Effectively classical quantum states for open systems

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Abstract

Notions of robust and "classical" states for an open quantum system are introduced and discussed in the framework of the isometric-sweeping decomposition of trace class operators. Using the predictability sieve proposed by Zurek, "quasi-classical" states are defined. A number of examples illustrating how the "quasi-classical" states correspond to classical points in phase space connected with the measuring apparatus are presented.
1. Introduction

Quantum mechanics, whose basic laws were formulated in the twenties, still remains the most fundamental theory we know. Although, it was originally conceived as a theory of atoms, it has shown a wide range of applicability, making it more and more evident that the formalism describes some general properties of Nature. However, despite its successes, there is still no consensus about its interpretation with the main questions being centered around the quantum measurements. Clearly, the existence of classical quantities which would allow to express the measurement results and explain the classical appearance of the macroscopic world is a fundamental problem in this matter. The standard explanation which states that, for example, the center-of-mass motion of a macroscopic object should be described by a narrow wave packet, well localized in both position and momentum, is not satisfactory. It still remains unanswered why such objects are represented by narrow wave packets, while the superposition principle allows the emergence of non-classical states as well. Moreover, measurement-like processes would necessarily produce such non-classical states, as in the infamous example of Schrödinger’s cat. A superposition of being dead and alive should produce an entirely new state, in the same sense as the superposition of $K$ meson and its antiparticle does.

The main reason for this annoying situation seems to be based on the assumption that it is possible to isolate systems from their environment. When we drop it as unjustified and consider quantum systems as open ones we obtain a new perspective for the understanding of the emergence of classical properties within the framework of open systems theory. This is the basic objective of the program of decoherence proposed by Zurek [1,2] and further developed in [3,4,5]. For a recent review of the subject and a wide range of references up to 1996 see [6]. Decoherence is a process of continuous measurement-like interaction between a system and its environment which results in limiting the validity of the superposition principle in the Hilbert space of the system. In other words, the environment destroys the vast majority of superpositions in short time, and, in the case of macroscopic objects, almost instantaneously. This leads to the appearance of environment-induced superselection rules, which precludes all but a particular subset of states from stable existence. On the other hand, it singles out a preferred set of states which behave in an effectively classical, predictable manner. Generalizing
this notion the predictability sieve was introduced [7,8] (see also [9]). It is a procedure, which systematically explores states of an open quantum system in order to arrange them and next put on a list, starting with the most predictable ones and ending with those, which are most affected by the environment. Clearly, the states being on the top of the list can be thought of as "classical" or "quasi-classical" ones.

In order to study decoherence, the analysis of the evolution of the reduced density matrix obtained by tracing out the environment variables is the most convenient strategy. If the interaction is such that the reduced density matrix becomes approximately diagonal in a particular basis (in the simplest case), then it is said that an environment-induced superselection structure has emerged. Generally, the procedure of tracing out environment variables, being the composition of a unitary automorphism with a conditional expectation, leads to a complicated integro-differential equation for the reduced statistical operator. However, for a large class of interesting physical phenomena we can derive, using certain limiting procedures, an approximate Markovian master equation for the reduced density matrix [10]. More recently, the derivation of the master equation for the reduced density matrix of a system coupled linearly to an ohmic, subohmic and supraohmic environment at arbitrary temperature has been obtained in [11].

Usually, when deriving the master equation for the reduced density matrices it is assumed that the quantum system interacts with the environment, which is another quantum system and hence is also described in terms of quantum mechanics. However, it should be pointed out here that such an assumption stemming from the thinking of quantum mechanics as a universal theory, can be replaced by a more general one. Sometimes, it is more useful and natural to treat the external degrees of freedom as a classical system described by a commutative algebra of functions. In this approach (see [12,13] for discrete classical systems and [14,15] for continuous ones) the evolution equation for the classical part is modified by the expectation value of some quantum observable while, at the same time, the Schrödinger unitary dynamics for the quantum subsystem is replaced by a dynamical semigroup of completely positive maps. Therefore, when we allow the quantum system to interact with its environment, then, regardless of the nature of this interaction, the evolution becomes dissipative, given by the Markovian master equation.

