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Spinors, Dirac Operator and
de Rham Algebra**

by

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THE COMMUTATIVE CASE: SPINORS, DIRAC OPERATOR AND DE RHAM ALGEBRA

MICHAEL FRANK

ABSTRACT. The present paper is a short survey on the mathematical basics of Classical Field Theory including the Serre-Swan' theorem, Clifford algebra bundles and spinor bundles over smooth Riemannian manifolds, $\text{Spin}^{\mathbb{C}}$ -structures, Dirac operators, exterior algebra bundles and Connes' differential algebras in the commutative case, among other elements. We avoid the introduction of principal bundles and put the emphasis on a module-based approach using Serre-Swan's theorem, Hermitian structures and module frames. A new proof (due to Harald Upmeyer) of the differential algebra isomorphism between the set of smooth sections of the exterior algebra bundle and Connes' differential algebra is presented.

The content of the present paper reflects a talk given at the Workshop 'The Standard Model of Elementary Particle Physics from a mathematical-geometrical viewpoint' held at the Ev.-Luth. Volkshochschule Hesselberg near Gerolfingen, Germany, March 14-19, 1999.

In the first two sections we explain the Gel'fand and the Serre-Swan theorems to explain the background of ideas leading to noncommutative geometry. In section three Hermitian structures on vector bundles and generalized module bases called frames are introduced to have some more structural elements for proving. Furthermore, we give a short introduction to the theory of Clifford and spinor bundles over compact smooth Riemannian manifolds M . Following J. C. Várilly [24] we use the duality between vector bundles and projective finitely generated $C^{\infty}(M)$ -modules as described by the Serre-Swan theorem to give a comprehensive account to the commutative theory. The spectral triple is derived and the crucial properties of the Dirac operator are listed without proof. Further, we define the differential algebra of Connes' forms in the commutative setting and compare it to the set of all smooth sections of the exterior algebra bundle which forms also a differential algebra. The isomorphism of both these differential algebras is demonstrated by a new proof appearing here with the kind permission of its inventor Harald Upmeyer.

1. THE THEOREMS BY GEL'FAND AND SERRE-SWAN

One of the corner stones of the beginning of noncommutative geometry was I. M. Gel'fand's theorem published in 1940. He established an equivalence principle between some topological objects and algebraic-axiomatic structures that can be expressed in the following way (cf. [2, 18]):

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Theorem 1.1. (I. M. Gel'fand)

Let A be a commutative C^* -algebra and X the set of its characters. The topology on X should be that one induced by the weak* topology on the dual space A^* .

Then X is a locally compact Hausdorff space, and X is compact iff A is unital.

The C^* -algebra A is $*$ -isomorphic to the commutative C^* -algebra $C_0(X)$ of all continuous functions on X vanishing at infinity.

In a more contemporary language this bijection can be expressed as a categorical equivalence. We have to add a set of suitable morphisms to the sets of objects 'commutative C^* -algebras' and 'locally compact Hausdorff spaces'. They are called proper morphisms: for C^* -algebras we have to take $*$ -homomorphisms that map approximate identities to approximate identities, and for locally compact Hausdorff spaces we have to select those continuous maps for which the pre-image of a compact set is always compact. Then we can summarize the categorical equivalence:

$$\begin{array}{ll} \text{commutative } C^*\text{-algebras } C(X) & \Leftrightarrow \text{locally compact Hausdorff spaces } X \\ \text{proper } *\text{-homomorphisms} & \text{proper continuous homomorphisms} \end{array}$$

The noncommutative viewpoint enters the picture removing the commutativity condition on the multiplication in C^* -algebras. Algebraically the left side is still a proper category, and many theorems for commutative C^* -algebras can be generalized to the noncommutative situation. (But, there are also pure noncommutative structures like those described by Tomita-Takesaki theory.) However, the right side possesses no obvious candidate for a counterpart of the left side generalization to preserve the categorical equivalence. One reason is that the notion of a point that is crucial for any geometry becomes a vacuous notion under such an extension of the theory. Consequently, what we are left with is the algebraic noncommutative picture on the left side.

Looking for further topological and geometrical structures that can be categorically replaced by appropriate algebraic structures J.-P. Serre [21](1957/58) and R. G. Swan [22](1962) independently established a categorical equivalence between projective finitely generated $C(X)$ -modules and locally trivial vector bundles over X for compact Hausdorff spaces X . To describe it in greater detail some preparation is necessary.

To introduce both the notions, first, define a (left) unital A -module $\mathcal{H}$ over a unital algebra to be *projective finitely generated* if it is a direct summand (in an A -module sense) of a free A -module A^n for $n \in \mathbb{N}$, where A^n consists of all n -tuples of elements of A equipped with coordinate-wise addition and an action of A on A^n given as (left) multiplication of any n -tuple entry by fixed elements of A . The set of projective finitely generated A -modules can be equipped with the structure of direct sums $\oplus$ of A -modules. To introduce the structure of a module tensor product we have to consider them as A -bimodules defining another (right) action of A on A^n as a (right) multiplication of any n -tuple entry by fixed elements of A . The module tensor product $\mathcal{H}_1 \otimes_A \mathcal{H}_2$ is the algebraic tensor product of the linear spaces $\mathcal{H}_1, \mathcal{H}_2$ factored by the module ideal generated as the linear hull

$$\text{Lin}\{h_1 \otimes ah_2 - h_1a \otimes h_2 : h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2, a \in A\}.$$

We have associative and distributive laws for the addition and the tensor products and a commutative law for addition. The neutral element of addition is the A -module consisting

only of the zero element, and the neutral element of the module tensor product is the bimodule $A^1 = A$.

