We show how shared entanglement, together with classical communication and local quantum operations, can be used to perform an arbitrary collective quantum operation upon \( N \) spatially-separated qubits. A simple teleportation-based protocol for achieving this, which requires \( 2(N - 1) \) ebits of shared, bipartite entanglement, is introduced. In terms of the total required entanglement, this protocol is optimal for even \( N \) in both the asymptotic limit and for one ‘one-shot’ applications. It is also optimal for odd \( N\neq3 \) under one-shot circumstances where only integer entanglement is allowed. We also obtain a lower bound on the asymptotically required entanglement for odd \( N \).

Interactions between physical systems involve the transmission of information between them. The future state of any subsystem will depend not only upon its own history, but also those of the remaining subsystems. This influence naturally involves the transmission of information.

In classical physics, this information is purely classical. In quantum physics, it is quantum information that is exchanged by the subsystems. Unlike classical information, quantum information cannot be copied \([1,2]\). This implies that any quantum information transferred to some system must be lost by its source in the process. If this transfer of information is incomplete, which it often is, the result is entanglement between the systems.

Entanglement forms a crucial link between classical and quantum information. Nowhere is this made more explicit than in the transmission of quantum information by teleportation \([3]\). As is well-known, this can be achieved by sending classical information and making use of entanglement shared by the sending and receiving locations.

Interactions between quantum subsystems are represented as collective operations on the state space of the entire system. In this Letter, we show how shared entanglement (SE), together with classical communication (CC) and local quantum operations (LQ), can be used to perform an arbitrary collective operation upon \( N \) 2-level quantum systems (qubits), using a simple teleportation-based protocol. This requires \( 2(N - 1) \) ebits of bipartite entanglement to be shared between the locations of the qubits.

Large amounts of entanglement are difficult to produce under controlled circumstances, so it is natural to enquire as to whether or not this figure is optimal. We show that, for even \( N \), it indeed is for ‘one-shot’ applications, where the operation is carried out only once. It is also optimal for odd \( N\neq3 \) when only integer amounts of entanglement may be used. We also examine this issue in the asymptotic limit \([6]\), where the operation is carried out a large number of times and we are interested in the average entanglement required per run of the operation. Here, the teleportation protocol is still optimal for even \( N \), and we obtain a lower bound on the entanglement required to carry out an arbitrary operation for odd \( N \).

We begin by considering the following scenario: there is a network of \( N \) laboratories, \( A_j \), where \( j = 1,\ldots,N \), each of which contains a qubit. We label these \( q_j \). The laboratories also share a certain amount of bipartite entanglement with each other. We shall refer to this as the resource entanglement. Each one also contains auxiliary quantum systems, allowing arbitrary local collective operations to be carried out upon the \( q_j \) and the local parts of the entangled systems. The laboratories can also send classical information to each other.

Let us define the resource entanglement matrix \( E_R = \{ E_{ij} \} \), where \( E_{ij} \) is the number of ebits shared by \( A_i \) and \( A_j \). This matrix is clearly symmetric, has non-negative real elements and zeros on the diagonal.

From this matrix, we can construct a graph, which we term the resource entanglement graph, \( G_E(V,E) \). The vertex set \( V \) is that of the laboratories \( A_j \), and the edge set \( E \) represents the bipartite entanglement shared among them. The edge joining vertices \( A_i \) and \( A_j \) has weight \( E_{ij}^R \), and an edge of weight zero is equivalent to no edge. The total resource entanglement is

\[
E_R = \frac{1}{2} \sum_{ij} E_{ij}^R \tag{1}
\]
We wish to use these resources to carry out an arbitrary collective operation upon the $q_j$. Perhaps the most natural way doing so is by teleportation. Teleportation of a qubit from one location to another costs 1 ebit of entanglement and requires 2 classical bits to be sent from the origin to the destination of the qubit [3].

We can consider the situation in which all laboratories share entanglement and have the resources for two-way classical communication with one particular laboratory. Let this laboratory be $A_1$. The other laboratories can teleport the states of their qubits to $A_1$. The operation can then be carried out locally at $A_1$. The final states of the other qubits can then be teleported back to their original laboratories, completing the operation.