The loss of quantum coherence in the Markovian regime was established
in a number of open systems [16,17], giving a clear evidence of environment-
induced superselection rules. In a recent paper [18] a thorough mathematical analysis of the superselection structure associated to an environment-
induced semigroup was presented. It was achieved by the use of the isometric-
sweeping decomposition, which singles out a subspace of density matrices, 
on which the semigroup acts in a reversible, unitary way, and sweeps out the 
rest of statistical states. The purpose of this paper is to pursue that investiga-
tion with a particular emphasis on the analysis of the classicality of states. 
We put the notion of pointer states, previously introduced and discussed 
by Zurek and other authors, into a general framework, and examine their 
properties. It is worth noting that proposed definitions are expressed solely 
in terms of the dynamical semigroup and does not refer to any additional 
conditions like that one involving the knowledge of the state of the quantum 
system before the beginning of interaction. A number of examples including 
that of a quantum stochastic process of Davies, and illustrating how the 
“quasi-classical” states correspond to classical points in the underlying phase 
space are also presented.

2. ”Classical” states

One can show that purely unitary evolution can never resolve the apparent 
conflict between predictions it implies and perception of the classical reality. 
Therefore, in order to explain the appearance of classical (non-quantum) 
properties of a quantum system, we have to open the system and allow it 
to interact with the environment. As was mentioned in Introduction we 
restrict our considerations to the Markovian regime and thus assume that 
the evolution of the reduced density matrix is given by an environment-
induced semigroup. The concept of the environment-induced semigroup was 
introduced in [19], and the justification of the name was also given there. 
They form a subclass of dynamical semigroups (completely positive, trace 
preserving and contractive in the trace norm $\| \cdot \|_1$), which are also contractive 
in the operator norm $\| \cdot \|_\infty$. This additional property ensures that both the 
linear and statistical entropy of the open quantum system never decrease in 
the course of interaction [19].

In this section we search for quantum states which can correspond to the 
classical points in phase space once the interaction with the environment is 
acknowledged. Our strategy is as follows. We start with a quantum system
whose evolution is given by an environment-induced semigroup $T_t$ without asking question where it comes from. Then, using general principles and properties of that semigroup we determine sets of ”classical” and “quasi-classical” (see the next section) states. If they are empty and all states are stable, the system can be thought of as closed, evolving in a unitary way, and hence the semigroup may be extended to a one parameter group of unitary automorphisms. However, if one of them is non-trivial, then the system is open and we conclude that the selected states correspond with points in a classical phase space. Therefore, these sets contain the information about the type of interaction with the environment (measuring apparatus) which led to the appearance of the semigroup $T_t$.

At first, let us comment on crucial differences between states of classical and quantum systems. One of the most characteristic features distinguishing classical from quantum states is their sensitivity to measurements. In classical physics we could perform many kinds of measurements which would not disturb the system in an essential way. A measurement can increase our knowledge of the state of the system but, in principle, it has no effect on the system itself. By contrast, in quantum mechanics it is impossible to find out what the state is without, at the same time, changing it in the way determined by the measurement. According to the von Neumann projection postulate the outcome will be, in general, represented by a density matrix. Therefore, as a convenient measure of the influence of the environment on the state, we take the measure of the loss of its purity expressed in terms of the linear entropy $S_{\text{lin}}(\rho) = \text{tr}(\rho - \rho^2)$ [5,8].

Let $S$ denote the set of all states of the quantum system. By a state we always mean a pure state, whereas for a mixed state we reserve such notions like density matrix or statistical state. Hence $S$ consists of unit vectors from a Hilbert space $\mathcal{H}$ determined up to the phase factor $|\langle \psi \rangle|$ or, in other words, of one-dimensional projectors in $\mathcal{H}$. Hence $|\langle \psi \rangle|$ is the abstract class of unit vectors with respect to the following equivalence relation: $|\langle \psi \rangle| \equiv |\langle \psi' \rangle|$ if $|\langle \psi \rangle| = e^{i\alpha} |\langle \psi' \rangle|$ for some $\alpha \in \mathbb{R}$. Let us notice that the scalar product of two distinct states is not well defined but its absolute value is. Also the one-dimensional projector $|\langle \psi \rangle| < |\psi\rangle$ does not depend on the choice of a state vector $|\psi\rangle$.

