Massive Scalar Particles in a Modified Schwarzschild Geometry

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Abstract

Massive, spinless bosons have vanishing probability of reaching the sphere $r = 2M$ from the region $r > 2M$ when the original Schwarzschild metric is modified by maximal acceleration corrections.

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It is commonly believed that the construction of a geometrical theory of quantum mechanics would lend perspective to a variety of problems, from the unification of general relativity and quantum mechanics to the regularization of field equations. In response to this need Caianiello and collaborators [1], [2] developed a model in which quantization is interpreted as curvature of the eight-dimensional space-time tangent bundle $TM$. The model incorporates the Born reciprocity principle and the notion that the proper acceleration of massive particles has an upper limit $A_m$.

Classical and quantum arguments supporting the existence of a maximal acceleration (MA) have long been adduced [3]. MA also appears in the context of Weyl space [4] and of a geometrical analogue of Vigier’s stochastic theory [5].

Some authors regard $A_m$ as a universal constant fixed by Planck’s mass [6],[7], but a direct application of Heisenberg’s uncertainty relations [8],[9] as well as the geometrical interpretation of the quantum commutation relations given by Caianiello, suggest that $A_m$ be fixed by the rest mass of the particle itself according to $A_m = 2mc^3/h$.

MA touches upon a number of issues. The existence of a MA would rid black hole entropy of ultraviolet divergencies [10],[11], and circumvent inconsistencies associated with the application of the point-like concept to relativistic quantum particles [12].

It is significant that a limit on the acceleration also occurs in string theory. Here the upper limit manifests itself through Jeans-like instabilities [13] which occur when the acceleration induced by the background gravitational field is larger than a critical value $a_c = (ma)^{-1}$ for which the string extremities become causally disconnected [14]. $m$ is the string mass and $\alpha$ is the string tension.

Frolov and Sanchez [15] have then found that a universal critical acceleration $a_c = (ma)^{-1}$ must be a general property of strings.

While in all these instances the critical acceleration is the result of the interplay of the Rindler horizon with the finite extension of the particle [16],[17], in the Caianiello model MA is a basic physical property of all massive particles which appears automatically in the physical laws. At the same time the model introduces an invariant interval in $TM$ that leads to a regularization of the field equations that does not require a fundamental length as in [18] and does therefore preserve the continuum structure of space-time.

Applications of the Caianiello model range from cosmology to the cal-
culation of corrections to the Lamb shift of hydrogenic atoms. A sample of pertinent references can be found in [19]. In all these works space-time is endowed with a causal structure in which the proper accelerations of massive particles are limited. This is achieved by means of an embedding procedure pioneered in [20] and discussed at length in [16],[21]. The procedure stipulates that the line element experienced by an accelerating particle is represented by

\[ dx^2 = \frac{1 + \ddot{x}^2}{A_m^2} \eta_{\mu \nu} dx^{\mu} dx^{\nu}, \]

and is therefore observer-dependent as conjectured by Gibbons and Hawking [22]. As a consequence, the effective space-time geometry experienced by accelerated particles exhibits mass-dependent corrections, which in general induce curvature, and give rise to a mass-dependent violation of the equivalence principle. The classical limit \((A_m)^{-1} = \frac{\hbar}{2mc^2} \to 0\) returns space-time to its ordinary geometry.

In the presence of gravity, we replace \(\eta_{\mu \nu}\) with the corresponding metric tensor \(g_{\mu \nu}\), a choice that preserves the full structure introduced in the case of flat space. We obtain

\[ d\tau^2 = \left( 1 + \frac{\dot{x}_{\mu} \dot{x}^\mu}{A_m^2} \right) g_{\alpha \beta} dx^\alpha dx^\beta \equiv \sigma^2(x) g_{\alpha \beta} dx^\alpha dx^\beta, \]

where \(\ddot{x}^\mu = d^2x^\mu / ds^2\) is the, in general, non-covariant acceleration of a particle along its worldline.

