We calculate for the first time the surface tension and curvature coefficient of a first order phase transition between two possible phases of cold nuclear matter, a normal nuclear matter phase in equilibrium with a kaon condensed phase, at densities a few times the saturation density. We find the surface tension is proportional to the difference in energy density between the two phases squared. Furthermore, we show the consequences for the geometrical structures of the mixed phase region in a neutron star.

I. INTRODUCTION

The possibility of different phase transitions taking place in the superdense interior of neutron stars has been the target of considerable interest during the last few decades, where pion and kaon condensation as well as quark deconfinement have been investigated. But only less than a decade ago was it realized that if the phase transition is of first order, then a geometrically structured extended region will form in the superdense interior of the neutron star, where the two phases are in equilibrium [1]. The reason for this richness in structure is that a neutron star has two globally conserved charges, baryon number and electric charge, and two chemical potentials associated with these charges. Previous studies using the Maxwell construction could only ensure one chemical potential was common in the two phases, whereas the general phase equilibrium criteria by Gibbs [2] ensure thermodynamical equilibrium for a system with any number of chemical potentials. The consequences are that the system is not locally charge neutral and a competition between Coulomb and surface energies are responsible for the geometrical structures. Moreover, the common pressure will vary with the proportion of the phases, and thus create an extended mixed phase region with structure in the neutron star.

For a first order deconfinement transition, studies of the detailed crystalline structure of the mixed phase region have always been hindered by the lack of a single good model describing both phases. This is in contrast to the first order transition to a kaon condensed phase described in [3,4], where both the normal nuclear matter phase and the kaon condensed phase are described by the same relativistic mean-field model, which allow us to calculate the profiles of all important quantities across the interface. From these profiles the Coulomb energy and the surface tension can be found, where, especially for the latter, only educated guesses were previously possible. A condensate consisting of negatively charged kaons is favored in neutron stars because they, contrary to other kaon types, can replace electrons as neutralizing agents [5–10].

Detailed knowledge of the structure of a possible mixed phase region at densities above saturation is important, irrespective of which first order phase transition is responsible for it, as it may have important consequences for transport and superfluid properties and rotation in the form of r-mode instabilities, non-canonical values of the braking index, and glitch phenomena in pulsars [11–14].

In the present paper we calculate the surface properties, i.e., the surface tension and the curvature coefficient, for a semi-infinite slab of normal nuclear matter in phase equilibrium with a semi-infinite slab of kaon condensed matter and show the resulting crystalline structure in the central part of the neutron star. Since we assure compliance with Gibbs phase equilibrium criteria, the two phases cannot be separately charge neutral, though overall the net charge vanishes. Thus for a infinite system we cannot explicitly take Coulomb interactions into account in the Lagrangian. However, it turns out that the typical radii of the geometrical structures are smaller than the Debye screening lengths of about 10 fm [15], and therefore it is a reasonable first approximation to ignore this effect in the calculation of the surface tension.

Section II contains a description of the non-uniform relativistic mean-field model used to describe both phases. In Sec. III the surface properties are described, whereas the consequences for the crystalline structure in a neutron star are illustrated in Sec. IV. Finally our results are summarized in Sec. V.
II. NON-UNIFORM RELATIVISTIC MEAN-FIELD MODEL

The relativistic mean-field model [16,17] used here is described in detail in [4]. In this model the nucleon and kaon interactions are treated on an equal footing which means both couple to the scalar meson Ψ, the vector meson ω, and the isovector meson ρ, denoted σ, V_μ, and \( \vec{R}_\mu \), respectively. The Lagrangian for the nucleons is given by

\[
\mathcal{L}_N = \overline{\psi}_N \left( \gamma^\mu \partial_\mu - m_N - g_{\omega N} V_\nu \gamma^\nu - g_{\rho N} \gamma^\mu \vec{R}_\mu \right) \psi_N + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} m_\omega^2 V_\mu V^\mu - \frac{1}{4} \vec{R}_\mu \cdot \vec{R}^{\mu} - \frac{1}{2} m_\rho^2 \vec{R}_\mu \cdot \vec{R}^{\mu},
\]

where \( m_N \equiv m_N - g_{\sigma N} \sigma \), \( V_\mu \equiv \partial_\mu \psi + \partial_\nu \bar{\psi} \gamma_\nu \), \( \vec{R}_\mu \equiv \partial_\mu \bar{\psi} \vec{\gamma}, \) and the scalar self-interactions are on the form \( U(\sigma) = (1/3) b m_N (g_{\sigma N} \sigma)^3 + (1/4) c (g_{\sigma N} \sigma)^4 \), where \( b \) and \( c \) are constants. \( \Psi_N \) is the nucleon field operator and \( \vec{\tau}_N \) is the isospin operator.