In order to define a subset of robust (completely stable) states let us notice that any environment-induced semigroup $T_t$ determines two linear closed and $T_t$-invariant subspaces $\text{Tr}(\mathcal{H})_{\text{iso}}$ and $\text{Tr}(\mathcal{H})_s$ in the Banach space of all
trace class operators $\text{Tr}(\mathcal{H})$. The subspace $\text{Tr}(\mathcal{H})_{\text{iso}}$ is called the isometric part and $\text{Tr}(\mathcal{H})_{\text{s}}$ the sweeping part. For the reader convenience we recall here some basic results of this isometric-sweeping decomposition. For proofs and a more detailed discussion see [18]. The isometric and sweeping subspaces have the following properties:

a) $\text{Tr}(\mathcal{H})_{\text{iso}}$ and $\text{Tr}(\mathcal{H})_{\text{s}}$ are $^*$-invariant,
b) $\text{Tr}(\mathcal{H})_{\text{iso}} \perp \text{Tr}(\mathcal{H})_{\text{s}}$ in the following sense: $\forall \phi_1 \in \text{Tr}(\mathcal{H})_{\text{iso}} \forall \phi_2 \in \text{Tr}(\mathcal{H})_{\text{s}}$ we have $\text{tr}\phi_1 \phi_2 = 0$,
c) $\text{Tr}(\mathcal{H}) = \text{Tr}(\mathcal{H})_{\text{iso}} \oplus \text{Tr}(\mathcal{H})_{\text{s}}$, $T_t = T_{1t} \oplus T_{2t}$,
d) $T_{1t}$ is an invertible isometry given by a unitary group, i.e. $T_{1t}\phi = U_t \phi U_t^*$ for any $\phi \in \text{Tr}(\mathcal{H})_{\text{iso}}$,
e) $T_{2t}$ is sweeping, i.e. $w^* - \lim_{t \to -\infty} T_{2t}\phi = 0$ for any $\phi \in \text{Tr}(\mathcal{H})_{\text{s}}$, where $w^*$ denotes the weak$^*$ topology.

Hence, to any environment-induced semigroup corresponds a space of statistical states $\text{Tr}(\mathcal{H})_{\text{iso}}$ and associated with it an algebra of observables such that the evolution, when restricted to these spaces, is given by the Schrödinger unitary dynamics. In addition, $\text{Tr}(\mathcal{H})_{\text{iso}}$ has the following properties:

(i) if $\phi_1$, $\phi_2 \in \text{Tr}(\mathcal{H})_{\text{iso}}$, then also $\phi_1 \cdot \phi_2 \in \text{Tr}(\mathcal{H})_{\text{iso}}$,
(ii) if projectors $e$, $f \in \text{Tr}(\mathcal{H})_{\text{iso}}$, then also $e \lor f \in \text{Tr}(\mathcal{H})_{\text{iso}}$, where $e \lor f$ denotes a projector onto the two-dimensional subspace spanned by the ranges of $e$ and $f$.

As a consequence, any one-dimensional projector $e \in \text{Tr}(\mathcal{H})_{\text{iso}}$ remains a projector during the evolution, and so $S_{\text{lin}}(T_t e) = 0$ for any $t \geq 0$. Therefore, we define a subset $S_0$ of robust states by

$$S_0 = S \cap \text{Tr}(\mathcal{H})_{\text{iso}}$$

or, equivalently,

$$S_0 = \{e \in S : S_{\text{lin}}(T_t e) = S_{\text{lin}}(T_t^* e) = 0\}$$

where $T_t^*$ denotes the adjoint semigroup. If $T_t^*$ commutes with $T_t$, then

$$\text{tr}(T_t^* e)^2 = \langle T_t^* e, T_t^* e \rangle_{HS} = \langle T_t e, T_t e \rangle_{HS} = \text{tr}(T_t e)^2$$