We have recently studied the modifications produced by MA in the motion of a test particle in a Schwarzschild field [21]. We have found that these account for the presence of a spherical shell, external to the Schwarzschild sphere, that is forbidden to any classical particle and hampers the formation of a black hole. Our aim here is to study the behaviour of a quantum, scalar particle in this modified Schwarzschild geometry. The calculations involve both classical and quantum behaviours of the particle together in a single framework. The first one determines, through the expectation value of the acceleration, the effective gravitational field which in turn defines the latter by altering the make-up of the Klein-Gordon equation.

Before embarking on this problem, some cautionary remarks are in order [21].
The effective theory presented is intrinsically non-covariant. Non-covariant is the quadri-acceleration that appears in $\sigma^2(x)$ and non-covariant is $\sigma^2(x)$ itself which is not, therefore, a true scalar. In addition $\sigma^2(x)$ could be eliminated from (2) by means of a coordinate transformation if one insisted on applying the principles of general relativity to this effective theory. On the contrary, the embedding procedure requires that $\sigma^2(x)$ be present in (2) and that it be calculated in the same coordinates of the unperturbed gravitational background. It is therefore desirable to check the results of a particular calculation in more than a single coordinate system. Nonetheless the choice of $\ddot{x}^\mu$ is supported by the derivation of $A_m$ from quantum mechanics, by special relativity and by the weak field approximation to general relativity. A fully covariant presentation of the ideas expounded is still lacking. The model is not intended, therefore, to supersede general relativity, but rather to provide a way to calculate the effect of MA on the quantum particle.

For convenience, the natural units $\hbar = c = G = 1$ are used below. The conformal factor can be easily calculated as in [21] starting from (2), with $\theta = \pi/2$, and from the well known expressions for $\ddot{t}, \ddot{r}$ and $\ddot{\phi}$ in Schwarzschild coordinates [23]. One obtains

$$
\sigma^2(r) = 1 + \frac{1}{A_m^2} \left\{ -\frac{1}{1 - 2M/r} \left( \frac{3M \tilde{L}^2}{r^4} + \tilde{L}^2 - \frac{M}{r^2} \right)^2 + \left( -\frac{4 \tilde{L}^2}{r^4} + \frac{4 \tilde{E}^2 M^2}{r^4(1 - 2M/r)^3} \right) \left[ \tilde{E}^2 - \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{\tilde{L}^2}{r^2} \right) \right] \right\},
$$

where $M$ is the mass of the source, $\tilde{E}$ and $\tilde{L}$ are the total energy and angular momentum per unit of test particle rest mass $m$.

As discussed in [21], the modifications introduced by Eq. (3) include the presence of a spherical shell, external to the Schwarzschild sphere, that is forbidden to classical particles. The radius of the shell is $2M < r < (2 + \eta)M$, where $\eta$ is much less than one and increases with the total energy per unit of test particle mass $\tilde{E}$. The question now arises whether quantum particles can penetrate the shell. This problem is tackled in the present work where the massive, quantum particle satisfies the Klein–Gordon equation.

In the effective curved space–time of metric (2), the wave equation for a scalar particle of rest mass $m$ is

$$
(\nabla_\mu \nabla^\mu + m^2) \psi(x) = 0,
$$

(4)
where $\nabla_\mu \nabla^\mu = (1/\sqrt{-g})\partial_\mu (\sqrt{-g} \tilde{g}^{\mu \nu} \partial_\nu)$, $\tilde{g}_{\mu \nu} = \sigma^2(r)g_{\mu \nu}$, and $\nabla_\mu$ is the covariant derivative.

Written explicitly, Eq. (4) takes the form

$$
\left\{ \frac{\partial^2}{\partial t^2} - \frac{e^\lambda}{\sigma^2 r^2} \frac{\partial}{\partial r} \left( \sigma^2 r^2 e^\lambda \frac{\partial}{\partial r} \right) - \frac{e^\lambda}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + m^2 \sigma^2 e^\lambda \right\} \psi(t, r, \theta, \phi) = 0. \tag{5}
$$