The kaon Lagrangian is given by

\[
\mathcal{L}_K = D_\mu K^* D^\mu K - m_K^2 K^* K,
\]

where \( D_\mu \equiv \partial_\mu + ig_{\omega K} V_\mu + ig_{\rho K} \vec{R}_\mu \) and \( m_K^2 \equiv m_K - g_{\rho K} \sigma \). \( K \) denotes the \( K^- \) field and \( \vec{\tau}_K \) the kaon isospin operator.

In the mean-field approximation only the 0'th component of the vector fields and the isospin 3-component of the isovector field have finite mean values. The equation of motion for the kaon is

\[
(D_\mu D^\mu + m_K^2) K = 0.
\]

In a semi-infinite system the kaon field is only translational invariant in the transverse direction \( z_\perp \) and can then be written as

\[
K = \phi(z) e^{i(\omega_K z_\perp + k_\perp \cdot z_\perp)},
\]

where \( k_\perp \) is the transverse momentum, \( \omega_K \) is the intermediate kaon energy, and \( \phi(z) \) is the kaon amplitude.

For s-wave condensation, i.e. \( k = 0 \), the kaon field equation is given by

\[
\nabla^2 \phi + \left( (\omega_K + g_{\omega K} V_0 + g_{\rho K} R_{03})^2 - m_K^2 \right) \phi = 0.
\]

In the semi-infinite case \( \nabla^2 = d^2/dz^2 \), where \( z \) is the direction perpendicular to the surface.

The kaon density is

\[
\rho_K = 2 (\omega_K + g_{\omega K} V_0 + g_{\rho K} R_{03}) K^* K,
\]

and only for an infinite system \( \rho_K = 2m_K^2 K^* K \) as seen from Eq. (5).

The equations of motion for the position dependent meson fields become

\[
\nabla^2 \sigma - m_\sigma^2 \sigma = g_{\sigma N} \left( -\rho_\sigma + bm_N (g_{\sigma N} \sigma)^2 + c (g_{\sigma N} \sigma)^3 \right) - 2g_{\sigma K} m_K^2 K^* K,
\]

\[
\nabla^2 V_0 - m_\omega^2 V_0 = -g_{\omega N} (\rho_\omega + \rho_p) + g_{\omega K} \rho_K,
\]

where the neutron and proton densities, \( \rho_\omega \) and \( \rho_p \), and the scalar density \( \rho_\sigma \) are calculated in the Thomas-Fermi (local density) approximation.

\[
\rho_i = \frac{1}{3\pi^2 k_i^3}, \quad i = n, p
\]

and

\[
\rho_s = \frac{m_N^2}{2\pi^2} \sum_{i=n,p} \left( k_i \sqrt{k_i^2 + m_N^2} - m_N^2 \ln \left( \frac{k_i + \sqrt{k_i^2 + m_N^2}}{m_N^2} \right) \right).
\]

Expressions for the local Fermi momenta \( k_i \) are obtained from the Dirac equation for the nucleons

\[
\mu_i = g_{\omega N} V_0 \mp g_{\rho N} R_{03} \mp \sqrt{k_i^2 + m_N^2},
\]

where the upper sign is used for neutrons. The electron chemical potential \( \mu_e = \mu_n - \mu_p \). The kaon amplitude is zero unless the condition \( \omega_K = \mu_K = \mu_e \) is fulfilled.

The model parameters \( g_{iN}/m_i \) \( (i = \sigma, \omega, \rho) \), \( b \), and \( c \) can be algebraically determined from the five bulk properties of nuclear matter, which we take as: \( E/A = -16.3 \text{ MeV}, \rho_0 = 0.153 \text{ fm}^{-3}, a_{sym} = 32.5 \text{ MeV}, K = 240 \text{ MeV}, \) and \( m_N/m_N = 0.78 \). Because of the Laplacian terms in the meson field equations explicit values for the masses are required. We use for the \( \omega \) and \( \rho \) their rest masses \( m_\omega = 782 \text{ MeV} \) and \( m_\rho = 768 \text{ MeV} \), while the \( \sigma \) mass is determined from the surface properties of symmetric nuclear matter as described in [17,18], \( m_\sigma = 390 \text{ MeV} \). The kaon coupling constants for the vector mesons are determined from the quark and isospin counting rule,