where $\langle \cdot, \cdot \rangle_{HS}$ is the scalar product in the Hilbert space of Hilbert-Schmidt operators, and so the condition $S_{\text{lin}}(T_t e) = 0$ implies that also $S_{\text{lin}}(T_t^* e) = 0$. Therefore, in such a case for a state $e \in S$ to be robust it is enough that its
linear entropy does not change in the course of evolution. It is worth noting that for quantum systems over finite dimensional Hilbert spaces it also turns out that the condition $S_{\text{lin}}(T_t e) = 0$ alone is sufficient for state $e$ to be in $S_0$, see Appendix. If $S_0 = S$, then the semigroup $T_t$ may be extended to a group of unitary automorphisms.

Obviously, any state from $S_0$ will remain pure during the evolution and so remain in $S_0$. Therefore, elements from $S_0$ are the most probable candidates for “classical” states. But the unitary evolution and thus perfect predictability alone does not suffice to accomplish our goal. Another feature distinguishing quantum from classical states, namely the validity of the superposition principle, has to be taken into account. In quantum mechanics it guarantees that any superposition of two distinct, and not necessarily orthogonal, states is again a legitimate quantum state. It means that for any pair of different one-dimensional projectors $e_1$ and $e_2$ we can associate a set of one-dimensional projectors $e$ given by $e = (e_1 \vee e_2) = e$. Equivalently, we may write that

$$e = \frac{(z_1|\psi_1 > + z_2|\psi_2 >)(z_1^* < \psi_1 | + z_2^* < \psi_2 |)}{\|z_1|\psi_1 > + z_2|\psi_2 >\|^2}$$

where $|\psi_1 > < \psi_1 | = e_1$, $|\psi_2 > < \psi_2 | = e_2$ and $z_1, z_2$ are complex numbers. By contrast, classical states do not combine into another state. The only situation when their combination can be considered is inevitably tied to probability distributions on the phase space. Therefore, it is natural to assume that any non-trivial superposition of ”classical” states cannot be robust.

**Definition 2.1.** A state $e \in S$ is called ”classical” if $e \in S_0$ and for any $f \in S_0$, $f \neq e$, $S(e, f) \cap S_0 = \emptyset$, where $S(e, f)$ denotes the collection of all states being non-trivial superpositions of $e$ and $f$. The collection of all ”classical” states we denote by $S_c$.

Hence, although ”classical” states remain pure during the evolution, any of their superpositions deteriorates into a mixture. Under a mild, technical assumption namely that $T_t$ admits a holomorphic extension to a sector $\Sigma_\epsilon = \{z: \text{Re}z > 0, |\text{arg}z| < \epsilon\}$, for some $\epsilon > 0$, the loss of the purity of their superpositions happens instantaneously.

We are now in position to describe the structure of set $S_c$.

**Theorem 2.2.** If $S_c \neq \emptyset$, then it consists of a family, possibly finite, of pairwise orthogonal states $\{e_1, e_2, \ldots\}$ such that $T_t e_i = e_i$ for all $t \geq 0$ and
any index $i$.

Proof: Let $e \in S_c$. We show that $e$ is orthogonal to any state $f \in S_0$, $f \neq e$. Suppose, on the contrary, that $e \cdot f \neq 0$. Then, because $e \lor f \in \text{Tr}(\mathcal{H})_{\text{iso}}$, the state $e' = e \lor f - e$ also belongs to $\text{Tr}(\mathcal{H})_{\text{iso}}$ and is orthogonal to $e$. All of these states can be considered as acting on a two-dimensional Hilbert space, the range of $e \lor f$. Choosing an appropriate coordinate system we represent them by

$$e = \frac{1}{2}(I + \vec{m}_1 \cdot \vec{\sigma}), \quad f = \frac{1}{2}(I + \vec{m}_2 \cdot \vec{\sigma})$$