This teleportation procedure requires each of the laboratories $A_2, \ldots, A_N$ to share 2 ebits of entanglement with $A_1$ and for 2 bits of classical information to be communicated each way between each of them and $A_1$. The elements of the corresponding resource entanglement matrix are

$$E^{ij}_R = 2(1 - \delta_{ij})(\delta_{i1} + \delta_{j1}).$$

The corresponding graph $G_E$ is depicted in figure (1). The total resource entanglement is

$$E_R = 2(N - 1).$$

Any quantum operation upon $N$ qubits can be performed using this method and thus, at least for the topology of entanglement in our protocol, the value of $E_R$ in Eq. (3) is sufficient.

This teleportation-based method for carrying out an arbitrary collective quantum operation upon $N$ spatially separated qubits requires $E_R = 2(N - 1)$ ebits of entanglement. Is this figure optimal, in the sense that no less bipartite entanglement will suffice?

We can pose this question in the following, alternative way: a network of laboratories $A_i$ possesses shared bipartite entanglement, described by the graph $G_E$. If the corresponding total resource entanglement is sufficient to enable any collective operation to be performed, then what lower bound must $E_R$ satisfy?

The first observation we shall make is that if any operation can be carried out using the entanglement described by $G_E$, then any graph obtained from $G_E$ by a permutation of the vertices also describes sufficient entanglement to carry out any operation. The permutation invariance of this sufficiency condition is intuitive. We will provide a proof of it elsewhere [4].

Consider the graph $\tilde{G}_E$ defined by

$$\tilde{G}_E = \sum_{P[V]} G_E(V).$$

This graph is obtained from $G_E$ by summing over all permutations $P$ of the vertex set $V$. By summing, we mean summing the entanglement represented by the weights of the edges. The resource entanglement matrix for this graph is easily obtained. The elements are

$$\tilde{E}^{ij}_R = \sum_{P[V]} E_R^{P(i),P(j)}.$$

This graph is regular and complete. These properties follow immediately from the fact that $G_E$, being defined as a sum over all vertex permutations, is itself permutation invariant.

The total resource entanglement for this graph, $\tilde{E}_R$, is easily evaluated in terms of the total resource entanglement of $G_E$. There are $N!$ permutations of the vertex set, so that $G_E$ describes $N!$ times as much entanglement as $G_E$, that is

$$\tilde{E}_R = N! E_R.$$  

All $N(N - 1)/2$ edges in this graph have the same weight. Denoting this weight simply be $e$, we obtain

$$e = 2(N - 2)! E_R.$$  

There are $N!$ permutations of the vertex set. The permutation invariance of the sufficiency condition then implies that the resources represented by $G_E$ can be used to perform any operation $N!$ times. By this, we mean the following: suppose that $A_i$ contains $N!$ qubits. We can then define $N!$ sets of qubits, where each contains one from each laboratory. It will be possible to perform the same operation separately upon each one of these sets.

We can solve for the minimum value of $E_R$ exactly when $N$ is even. Our approach makes use of the SWAP operation upon 2 qubits. Consider a pair of qubits, $\alpha$ and $\beta$, with respective initial states $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$. Then the SWAP operation, $U_S$, exchanges the states of these systems:

$$U_S |\psi_\alpha\rangle \otimes |\psi_\beta\rangle = |\psi_\beta\rangle \otimes |\psi_\alpha\rangle.$$  

FIG. 1. Resource entanglement graph for the teleportation protocol.
The main property of $U_S$ that will be important to us is that it can be used to create 2 ebits of entanglement. To see how, suppose that in the laboratory containing $\alpha$ there is another system, $\alpha_0$, and that these two systems are maximally entangled. Likewise, $\beta$ is maximally entangled with a neighbouring system $\beta_0$. If the SWAP operation is performed upon $\alpha$ and $\beta$, then $\alpha$ will become maximally entangled with $\beta_0$, and likewise with $\beta$ and $\alpha_0$. Two ebits of entanglement have been produced.