\[
g_{\omega K} = g_{\omega N}/3 \quad \text{and} \quad g_{\rho K} = g_{\rho N},
\]

and the scalar coupling constant is fixed to the optical potential of the \( K^- \) at \( \rho_0 \) from

\[
U_K(\rho_0) = -g_{\sigma K} \sigma(\rho_0) - g_{\omega K} V_0(\rho_0),
\]

where we for the optical potential use \( U_K(\rho_0) = -120 \text{ MeV} \).

The Gibbs conditions for an infinite system of a kaon condensed phase and a normal nuclear matter phase to be in thermodynamical equilibrium at zero temperature are
\[ \mu_{i,N} = \mu_{i,K} \quad i = n, e \]
\[ P_N(\mu_n, \mu_e) = P_K(\mu_n, \mu_e), \quad (15) \]

where \( P_N \) and \( P_K \) are the pressures of the normal nuclear matter phase and the kaon condensed phase, respectively. These conditions combined with the global condition of electric charge conservation,

\[ q_{total} = (1 - \chi)q_N(\mu_n, \mu_e) + \chi q_K(\mu_n, \mu_e), \quad (16) \]

where \( q \) denotes the charge of the corresponding phase, and the field equations (without the Laplacian terms) allow us to solve for the bulk values of the fields, densities, and chemical potentials in each phase for any volume fraction of kaon phase \( \chi \) between zero and one.

In order to calculate the surface tension between the kaon condensed phase and the normal nuclear matter phase, the profiles of the fields and densities have to be determined across the interface between semi-infinite slabs of each phase. This is done by simultaneously solving the four coupled differential equations for the \( K^- \) and the meson fields Eqs. (5,7-9) through a relaxation procedure, where initial guesses for the different profiles are relaxed to their equilibrium values. The boundary conditions at \( \pm \infty \) are provided by the bulk values of the kaon amplitude and meson fields for each phase at a fixed \( \chi \). In practice a 30 fm region with the interface placed approximately in the center is sufficient to fulfill the boundary conditions, see Fig. 1.

**FIG. 1.** Profiles of the meson fields \( \sigma \), \( V_\theta \), and \( R_{0,2} \) and the neutron, proton, and kaon densities, \( \rho_n \), \( \rho_p \), and \( \rho_k \), respectively, across the interface. The boundary conditions at \( z = 0 \) and 30 fm correspond to \( \chi = 0.107 \).

It is a surprisingly good approximation and a significant simplification to neglect the Laplacian terms in the meson field equations, and only keeping the term in the kaon field equation. Differentiating twice brings the kaon equation to the form \( d^2\phi/dz^2 = X/\left( \omega_K + g_\omega K V_\theta + g_\rho K R_{0,3} \right)^2 - m_K^2 \). The denominator vanishes in the bulk kaon condensed phase in contrast to the meson field equations, which take the form \( d^2\sigma/dz^2 \simeq Y/m_k^2 \) (\( X \) and \( Y \) denote two complicated functions). Thus the Laplacian term is generally most important for the kaon field equation. In Ref. [13] the approximation was used to calculate the weak charge of droplets of kaon condensed phase to study effects on the neutrino opacity. We do both calculations and use the results from the approximative procedure as initial guess for the solution to the full set of four differential equations. This procedure ensures rapid convergence.

**III. SURFACE PROPERTIES**

In the transition zone between two phases the pressure tensor is no longer isotropic as it is for the homogeneous bulk phases. For a plane interface with the normal in the \( z \)-direction, the pressure tensor can be split into a normal and a tangential component. The normal component of the pressure \( P \) stays constant across the interface, cf. to Gibbs phase equilibrium conditions, whereas the tangential pressure component \( P_T(z) \) changes as function of \( z \) across the interface. The mechanical definition of the surface tension \( \sigma_G \) of a plane interface is [19]

\[ \sigma_G = \int_{-\infty}^{\infty} (P - P_T(z))dz, \quad (17) \]

where \( P_T(-\infty) = P_T(\infty) = P \). In nuclear physics this definition of the surface tension is denoted Gibbs definition [20,21].