with $\vec{m}_1 = (0, 0, 1)$ and $\vec{m}_2 = (\cos \theta, 0, \sin \theta)$, $\theta \in [0, \pi/2)$. Because the hermitian matrix $i[e, f] \in \text{Tr}(\mathcal{H})_{\text{iso}}$ is non-zero so its spectral projectors $\frac{1}{2}(I \pm \vec{m} \cdot \vec{\sigma})$, where $\vec{m} = (0, 1, 0)$, also belong to $\text{Tr}(\mathcal{H})_{\text{iso}}$. On the other hand, they are superpositions of $e$ and $e'$. Therefore, $S(e, e') \cap S_0 \neq \emptyset$, what contradicts the assumption that $e$ is "classical". Hence $e \perp f$ for any $f \in S_0$, $f \neq e$ and so $S_c = \{e_1, e_2, \ldots\}$ with $e_i \cdot e_j = \delta_{ij}e_i$. Finally, we show that $T_t e = e$ for all $t$. If not so, then for any $\epsilon > 0$ we find an instant $s$ such that $T_s e \neq e$ and $\|T_s e - e\|_1 < \epsilon$. However, by the above argument, $T_s e$ is orthogonal to $e$, so $\|T_s e - e\|_1 = 2$, the contradiction. □

Therefore, it turned out that "classical" states, which are defined in a general way, form so-called pointer basis being introduced so far only on the operational level. Let us recall that pointer basis arises in a specific situation when before the measurement the quantum system was in an eigenstate of the measured observable. Such states are completely predictable since they do not evolve at all. By Theorem 2.2, they always correspond to points in a discrete classical phase space.

It is also clear that a unitary evolution $T_t = e^{-itH} \cdot e^{itH}$ with $H = H^*$, does not lead to the appearance of "classical" states at all. Although $S_0 = S$ in this case, $S_c = \emptyset$ since any superposition of robust states is again robust. It is worth noting that, in general, even if "classical" states exist, they may form an incomplete set of one-dimensional projectors.

3. “Quasi-classical” states

In this section we continue the investigation of states of an open quantum system which offer optimal predictability of their own future values. In the case when "classical" states exist, they are the best candidates for states corresponding to classical points of phase space. If they are absent, it is
natural to consider the states which are least affected by the interaction with the environment, that is, which are least prone to deteriorate into mixtures. Since linear entropy is a convenient measure of the loss of purity, we take its increase for initial states as a basic criterion. For a more complete analysis one should search for states which minimize the linear entropy over some finite period of time characteristic for the evolution of the system. To start with we define a quadratic form on the Hilbert space $\text{HS}(\mathcal{H})$ of Hilbert-Schmidt operators

$$B(\phi) = -<\phi, L(\phi)>_{HS}$$

where $\phi \in D(L) \subset \text{Tr}(\mathcal{H})$ and $L$ denotes the generator of semigroup $T_t$. Since $\text{Tr}(\mathcal{H})$ is dense in $\text{HS}(\mathcal{H})$ so $B$ is densely defined. The closure of its symmetric part we denote by $\lambda$. By the Hille-Yosida theorem, the Lumer-Philips form, $\lambda$ is positive definite. It is clear that

$$\lambda(\phi) = \frac{1}{2\frac{d}{dt}S_{\text{lin}}(T_t\phi)|_{t=0}}$$

whenever the corresponding derivative exists. Let

$$S(a) = \{e \in \mathcal{S}\cap D(\lambda) : \lambda(e) = a\}$$

for $a > 0$, and put $a_0 = \inf\{a : S(a) \neq \emptyset\}$. Guided by the previous considerations we define the set $S_s$ of most stable states by $S_s = S(a_0)$, if it is non-empty. When $S(a_0) = \emptyset$, then, in general, $S_s \subset \bigcup_{a < a_0 + \epsilon} S(a)$. The choice of $\epsilon$ is somewhat arbitrary as it serves as the border between the preferred “quasi-classical” states and the ”non-classical” remainder. In this case, as was mentioned above, a further analysis examining the behavior of $T_t e$ also for $t > 0$ may be inevitable in order to select the set $S_s$.

By combining predictability with the previously exploited principle expressing the fact that any superposition of two distinct preferred states cannot belong to the same class of stability we obtain the following.

**Definition 3.1.** A state $e$ is called “quasi-classical” if $e \in S_s$ and for any $f \in S_s$, $f \neq e$, $S(e, f) \cap S_s = \emptyset$. The space of “quasi-classical” states will be denoted by $S_{qc}$.

