetry breaking terms by assuming that they respect the $Z_2$ symmetry which is defined by

$$\Psi_1 \rightarrow -\Psi_1, \quad \nu_{eR} \rightarrow -\nu_{eR}, \tag{29}$$

and all other fields are unchanged. Then, the $Z_2$ invariant Lagrangian is given by

$$\mathcal{L}_{SB} = -\frac{1}{u} (a\bar{\nu}_{eR} H \Psi_1 + c\bar{\nu}_{\mu R} H \Psi_2 + c\bar{\nu}_{\tau R} H \Psi_3)$$

$$-\frac{1}{u} (f\bar{\nu}_{\mu R} H \Psi_3 + f'\bar{\nu}_{\tau R} H \Psi_2) \tag{30}$$

together with the Majorana mass term for the right-handed neutrinos given in Eq.(21).

This model gives the mass matrix in the $\psi$ basis in Eq.(12) with non-zero $m_1^0$, $\tilde{m}_1$, $m_2^0$ and $m_3^0$. In other words, in the flavor eigenstate basis, it is

$$m_\nu = \frac{m_1^0}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{pmatrix} + \tilde{m}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$$+ \frac{m_2^0}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} + \frac{m_3^0}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{31}$$

In the new basis which is obtained from the transformation by $V_1$ in Eq.(28), the mass matrix is

$$V_1^T m_\nu V_1 = \begin{pmatrix} m_1^0 + \tilde{m}_1 & 0 & 0 \\ 0 & \tilde{m}_1 + \frac{1}{2}(m_3^0 + m_2^0) & \frac{1}{2}(m_3^0 - m_2^0) \\ 0 & \frac{1}{2}(m_3^0 - m_2^0) & -\tilde{m}_1 + \frac{1}{2}(m_3^0 + m_2^0) \end{pmatrix}. \tag{32}$$
Thus, the mixing matrix which diagonalize $m_\nu$ is given by

$$V = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}
\begin{pmatrix}
\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & i
\end{pmatrix}
\begin{pmatrix}
c' & -s' \\
s' & c'
\end{pmatrix},$$

(33)

where $c' = \cos \theta'$ and $s' = \sin \theta'$ and

$$\tan \theta' = \frac{m_1^0 - m_2^0}{\tilde{m}_1 + \sqrt{\tilde{m}_1^2 + (\frac{m_1^0 - m_2^0}{2})^2}},$$

(34)

and neutrino masses are given by

$$m_1 = m_1^0 + \tilde{m}_1,$$

$$m_2 = \frac{m_3^0 + m_2^0}{2} + \sqrt{\tilde{m}_1^2 + \left(\frac{m_3^0 - m_2^0}{2}\right)^2},$$

$$m_3 = \frac{m_3^0 + m_2^0}{2} - \sqrt{\tilde{m}_1^2 + \left(\frac{m_3^0 - m_2^0}{2}\right)^2}.$$  

(35)

Here we take the convention, $\tilde{m}_1 > 0$, and also consider that $\tilde{m}_1 > |(m_3^0 + m_2^0)/2|$, $|(m_3^0 - m_2^0)/2|$, because $m_3^0$ and $m_2^0$ are the $Z_3$ symmetry breaking parameters. Then, we find $m_2 > 0$ and $m_3 < 0$. The parameter $\tilde{m}_1$ controls the average size of neutrino masses, and $m_3^0 + m_2^0$ does the mass splitting between $m_2$ and $m_3$, and the parameter $m_3^0 - m_2^0$ does the size of $(V_{MNS})_{13}$.

The MNS mixing matrix is explicitly given by

$$V_{MNS} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}}c' & i\frac{\sqrt{2}}{\sqrt{3}}s' \\
\frac{1}{\sqrt{6}}(c' + i\sqrt{3}s') & \frac{1}{\sqrt{6}}(\sqrt{3}c' + is') & 0 \\
\frac{1}{\sqrt{6}}(c' - i\sqrt{3}s') & \frac{1}{\sqrt{6}}(\sqrt{3}c' - is') & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & i
\end{pmatrix}. $$

(36)

It should be noted that the model predicts $\delta = \frac{\pi}{2}$ and the angle $\theta_{12}$ which is relevant to the solar neutrino mixing. We have

$$\sin^2 2\theta_{sol} = \frac{8}{9} c'^2, \quad \sin^2 2\theta_{atm} = 1 - \frac{4}{9} s'^4.$$ 

(37)

If we impose the CHOOZ bound, $|\sqrt{2/3s'}| < 0.16$, we have

$$\frac{8}{9} > \sin^2 2\theta_{sol} > 0.87, \quad \sin^2 2\theta_{atm} > 0.999.$$ 

(38)
Below, we consider the special cases.