Because we have assumed the leptonic species are homogeneously distributed throughout the system, they do not contribute to the surface tension. The pressure of the hadronic species \( P_H \) is given by

\[ P_H = \rho \frac{\partial \epsilon_H}{\partial \rho} - \epsilon_H = \mu_n \rho_n + \mu_p \rho_p + \mu_e \rho_e - \epsilon_H. \quad (18) \]

Therefore the surface tension of a plane interface between a kaon condensed phase and a normal nuclear matter phase can be written as

\[ \sigma_G = \int_{-\infty}^{\infty} dz (\epsilon_H(z) - \epsilon_H,N - \mu_n (\rho_n(z) - \rho_n,N) - \mu_p (\rho_p(z) - \rho_p,N) - \mu_e \rho_e(z)), \quad (19) \]

where the \( z \) dependent quantities at minus infinity take the bulk values of the kaon condensed phase and at plus infinity take the bulk values of the normal nuclear matter phase. The energy density of the hadrons across the interface is given by
\[ \epsilon_H(z) = \frac{1}{2} (\nabla \sigma)^2 + m_s^2 \sigma^2 + \frac{1}{2} (\nabla V_0)^2 + m_b^2 V_0^2 \]
\[ + \frac{1}{2} (\nabla R_{0,3})^2 \]
\[ + 2 (\omega K + g_{\omega K} V_0 + g_{\rho K} R_{0,3})^2 \phi^2 \]
\[ + U(\sigma) + \sum_{i=n,p} \frac{1}{\pi^2} \int_0^{\kappa_i} \sqrt{k^2 + m_N^2 k^2} \, dk. \quad (20) \]

Notice, if the Laplacian terms are neglected except for the kaon field, the only consequence is that the squared gradient terms of \( V \) and the prime denotes the derivative with respect to \( z \).

The curvature coefficient is in Gibbs definition given by [22,21]
\[ \gamma_G = \int_{-\infty}^{\infty} dz (z - z_0) \left( \epsilon_H(z) - \epsilon_{H,N} - \mu_n (\rho_n(z) - \rho_{n,N}) \right. \]
\[ \left. - \mu_p (\rho_p(z) - \rho_{p,N}) - \mu_c \rho_c(z) \right), \quad (21) \]
where the surface location, i.e., the position of the equivalent sharp surface,
\[ z_0 = \frac{\int_{-\infty}^{\infty} z \rho'(z) \, dz}{\int_{-\infty}^{\infty} \rho'(z) \, dz}, \quad (22) \]
and the prime denotes the derivative with respect to \( z \).

In Fig. 2 the surface tension and curvature coefficient densities, \( \epsilon_s \) and \( \epsilon_c \), respectively, across the interface. The solid curves are for the approximative calculation and the dotted curves for the exact calculation.

The overall consequences for the surface tension and curvature coefficient are shown in Fig. 3, where \( \sigma_G \) and \( -\gamma_G \) have been plotted as a function of \( \chi \). Notice, the curvature coefficient is negative. The surface tension decreases monotonically with increasing volume fraction of the kaon phase, whereas the curvature coefficient increases. The values of \( \sigma_G \) from the exact and approximative procedure deviate less than 0.4%. Thus the two curves lie on top of each other in this plot. This is in contrast to the curvature, where the difference is about 33% between the exact and approximative calculation. Furthermore, the absolute value of the curvature coefficient is comparable to the surface tension. This means the curvature energy is not negligible compared to the surface energy. But of course even higher order effects may mask the effect of the curvature energy as in the nuclear mass formula, where the apparent absence of a curvature term seem to be caused by higher order effects [23]. For a spherical droplet of kaon phase with radius \( R = 5 \) fm in a system where \( \chi \rightarrow 0 \), the surface energy is reduced about \( 2 \gamma_G/(\sigma_G R) = 18 \% \), whereas the surface energy increases, since the curvature is a signed quantity, about 40% for a spherical bubble of normal nuclear matter phase in the \( \chi \rightarrow 1 \) limit. We have here assumed, that both \( \sigma_G \) and \( \gamma_G \) calculated from a plane interface remain unaltered.
In Fig. 4 the exact calculation of $\sigma_G$ and $-\gamma_G$ as a function of $\chi$ have been compared with the fits

$$\sigma_{fit} = \left(0.00786 \frac{\epsilon_K - \epsilon_N}{\text{MeV/fm}^3}\right)^2 \text{MeV/fm}^2$$

$$\gamma_{fit} = -0.0976 \left(\frac{\epsilon_K - \epsilon_N}{\text{MeV/fm}}\right)^{3/4} \text{MeV/fm},$$