It should be pointed out that “quasi-classical” states can form an overcomplete set in contrast to the ”classical” states. On the other hand they may not exist at all. A simple example illustrating such a case is given by a dynamical semigroup on $2 \times 2$ complex matrices with the following generator:

$$L(\rho) = -i[H, \rho] + (\text{tr}\rho)I - 2\rho$$
Then $S(a) = \emptyset$ if $a \neq 1$ and $S(a) = S$ for $a = 1$. Hence $S_s$ consists of all states and so any superposition of its two states again belongs to $S_s$. Therefore $S_{qc} = \emptyset$. In this case all states deteriorate into a completely mixed state in a uniform way.

4. Examples

Having discussed theoretical properties of ”classical” and “quasi-classical” states, let us now consider some physical examples.

4.1. Pointer states

Pointer states have been thoroughly discussed, see [6] and references therein. They arise, for example, when the dynamical generator for the reduced density matrix is given by (see [12,13])

$$L(\rho) = -i[H, \rho] + \sum_i P_i \rho P_i - \frac{1}{2} \{P, \rho\}$$

where $P_i$ are one-dimensional orthogonal projectors, $P = \sum_i P_i$, and Hamiltonian $H$ commutes with all $P_i$. Then $S_c = \{P_i\}$.

4.2. Quantum Brownian motion

For quantum Brownian motion

$$\dot{\rho} = -i[H, \rho] - D[x, [x, \rho]]$$

which leads to an environment-induced semigroup, the rate of change of linear entropy for a state $e = |\psi><\psi|$ is given by

$$\frac{d}{dt}S_{\text{lin}}(T_t e)|_{t=0} = 4D(<x^2> - <x>^2)$$

where $<x> = <\psi|x|\psi>$ and $<x^2> = <\psi|x^2|\psi>$. Hence $\lambda(e)$ is proportional to the dispersion in position of $|\psi>$ and so $S(a) \neq \emptyset$ for any $a > 0$. However, $S(a = 0) = \emptyset$. A more detailed analysis shows that the space of the most stable states $S_s$ consists of coherent states of the quantum harmonic oscillator [7]. Because these states are represented by [20]

$$|\alpha> = \exp(-\frac{1}{2} |\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n>$$
where $\alpha$ is a complex number and $|n>$ denotes the energy eigenstate, hence for any two distinct states $|\alpha>$ and $|\beta>$, $\alpha \neq \beta$, none of their superpositions belongs to $S_s$. Therefore, $S_s \cap S(|\alpha><\alpha|, |\beta><\beta|) = \emptyset$, and so the coherent states are “quasi-classical”. It is worth noting that coherent states of a harmonic oscillator coupled with up and down spins are also selected as the preferred states (“quasi-classical” in our terminology) in a model of the joint system of a spin-$\frac{1}{2}$ particle and a harmonic oscillator interacting with a zero-temperature bath of harmonic oscillators [21].

4.3. GRW spontaneous localization

Let us now examine the behavior of pure states $e_\psi = |\psi><\psi|$ for a semigroup given by the master equation of the type discussed by Ghirardi, Rimini and Weber [22] (see also [23])

$$\dot{\rho} = -i[H, \rho] + \int_{-\infty}^{\infty} da \, \rho G_a - \kappa \rho \quad (5)$$

where $G_a$ is an operator of multiplication by a Gaussian function

$$g_a(x) = \kappa^{1/2} \left( \frac{2\alpha}{\pi} \right)^{1/4} e^{-\alpha(x-a)^2}$$

that is $G_a \psi(x) = g_a(x) \psi(x)$. Clearly, the above master equation leads to a dynamical semigroup which is also contractive in the operator norm. Hence $\lambda$ is well defined and

$$\lambda(e_\psi) = \kappa - \int_{-\infty}^{\infty} da (\text{tr} G_a e_\psi)^2$$

$$= \kappa - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdy |\psi(x)|^2 |\psi(y)|^2 \int_{-\infty}^{\infty} da g_a(x) g_a(y)$$

$$= \kappa [1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdy |\psi(x)|^2 |\psi(y)|^2 e^{-\alpha(x-y)^2/2}]$$

Therefore, for any $e_\psi$, $0 < \lambda(e_\psi) < \kappa$ and for any $0 < a < \kappa$, $S(a) \neq \emptyset$. Clearly, the most stable states are those of Dirac’s delta type, whereas states
which are uniformly distributed over large intervals are strongly affected by the interaction.