By separating variables, the wave function can be written as

$$
\psi(t, r, \theta, \phi) = T(t) R(r) \Theta(\theta, \phi) \tag{6}
$$

and Eq. (5) can be split into the following three equations

$$
\frac{\partial^2 T}{\partial t^2} + \omega^2 T = 0, \tag{7}
$$

$$
\frac{1}{\Theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Theta = -l(l + 1), \tag{8}
$$

$$
e^{2\lambda} R'' + \left( \frac{2 \sigma'}{\sigma} + \frac{2}{r} + \lambda' \right) e^{2\lambda} R' + \left[ \omega^2 - \frac{l(l + 1)}{r^2} + m^2 \sigma^2 \right] R = 0, \tag{9}
$$

where $\omega^2$ is a separation constant corresponding to the frequency of the wave, $l$ is the orbital angular momentum quantum number of the scalar particle and a prime indicates differentiation with respect to $r$.

The solution of Eq. (8) is

$$
\Theta_{lp}(\theta, \phi) = Y_l^p(\cos \theta)e^{ip\phi} \tag{10}
$$

where $Y_l^p(\cos \theta)$ are the usual spherical harmonics, and $p$, with $|p| \leq l$, is the magnetic quantum number. The general solution of eq. (7) is

$$
T(t) = C_1 e^{-i\omega t} + C_2 e^{i\omega t}, \tag{11}
$$

where $C_1$ and $C_2$ are arbitrary constants. It follows, from Eqs. (6), (10) and (11) that the eigenfunctions of the scalar wave equation (5) can be cast in the form

$$
\psi(t, r, \theta, \phi) = NR(r)Y_l^p(\cos \theta)e^{i(p\phi \pm \omega t)}, \tag{12}
$$
where \( N \) is a normalization constant and \( R(r) \) is the solution of the radial wave equation (9).

In order to derive from (9) the effective quantum potential in which the boson field propagates, one usually introduces the variable \( r^* = r^*(r) \) such that

\[
e^{2\lambda} \left( \frac{dr^*}{dr} \right)^2 = 1.
\]

Eq. (13) implies that \( r^*(r) = r + 2M \ln(r - 2M) \), which is defined for \( r \geq 2M \). After substituting \( R(r^*) = \alpha(r^*)\beta(r^*) \) into (9), one requires that the coefficient of \( d\alpha/dr^* \) vanishes \([24]\), i.e.

\[
\frac{d\beta}{dr^*} - G(r)\beta = 0,
\]

where

\[
G(r) \equiv - \left( \frac{g'}{\sigma} + \frac{1}{r} \right) e^\lambda.
\]

In the region \( r \geq 2M \), the equation linking \( r \) to \( r^* \) may be used to integrate Eq. (14). The result is \( \beta(r) = \beta_0(r\sigma(r))^{-1} \) where \( \beta_0 \) is an integration constant. \( \beta \) vanishes for \( r \to 2M \). The equation for \( \alpha(r) \) reduces to the Schroedinger–like equation

\[
-\frac{d^2\alpha}{dr^*^2} + V_{\text{eff}}(r)\alpha = \omega^2\alpha,
\]

where the effective potential \( V_{\text{eff}}(r) \) is given by

\[
V_{\text{eff}}(r) = G^2(r) + e^\lambda G'(r) + e^\lambda \left( \frac{l(l+1)}{r^2} + m^2\sigma^2 \right).
\]

As in [21], it is convenient to introduce the adimensional quantities \( \lambda = \tilde{L}/M = l/(mM) \), \( \epsilon = (MA_m)^{-1} = (2mM)^{-1} \) and \( \rho = r/M \). The behaviour of \( \tilde{V}_{\text{eff}}(\rho) = V_{\text{eff}}(\rho)/m^2 \) is shown in Fig. 1. The largest contribution comes from the \( e^\lambda m^2\sigma^2 \) term in (16). For \( \rho \sim 2 \) one finds \( \tilde{V}_{\text{eff}}(\rho) \sim \frac{E^4 \rho^2}{(\rho-2)^2} \), which, unlike \( \tilde{V}_{\text{eff}}(\rho) \) of Ref. [21], definitely diverges on the Schwarzschild sphere.