(23)

where the energy densities are in units of MeV/fm$^3$. The fit deviate less than 5% for the curvature coefficient and less than 0.5% for the surface tension over the region where $\epsilon_K - \epsilon_N$ decreases from 682 MeV to 358 MeV, i.e., about 50%. It is not exactly trivial to understand why the fits, especially for the surface tension, are so good. For the surface tension, Eq. (23) is equivalent to saying

$$(\epsilon(z) - \epsilon_N) \frac{de}{dz} \propto P_N - P(z),$$

whenever $\epsilon_s = P_N - P(z)$ is not negligible. This is indeed the case as long as $d^2 \epsilon/dz^2$ is negative but certainly not otherwise. We have not pursued the issue any further, but suspect this may be a generic property of phase equilibria described in mean-field theory in the Thomas-Fermi approximation.

IV. CRYSTALLINE STRUCTURE IN NEUTRON STARS

In Ref. [4] the surface tension was assumed to be proportional to the difference in energy density between the two phases and the constant chosen rather arbitrarily, but in a way that ensured the sum of surface and Coulomb energies was always much smaller than the bulk energy. Furthermore, the Coulomb energy was, as usual, calculated under the assumption that the phase boundary is sharp. Figure 18 of that paper show the diameter $D$ and the spacing $S$ of the geometrical structures as a function of the radial coordinate for a star at the mass limit. As we have now shown, the surface tension is proportional to the difference in energy density squared, and it is between a factor of 2-5 smaller than what was guessed at in [4]. Knowledge of the density profiles of the charged particles across the plane interface allow us to calculate the Coulomb energy for a soft boundary. It turns out that the Coulomb energy is reduced less than 15% for geometries with the typical small sizes encountered here. This translate into about a 5% reduction of the sum of surface and Coulomb energies. Therefore it is a good approximation to treat the phase boundary as being sharp with respect to calculating the Coulomb energy. Figure 5 is an updated version of Fig. 18 of Ref. [4]. (Details of this type of calculation can be found in [24].) The differences between the two figures are minor. Overall the sizes of the geometrical structures just decrease about 30%. This is due to the fact, that to first approximation the location of the transition from one geometry to another is independent of $\sigma_G$. Furthermore, the sum of surface and Coulomb energies scale only as $\sigma_G^{1/3}$. We have ignored the curvature term completely in these calculations, even though it is straight forward to include the curvature energy, see [25]. However, if the
standing of the mixed phase region for a first order tran-

sition involving a kaon condensate. We do, however, re-

alize there is still much room for improvement.

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V. SUMMARY

Little is known about the equation of state for nuclear

matter above saturation density, but it is expected that

at least a phase transition to deconfined quark matter

happens at a few times the saturation density. However,

this phase transition may not be the only one encoun-
tered in neutron stars. A transition to a phase with a

$K^-$ condensate is also a possibility. If any of these phase

transitions is of first order, a neutron star will have a

mixed phase region in its dense interior, which is very

likely to have some observable consequences.

For the first time the surface properties in the interface

between normal nuclear matter and kaon condensed mat-

ter have been calculated, which makes it possible to study

this mixed phase region in greater detail. Our calcula-
tions are only a first approximation, there are a number

of complicating aspects, which we have ignored or only

treated approximately - e.g., explicit consideration of the

Coulomb field which results in screening effects; the va-

lidity of the assumption that the surface tension is the

same for semi-infinite slabs as it is for small slabs, rods,

and drops; and the importance of the curvature and even

higher order terms compared to the surface term. Con-

cerning the first two points, we can generally say that

screening effects will reduce the Coulomb energy, likewise

the surface energy will decrease due to the decrease in the

surface tension when the system is squeezed. The reason

for the latter is that the surface region will dominate a

small system, so that only $\epsilon_s$ and not $d\epsilon_s/dz$ is zero at the

boundaries. For example a slab which is 10 fm thick and

with a volume fraction of kaon condensed phase of about

0.4, the surface tension is reduced about 12% compared

to the surface tension of the infinite system (30 fm thick),

while the absolute value of the curvature coefficient drops

about a factor of three. These two effects pull in opposite
directions with regard to the size of the geometry which

minimize the sum of Coulomb and surface energies, thus

the overall effect on the size is expected to be only minor.

We have taken another step towards a better under-

standing of the mixed phase region for a first order tran-