4.4. Quantum stochastic process

This example shows that coherent states can be also selected as the “quasi-classical” states for quantum stochastic processes introduced by Davies [24]. Quantum stochastic processes were introduced to describe rigorously certain continuous measurement processes. They can be constructed from two infinitesimal generators. The first is the generator $Z$ of a strongly continuous semigroup on a Hilbert space $\mathcal{H}$, and the second is a stochastic kernel $J$, describing how the measuring apparatus interacts with the system. Let us recall that a stochastic kernel is a measure defined on the $\sigma$-algebra of Borel sets in some locally compact space and with values in the space of bounded positive linear operators on $\text{Tr}(\mathcal{H})$. In this example we take Poincaré disc $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ as the underlying topological space, and define

$$Z = iH - \frac{\kappa}{2}1$$

where $H$ is the Hamiltonian of the system, $\kappa > 0$ is the coupling constant. For $E \subset D$ and $\rho \in \text{Tr}(\mathcal{H})$ the stochastic kernel is defined by

$$\text{tr}[J(E, \rho)A] = \kappa \int_E d\mu(\zeta)\text{tr}(e_\zeta \rho e_\zeta A)$$

where $A$ is a bounded linear operator on $\mathcal{H}$, $e_\zeta = |\zeta > < \zeta|$ with $|\zeta >$ being a SU(1,1) coherent state, i.e. a holomorphic function on $D$ [25]

$$|\zeta > (z) = (1 - |\zeta|^2)(1 - z\zeta)^{-2}$$

and

$$d\mu(\zeta) = \frac{1}{\pi} \frac{d\zeta d\bar{\zeta}}{(1 - |\zeta|^2)^2}$$

is a SU(1,1) invariant measure on $D$. In order to define a quantum stochastic process $Z$ and $J$ have to satisfy the following relation

$$\text{tr}[J(D, e_\psi)] = -2\text{Re} < \psi |Z|\psi >$$
\[ e_\psi = |\psi> <\psi|, \] for all normalized vectors \( \psi \in D(Z) \). It is straightforward to check that
\[
\text{tr}[J(D, e_\psi)] = \kappa \int_Q d\mu(\zeta) \text{tr}(e_\zeta e_\psi e_\zeta) = \kappa = -2\text{Re} <\psi|Z|\psi>
\]
The strongly continuous semigroup \( T_t \) associated with the process is given by
\[
T_t(\rho) = \exp(tZ^*)\rho \exp(tZ) + tJ(D, \rho) + o(t)
\]
and so its generator reads
\[
L(\rho) = -i[H, \rho] + \kappa \int_Q d\mu(\zeta) e_\zeta \rho e_\zeta - \kappa \rho \quad (6)
\]
Obviously, it generates an environment-induced semigroup. It is worth noting that the integral formula above is a straightforward generalization of the von Neumann projection postulate to the case in which the family of states is overcomplete. Such a generator was thoroughly discussed in [26].

We now search for the most stable states with respect to semigroup \( T_t \). Let us first note that no state is stable since, by Lemma 3.2 in [26], \( T_t \) is strictly positive, that is \( T_t e \) is a faithful density matrix for every \( t > 0 \) and any \( e \in \mathcal{S} \). Hence \( \mathcal{S}_0 = \emptyset \) and so, in this case, there are no "classical" states at all. However, the quadratic form \( \lambda \) is bounded and allows to classify all states in the following way.

**Proposition 4.1.** \( \frac{2}{3}\kappa \leq \lambda(e) < \kappa \) for every \( e \in \mathcal{S} \), and \( \mathcal{S}(a) \neq \emptyset \) if \( \frac{2}{3}\kappa \leq a < \kappa \). For \( a_0 = \frac{2}{3}\kappa \), \( \mathcal{S}(a_0) \) consists exactly of the coherent states \( e_\zeta, |\zeta| < 1 \).