This suggests that \( |\alpha|^2 \to 0 \) as \( r \to 2M \).

In order to get a clear indication of the behaviour of \( |R(r)|^2 \), we calculate the asymptotic solution of the radial wave equation near the Schwarzschild horizon by writing

\[
\rho = 2 + x,
\]

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Figure 1: Solid line: effective potential per unit of test particle rest mass $m$ for $E = 9$, $\epsilon = 0.001$ and $\lambda = 0$. Dotted line: $E = 9$, $\epsilon = 0.001$ and $\lambda = 5$

where $x \ll 1$, and by expanding the coefficients of the radial wave equation in a power series. To leading order, one obtains $\sigma \sim \sqrt{8/\epsilon^2 x^3}$ and $\sigma'/\sigma \sim -3/(2Mx)$. The radial wave equation (9) then reduces to the Bessel equation

$$x^2 \frac{d^2 R}{dx^2} - 2x \frac{dR}{dx} - \frac{4}{x^2} R = 0 \quad (19)$$

and its solution is

$$R(x) = x^{3/2} e^{\pm i(3/2)\pi} Z_{\pm 3/2} \left(\frac{2}{\sqrt{x}}\right), \quad (20)$$

where $Z_\nu(z)$ is the Bessel function with half integer index. In the limit $x \to 0$, Eq. (20) reads

$$R(x) \sim \sqrt{\frac{1}{\pi}} x^{2} e^{i(1/x+\gamma)}, \quad (21)$$

where $\gamma$ is a constant phase factor. The radial probability density $P(x)$ in proximity of the event horizon is then

$$P(x) = |R(x)|^2 \sim x^4, \quad (22)$$
which vanishes as \( x \to 0 \). It must be emphasized that our result differs from that derived by Kofinti [24]. In fact, the corresponding probability density for propagation of a quantum particle in an unmodified Schwarzschild geometry does not vanish at \( \rho = 2 \) [24].

Let us now ascertain that the results obtained persist in isotropic coordinates. These are related to \( r \) by the non-linear transformation \( r = (1 + a/4u)^2 u \) and yield the metric tensor [25]

\[
g_{\mu \nu} = \text{diag}(e^\lambda, -e^\mu, -e^\mu u^2, -e^\mu u^2 \sin^2 \theta),
\]

where

\[
e^\lambda = \frac{(1 - a/4u)^2}{(1 + a/4u)^2}, \quad e^\mu = \left(1 + \frac{a}{4u}\right)^4, \quad a = 2GM/c^2.
\]

In these coordinates, therefore, the weak field limits of (23) and of the Schwarzschild metric coincide. The isotropic coordinates also leave the element of spatial distance in conformal form.

We start again from the expressions for the components of the four-velocity

\[
i = \frac{\tilde{E}(1 + a/4u)^2}{(1 - a/4u)^2},
\]

\[
\dot{u} = \frac{1}{(1 + a/4u)^2} \left\{ \frac{\tilde{E}^2(1 + a/4u)^2}{(1 - a/4u)^2} - \frac{\tilde{L}^2}{u^2(1 + a/4u)^4} - 1 \right\}^{1/2},
\]

\[
\dot{\phi} = \frac{\tilde{L}}{u^2(1 + a/4u)^4}.
\]

and that of the conformal factor

\[
\sigma^2(r) = 1 + \frac{1}{A_m^2} \left[ \frac{(1 - a/4u)^2}{(1 + a/4u)^2} i^2 - \left(1 + \frac{a}{4u}\right)^4 \dot{u}^2 - u^2 \left(1 + \frac{a}{4u}\right)^4 \dot{\phi}^2 \right].
\]

The classical, repulsive shell still exists in proximity of the horizon \( u = a/4 \) as indicated by Fig. 2 (compare with Fig. 1 of Ref. [21]). Its existence confirms the result of [21]. The analogous occurrence of a classically impenetrable shell was also derived by Gasperini [26] as a consequence of the breaking of the \( SO(3, 1) \) symmetry.
Repeating the same calculations as in the Schwarzschild case, we find that the radial wave function $R(u)$ satisfies the equation

$$e^{-\lambda u} R'' + e^{-\lambda u} \left( \frac{\sigma'}{\sigma} + \frac{2}{u} + \frac{\lambda'}{2} + \frac{\mu'}{2} \right) R' +$$

$$+ \left[ \omega^2 - e^{-\lambda} \left( e^{-\lambda} \frac{l(l+1)}{u^2} + m^2 \sigma^2 \right) \right] R = 0.$$  

The functions $\Theta(\theta, \varphi)$, $T(t)$ and the constants are as defined in (10) and (7).