(a-1) The model with $m_3^0 = m_2^0$

This model is realized by imposing the invariance under the permutation in addition to $Z_2$ symmetry as

$$
\nu_{\mu R} \leftrightarrow \nu_{\tau R}, \quad \Psi_2 \leftrightarrow \Psi_3.
$$

Then, the parameters in Eq.(30) are restricted as $f = f'$ and the condition $m_3^0 = m_2^0$ is realized. From Eq.(34), we have $\theta' = 0$ and neutrino masses are

$$
m_1 = m_1^0 + \tilde{m}_1, \quad m_2 = m_3^0 + \tilde{m}_1, \quad m_3 = m_3^0 - \tilde{m}_1.
$$

Thus, the model predicts the mixing matrix defined in Eq.(28). By parameterizing as $V = \text{diag}(1, \omega, \omega^2)V_{MNS}$, we find the MNS matrix is

$$
V_{MNS} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{3} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & i
\end{pmatrix},
$$

where the matrix diag$(1, -1, i)$ is the Majorana phase matrix.

This model predicts

$$
\sin^2 2\theta_{atm} = 1, \quad \sin^2 2\theta_{sol} = \frac{8}{9}.
$$

This mixing matrix is in contrast to the so called democratic mixing matrix$[7]$ which predicts $\sin^2 2\theta_{atm} = 8/9$ and $\sin^2 2\theta_{sol} = 1$.

(a-2) The model with $m_2^0 = 0$ or $m_3^0 = 0$

One way to realize this model is to consider two Higgs model by keeping $H_1$ and $H_2$ in Eq.(20). Then, we impose $Z_3 \times Z_2$ symmetry. By the $Z_3$ symmetry, we have $b' = d' = f' = 0$. Then, we impose the $Z_2$ symmetry given in Eq.(29) and we find that $b = e = 0$. These conditions give $m_2^0 = 0$. Similarly, $m_3^0 = 0$ case is realized. Since both cases give the same mixing matrix so that we consider $m_2^0 = 0$ case. Then, the mixing angle $\theta'$ is completely fixed by the ratio of neutrino masses, $m_2 > 0$ and $m_3 < 0$. 

10
Explicitly, we find with $V_{MNS} = V_{SF} \text{diag}(1, -1, i)$

$$V_{SF} = \left( \begin{array}{ccc}
\frac{1}{\sqrt{3}} & \frac{1}{\beta^{+2}} & \frac{1}{\beta^{+2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\beta^{+2}} & \frac{1}{\beta^{+2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\beta^{+2}} & \frac{1}{\beta^{+2}} \\
\end{array} \right) \left( \begin{array}{ccc}
1 \pm i & \sqrt{3} & 1 \\
1 \pm i & \sqrt{3} & 1 \\
1 \pm i & \sqrt{3} & 1 \\
\end{array} \right) \left( \begin{array}{ccc}
\beta^{+2} & \beta^{+2} & \beta^{+2} \\
\beta^{+2} & \beta^{+2} & \beta^{+2} \\
\beta^{+2} & \beta^{+2} & \beta^{+2} \\
\end{array} \right) \left( \begin{array}{ccc}
1 \pm i & \sqrt{3} & 1 \\
1 \pm i & \sqrt{3} & 1 \\
1 \pm i & \sqrt{3} & 1 \\
\end{array} \right)$$

(43)

where

$$\beta = \sqrt{\left| \frac{m_2}{m_3} \right| + \sqrt{\frac{m_3}{m_2}}} \geq 2$$

(44)

In the limit $m_2 = -m_3$, i.e., $\beta = 2$, the mixing matrix reduces to the mixing scheme in Eq.(41). Due to the CHOOZ bound, $\beta$ must be close to 2.

This model predicts

$$\sin^2 2\theta_{sol} = \frac{4\beta + 2}{9\beta} ,$$

$$\sin^2 2\theta_{atm} = \frac{4(\beta + 1)(2\beta - 1)}{9\beta^2} .$$

(45)

If we impose the CHOOZ bound

$$| (V_{MNS})_{13} | = \sqrt{\frac{1}{3} \beta - \frac{2}{3\beta}} < 0.16 ,$$

(46)

$\beta$ is restricted by $2 < \beta < 2.17$ which means

$$0.85 < \sin^2 2\theta_{sol} < \frac{8}{9} , \quad 0.999 < \sin^2 2\theta_{atm} < 1 ,$$