Let us introduce the pairwise-SWAP (PS) operation. This swaps the states of qubits $q_j$ and $q_{j+1}$, for each odd $j$. If we write $U_S^{j+1,j}$ as the two-particle SWAP operation exchanging the states of qubits $q_j$ and $q_{j+1}$, then the PS operation may be written as

$$U_{PS} = U_S^{N,N-1} \otimes U_S^{N-2,N-3} \otimes \ldots \otimes U_S^{2,1}. \quad (9)$$

This operation is depicted in figure (2). From the properties of the SWAP operation, we see that the PS operation be used to establish $N$ ebits of entanglement. Our aim is to use the resources contained in the graph $G_E$ to perform this operation $N$ times. This means producing $N!N$ ebits of entanglement. We wish to find the minimum value of $\epsilon$, and using Eq. (7), that of $E_R$, required to do so.

![Diagram of pairwise-SWAP (PS) operation for $N=4$.](image)

The PS operation can establish $2N!$ ebits between the locations of each swapped pair. Since entanglement cannot increase under LQCC operations, we require that the resource entanglement can be used to establish $2N!$ ebits between these pairs. Each pair already shares $Ne/2$ ebits of entanglement, and there is nothing to be gained by manipulating this entanglement in any way. The remaining shared entanglement, which is represented by the remaining edges in $G_E$, is the resource to be manipulated.

None of these ebits need contain any entanglement following the manipulation, and so we shall term this resource the *expendable entanglement*, $E_E$. By counting $e$ ebits for each of these edges, we find that the total expendable entanglement is

$$E_E = \epsilon \left[ \frac{N(N-1)}{2} - \frac{N}{2} \right]. \quad (10)$$

The first term here is the total entanglement, $E_R$, of the graph $G_E$, and the second is the resource entanglement between pairs whose states are to be swapped by the PS operation.

It is possible to move entanglement from the expendable edges to those joining the systems to be swapped by the PS operation. However, a cost is necessarily incurred in the process. In fact, the transfer of any given amount of entanglement in this manner, in the asymptotic limit, requires at least as much entanglement to be consumed.

A formal proof of this result, which holds even if the expendable entanglement is manipulated into multiparticle entangled states, will be given in forthcoming, more detailed publication [4]. It can be seen as a generalisation of the entanglement loss associated with entanglement swapping [5]. Here, there are 3 qubits, $\alpha$, $\beta$, and $\gamma$. The pairs $\alpha\beta$ and $\alpha\gamma$ are initially entangled. If, using LQCC operations, we wish to establish $\epsilon$ ebits of entanglement between $\beta$ and $\gamma$, then at least $2\epsilon$ ebits must be lost from the bipartite entanglement which initially exists between the other two pairs.

There are $Ne/2$ ebits of entanglement initially shared by the pairs of laboratories whose qubits are to be swapped. The minimum value of $\epsilon$ which is sufficient to create the required $N!N$ ebits saturates the inequality

$$N!N \geq \frac{Ne}{2} + \frac{E_E}{2}. \quad (11)$$

The first term on the r.h.s. is the initial entanglement between the pairs to be swapped, and the second, which is half of the expendable entanglement, is an upper bound on that which can be added to these pairs. Using Eq. (7), we find that

$$\epsilon \geq 4(N-1)! \! . \quad (12)$$

From this, and Eq. (7), we obtain the following inequality for the total resource entanglement of the original graph $G_E$:

$$E_R \geq 2(N-1). \quad (13)$$

This is precisely the total resource entanglement required by the teleportation protocol in Eq. (3).

Let us now examine the case of odd $N$. As before, we assume that the graph $G_E$ represents sufficient resources to perform any operation $N!$ times. The specific operation we shall consider here is the PS operation upon $N-3$ qubits, and a separate, cyclic permutation of the remaining three. We shall refer to this as the PS+CP operation, and it is illustrated in figure (3). For $N=3$, the operation has no pairwise-SWAP part, and consists of only a (cyclic) permutation among the three qubits.
Like the PS operation, the PS+CP can be used to establish \( N \) ebits of entanglement. The PS part can create \( N - 3 \) ebits. A further 3 ebits can be created using the cyclic permutation. With reference to figure (3), where this permutation takes \( \{5, 6, 7\} \) to \( \{7, 5, 6\} \), if the systems whose states are to be permuted are initially in the form of local maximally entangled states, then each pair of laboratories will share one ebit of entanglement following the cyclic permutation. This creates another 3 ebits, giving \( N \) in total.