**Proof:** Let \( e \in \mathcal{S} \). Then, by definition,
\[
\lambda(e) = \kappa [1 - \int_Q d\mu(\zeta) (\text{tr} e e_\zeta)^2]
\]
Suppose \( e_n = |n><n| \), where \( |n> (z) = \sqrt{n + 1}\ z^n, \ n \in \mathbb{N} \cup \{0\} \), is an orthonormal basis in \( \mathcal{H} \). It was shown in [15] that
\[
\int_Q d\mu(\zeta) (\text{tr} e_n e_\zeta)^2 = \frac{n + 1}{(2n + 1)(2n + 3)}
\]
Therefore,
\[ 0 < \int d\mu(\zeta)(\text{tr} e_{\zeta})^2 \leq \frac{1}{3} \]
for any \( e \), and so \( \frac{2}{3}\kappa \leq \lambda(e) < \kappa \). Finally, notice that \( \lambda \) is SU(1,1) invariant. Hence \( \lambda(e_0) = \lambda(\pi(g)e_0\pi(g)^*) = \frac{2}{3}\kappa \) for any \( g \in \text{SU}(1,1) \). However, the set \( \{\pi(g)e_0\pi(g)^*\} \) coincides with the set of coherent states \( e_{\zeta}, |\zeta| < 1 \).

Hence, by definition, \( S_s = S(a_0) \). Not surprisingly, the most stable states are the coherent ones. Next we show that they are “quasi-classical”. Because any \( |\zeta> \) has the following representation
\[ |\zeta> = (1 - |\zeta|^2) \sum_{n=0}^{\infty} \sqrt{n + 1}|\zeta|^n|n> \]
it follows that any superposition of two distinct coherent states \( |\zeta> \) and \( |\zeta'> \) is not coherent, and thus \( S_s \cap S(e_{\zeta}, e_{\zeta'}) = \emptyset \). Hence the coherent states are “quasi-classical” and make up an analog of the pointer basis. They correspond to points in Poincaré disc, which is the underlying space of the stochastic kernel representing the measuring apparatus.

In this section we have presented different types of examples which demonstrate how robust states can be selected and how non-stable states decohere to mixtures. Moreover, the rate of their deterioration to density matrices was examined. Such an analysis can be useful, for example, in quantum computing, where it is essential to control the process of decoherence in order to allow quantum bits to compute in parallel.

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Appendix

Suppose \( \dim\mathcal{H} < \infty \) and let \( S_{\text{lin}}(T_e) = 0 \) for some \( e \in S \). Then \( e \in S_0 \).

Proof: Let \( P \) be the projection onto \( \text{Tr}(\mathcal{H})_{\text{iso}} \) along \( \text{Tr}(\mathcal{H})_s \). Because \( P \) extends to an orthogonal projection in the Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H} \) so
\[ \|T_e\|_2^2 = \|P(T_e)\|_2^2 + \|(id - P)T_e\|_2^2 \]
where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. Since $P$ commutes with $T_t$ and $T_te \in \mathcal{S}$, we obtain that

$$1 = \|T_t(Pe)\|_2^2 + \|(id - P)T_te\|_2^2.$$

Because $\dim \mathcal{H} < \infty$ so $T_t$ is relatively compact in the strong operator topology and so, by Theorem 24 in [18], $\lim_{t \to \infty} \|T_t\phi - P(T_t\phi)\|_1 = 0$ for any trace class operator $\phi$. Since $\| \cdot \|_2 \leq \| \cdot \|_1$ we obtain that $\lim_{t \to \infty} \|T_t(Pe)\|_2 = 1$. However, $T_t$ is also contractive in the norm $\| \cdot \|_2$, so $\|Pe\|_2 = 1$ and hence $Pe = e$. It means that $e \in \text{Tr} (\mathcal{H})_{iso}$, and thus $e \in \mathcal{S}_0$. \hfill \Box

References


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