(47)

$$0.44 < \left| \frac{m_2}{m_3} \right| < 2.27 .$$

(48)

The effective mass for the neutrinoless double beta decay is given by

$$< m_\nu > = \frac{1}{3} \left| m_1 + \frac{\beta + 2}{\beta} m_2 + \frac{\beta - 2}{\beta} m_3 \right| \sim \frac{1}{3} | m_1 + 2m_2 | .$$

(49)

If we take the constraint $| < m_\nu > | < 0.3$ eV, then we find $< m_2 > \sim m_2 < 0.3$ eV for $m_1 m_2 > 0$ and $< m_2 > \sim m_2 < 0.9$ eV for $m_1 m_2 < 0$. Because $| (V_{MNS})_{13} | = \sqrt{(\beta - 2)/3\beta}$, the CP violation becomes larger when the mass splitting increases. The $m_3$ could be as large as 2 eV if $m_1 m_2 < 0$. 

4 Summary

In our previous paper[8], we proposed the democratic-type neutrino mass matrix and showed that this model predicts the quite interesting relations which are crucial to give the large atmospheric neutrino mixing and the large CP violation which are given in Eq.(9).

In this paper, we explored the mass matrices which inherit the attractive features we found, but give more predictions. In order to examine the origin of the democratic-type mass matrix further, we considered the Yukawa interaction and the Majorana mass matrix for the right-handed neutrinos and derive the neutrino mass matrix for the left-handed neutrinos.

We gave the most general $Z_3$ invariant Lagrangian with three Higgs doublets and showed that this Lagrangian gives the democratic-type mass matrix. Then, we restricted the number of the Higgs doublet to be one. By introducing the $Z_3$ symmetry breaking terms but keeping the $Z_2$ symmetry, we found the interesting model that predicts the large solar neutrino mixing also. The most prominent feature of the present model is the prediction that $\sin^2 2\theta_{atm}$ is very close to unity and also the large CP violation (the CP violation phase in the standard form $\delta = \pi/2$). These two features will be tested as unambiguous signals of the model.

We expect that our mass matrix is obtained at the right-handed mass $M_R$ scale. We did not discuss the effect of the renormalization in this paper. If $m_1 m_2 < 0$, our predictions are stable, while the substantial dependence is expected for $\sin^2 2\theta_{sol}$ for $m_1 m_2 > 0$ because of the degeneracy. The possibility that the large $\sin^2 2\theta_{sol}$ at $M_R$ reduces to the small value which explains the small mixing angle MSW solution is under estimation and will be published elsewhere.

Acknowledgment This work is supported in part by the Japanese Grant-in-Aid for Scientific Research of Ministry of Education, Science, Sports and Culture, No.11127209.
Appendix A: Derivation of Eq.(8)

Let us consider

\[ V \equiv V_T O \]
\[ = \frac{1}{\sqrt{3}} \begin{pmatrix}
O_{11} + O_{21} + O_{31} & O_{12} + O_{22} + O_{32} & O_{13} + O_{23} + O_{33} \\
\omega O_{11} + \omega^2 O_{21} + O_{31} & \omega O_{12} + \omega^2 O_{22} + O_{32} & \omega O_{13} + \omega^2 O_{23} + O_{33} \\
\omega^2 O_{11} + \omega O_{21} + O_{31} & \omega^2 O_{12} + \omega O_{22} + O_{32} & \omega^2 O_{13} + \omega O_{23} + O_{33}
\end{pmatrix}, \]

\hspace{1cm} (A.1)

where \( V_T \) is the tri-maximal mixing matrix in Eq.(6) and \( O \) is the orthogonal matrix.

Then, we find the constraints

\[ V_{2j} = V_{3j}^*, \quad (j = 1, 2, 3) \]

\hspace{1cm} (A.2)

because \( O_{ij} \) are real parameters. Now we define the standard form of the mixing matrix as

\[ V_{SF} = \begin{pmatrix}
 c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
 -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
 s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}, \]

\hspace{1cm} (A.3)

where \( s_{ij} = \sin \theta_{ij} \) and \( c_{ij} = \cos \theta_{ij} \). Since \( V \) is related to \( V_{SF} \) by the phase transformation as \( V = PV_{SF}P' \), the constraints are given by \( |(V_{SF})_{2j}| = |(V_{SF})_{3j}| \). From the constraint with \( j = 3 \), we obtain \( |s_{23}| = |c_{23}| \). From \( j = 1 \) and \( j = 2 \), together with \( |s_{23}| = |c_{23}| \), we find \( \cos \delta = 0 \).
References


   L. Wolfenstein, Phys. Rev. **D18** (1978) 958;