Our aim, as before, is to obtain lower bounds on the minimum value of \( e \) from the assumption that \( G_E \) represents sufficient resources to perform this particular operation \( N! \) times.

The expendable entanglement \( E_E \) is then all entanglement between laboratories that need not exist after the PS+CP operation. This is a sum of two terms: the expendable entanglement within the subset of laboratories whose qubits will undergo the PS part of the operation, and all of the entanglement shared by this subset and the remaining 3 laboratories whose resident qubits will undergo the independent cyclic permutation. We find that, in total,

\[
E_E = e \left[ \frac{N^2}{2} - N - \frac{3}{2} \right].
\]  

Again, we find that at most half of this entanglement can be transferred, using LQCC operations, to the pairs of laboratories which can gain shared entanglement following the PS+CP operation. Thus, by performing this operation \( N! \) times, we find, by analogy with Eq. (11), that

\[
N!N \leq e \left[ \frac{N - 3}{2} + 3 \right] + \frac{E_E}{2}.
\]  

The inequality in (16) gives a lower bound on the entanglement resources required to perform any operation upon an odd number of qubits. Taking the limit as \( N \to \infty \), it is clear that it is asymptotic, from below, to that required by the teleportation protocol. It is also intermediate between the minimum resources required for \( (N+1) \) and \( (N-1) \) qubits.

The bounds on the resource entanglement in (13) and (16) were derived as asymptotic results: the fact that at most half of the resource entanglement can be transferred is an asymptotic result. By asymptotic [6], we mean that, given a very large number of sets of separated qubits, where the same, arbitrary operation is to be carried out on each one, these bounds give the minimum average entanglement that is required per run of the operation.

In practical situations, it is often the resources required to carry out an operation successfully just once that will be of interest. In general, the lower bound on the resource entanglement for this ‘one-shot’ scenario must be greater than or equal to the corresponding asymptotic lower bound.

For even \( N \), the bounds for both scenarios are easily seen to be equal. This follows from the fact that the asymptotic minimum resource entanglement, when distributed according to the resource entanglement graph \( G_E \) in figure (1), can be used to perform any collective operation once using the teleportation protocol.

When \( N \) is odd, the situation is slightly more complicated. The true minimum resource entanglement which is sufficient to carry out any collective operation upon an odd number of qubits will be at least as large as the bound in (16), in both the asymptotic and one-shot sce-
narios. If, however, in the one-shot case, we are con-
strained to use entangled systems which contain integer
numbers of ebits, then the bound in (16) must be rounded
up to the nearest integer, which, for \( N \neq 3 \), gives \( 2(N - 1) \)
ebits, which is the teleportation bound. This follows from
the inequality
\[
\frac{2(N - 1)}{1 + 3/N^2} \geq 2(N - 1) - 1, \quad (17)
\]
for \( N \geq 3 \), with the equality only being attained when
\( N = 3 \). Thus, for one-shot applications which use integer
entanglement, the teleportation protocol is also optimal
for odd \( N \neq 3 \), and thus for all \( N \neq 3 \). Whether or not the
teleportation protocol is optimal for \( N = 3 \) is unclear.

ACKNOWLEDGEMENTS.

We would like to thank Dr. N. Linden, Prof. O. Hirota
and Dr. M. Sasaki for interesting discussions. We also
wish to thank Dr. J. A. Vaccaro for help with some of the
figures. Part of this work was carried out at the Japanese
Ministry of Posts and Telecommunications Communications Research Laboratory, Tokyo, and we would like to
thank Prof. M. Izutsu for his hospitality. This work was
funded by the UK Engineering and Physical Sciences Re-
search Council and by the British Council.

Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895
(1993).
glement, Information and Multiparticle Quantum Oper-
ations’, In preparation.