In order to derive the effective potential from (28), we now introduce the variable $u^* = u^*(u)$ such that

$$e^{\lambda - \mu} \left( \frac{du^*}{du} \right)^2 = 1.$$  

Eq. (29) implies that

$$u^*(u) = \frac{a^2}{16} + u - a \ln u + 2a \ln(4u - a),$$
which is defined for $4u \geq a$. We again substitute $R(u^*) = \alpha(u^*)\beta(u^*)$ into (28) and require that the coefficient of $d\alpha/du^*$ vanishes, i.e.

$$\frac{d\beta}{du^*} + G(u)\beta = 0,$$

(31)

where

$$G(r) \equiv e^{\lambda/2-\mu/2} \left( \frac{2\sigma'}{\sigma} + \frac{2}{u} + \mu' \right).$$

(32)

The integration of (31) yields the result

$$\beta(u^*) = \frac{\beta_1 e^{-\mu/2}}{\sigma u},$$

(33)

where $\beta_1$ is an integration constant. The equation for $\alpha(u^*)$ reduces to the Schroedinger–like equation, where now the effective potential $V_{eff}(u)$ is given by

$$V_{eff}(u) = G^2(u) + e^{\lambda/2-\mu/2} \frac{dG(u)}{du} + e^\lambda \left( e^{-\mu} \frac{l(l+1)}{r^2} + m^2 \sigma^2 \right).$$

(34)

It is a simple task to calculate the behaviour of the potential (34) in proximity of the singularity point $4u = a$. In fact, setting $4u = a + x$, with $x \approx 0$, one gets

$$V_{eff}(u) \sim \frac{m^2}{x^6} \to \infty \quad \text{as} \quad x \to 0,$$

(35)

where the dominant contribution is represented, once again, by $e^\lambda m^2 \sigma^2$. In analogy to the foregoing, we can therefore conclude that the probability to find the quantum particle near the horizon vanishes as $u \to a/4$. In fact, in the limit $x \to 0$, Eq. (28) reduces to the form $(R \equiv y)$

$$x^2y'' - 9xy' - \frac{4\tilde{E}^2m^2}{4a^2A^2x^6}y = 0,$$

(36)

whose solution is a Bessel function. In the limit $x \to 0$, we get

$$y \sim x^4 \cos \left( \frac{\tilde{E}m}{2aAx^3} \right).$$

(37)

Then the probability vanishes as $x \to 0$, as expected. The choice of isotropic coordinates does not alter the fact that massive scalar particles cannot cross the horizon when propagating in a space-time modified by MA corrections.
In conclusion, we have determined the behaviour of a spinless boson in the neighborhood of the Schwarzschild sphere $\rho = 2$ when the Schwarzschild metric is modified by maximal acceleration corrections according to the model of Refs. [1], [2], [16]. Though the effective potentials experienced by classical and quantum particles are different, their effects are similar. Classical particles can not penetrate the shell of radius $2 < \rho < 2 + \eta$ where their kinetic energy becomes negative. Similarly, the probability density to find massive spinless bosons in the region $\rho = 2 + x$ with $x << 1$, vanishes with $x$ at $\rho = 2$ where $\sigma^2(x)$ and the quantum potential diverge. In both instances, maximal acceleration corrections strongly suppress the absorption of particles in proximity of the horizon. Quantum tunneling of scalar particles through the shell is not therefore a viable process for black hole formation in the model, unless matter is transmuted first into massless particles, as discussed in [21]. These would then have to be absorbed by the interior of the star at a rate higher than the corresponding re-emission rate.

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