As the set of *homomorphisms* we consider all A -(bi-)module homomorphisms of projective finitely generated A -(bi-)modules.

The second structure involved in the stressed for categorical equivalence consists of locally trivial vector bundles over compact Hausdorff spaces X .

Definition 1.2. Given a topological space E , a compact Hausdorff space X and a continuous mapping $p : E \rightarrow X$. Then E is a *locally trivial vector bundle* (E, p, X) over X if for every $x \in X$ there exists a finite-dimensional vector space E_x (equipped with the Euclidean topology) and a neighborhood $U_x \subseteq X$ such that a homeomorphism $\phi : U_x \times E_x \rightarrow p^{-1}(U_x)$ exists and $p \circ \phi(x, e) = x$ for any $x \in U_x$. In case $U_x \equiv X$ the vector bundle is (*globally*) *trivial*.

A map $\phi : (E, p, X) \rightarrow (F, q, X)$ is a *vector bundle isomorphism* in case ϕ is bijective, ϕ and ϕ^{-1} are continuous and $\phi|_{E_x} : E_x \rightarrow F_x$ is linear for any $x \in X$. The map ϕ is a *vector bundle homomorphism* if ϕ is continuous and $\phi|_{E_x} : E_x \rightarrow F_x$ is a linear embedding as a subspace.

We call X the *base space*, E the *total space*, $E_x = p^{-1}(\{x\})$ the *fibre over x* and p the projection map.

Note, that the compactness of X implies $\sup(\dim(E_x)) < \infty$. As one of the alternative descriptions of vector bundles in geometry we can describe them in local terms : a vector bundle (E, p, X) is given by an atlas $\{U_\alpha\} \subset X$ of (open) charts and of coordinate homeomorphisms $\{f_\alpha : U_\alpha \times E_x \rightarrow p^{-1}(U_\alpha)\}$ ($x \in U_\alpha$) such that the transition functions

$$f_{\alpha\beta} := f_\beta^{-1} f_\alpha : (U_\alpha \cap U_\beta) \times E_x \rightarrow (U_\alpha \cap U_\beta) \times E_x$$

are described by $f_{\alpha\beta}(x, e) = (x, \overline{f_{\alpha\beta}}(x)e)$ with continuous functions $\overline{f_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{C})$ fulfilling the law

$$\overline{f_{\alpha\alpha}} = \text{id}_{U_\alpha}, \quad \overline{f_{\alpha\gamma} f_{\gamma\beta} f_{\beta\alpha}} = \text{id}_{U_\alpha \cap U_\beta \cap U_\gamma}.$$

We can show that the condition $\overline{f_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{C})$ can be always reduced to $\overline{f_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow U(n)$ (or, for real vector spaces, $\overline{f_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow O(n)$) changing the coordinate functions in a suitable way, cf. [17, 15]. The group $U(n)$ (or $O(n)$) is said to be the *structural group* of the vector bundle.

For further use we introduce the notion of an orientation on vector bundles over orientable compact manifolds.

Definition 1.3. Let M be an orientable compact manifold. The vector bundle (E, p, M) is *orientable* if there exists an atlas $\{U_\alpha\}$ describing E with transition functions $\{\overline{f_{\alpha\beta}}\}$ taking values in $GL^+(n, \mathbb{C})$. The corresponding atlas is said to be an *orientation of the vector bundle* (E, p, M) .

For a fixed compact Hausdorff space X the set of vector bundles with base space X can be equipped with some algebraic structure. The *Whitney sum of two vector bundles* (E, p, X)

and (F, q, X) is the vector bundle $(E \oplus F, p \oplus q, X)$, where

$$\begin{aligned} E \oplus F &:= \{(e, f) \in E \times F : p(e) = q(f) \in X\}, \\ (p \oplus q)(e, f) &:= p(e) = q(f) \in X. \end{aligned}$$

Local triviality is preserved under Whitney sums. The fibres are the vector spaces $E_x \oplus F_x$. The *tensor product of two vector bundles* (E, p, X) and (F, q, X) is the vector bundle $(E \otimes F, p \otimes q, X)$ with the fibres $E_x \otimes F_x$ for $x \in X$ and the transition functions $\overline{f_{\alpha\beta}}(x) := \overline{f_{\alpha\beta, E}(x)} \otimes \overline{f_{\alpha\beta, F}(x)}$ coming from a common atlas $\{U_\alpha\} \subset X$ of the vector bundles (E, p, X) and (F, q, X) . We observe that for trivial vector bundles $X \times \mathbb{C}^n =: \bar{n}$ the two operations are related by the isomorphism $E \otimes \bar{n} = \bigoplus_{i=1}^n E_{(i)}$, where $n \in \mathbb{N}$ is arbitrary. Concerning the algebraic properties of the two operations both they are associative and fulfil the obvious distributivity laws, and the Whitney addition is commutative in the sense of an appropriate isomorphism of vector bundles. The neutral elements are $\bar{0}$ and $\bar{1}$, respectively.

One of the central observations is Swan's theorem:

Theorem 1.4. (R. G. Swan, 1962)

Let (E, p, X) be a locally trivial vector bundle over a compact Hausdorff base space X . There exists a locally trivial vector bundle (F, q, X) over X such that $(E \oplus F, p \oplus q, X)$ is trivial (with finite-dimensional fibre).

The proof is elaborated, and we refer to R. G. Swan's paper [22] or to [8, 17, 15] for different versions of proofs.

Definition 1.5. A *section* in a vector bundle (E, p, X) is a continuous map $s : X \rightarrow E$ such that $(p \circ s)(x) = x$ for every $x \in X$. The set of sections of (E, p, X) is denoted by $\Gamma(E)$.

Proposition 1.6. *Let X be a compact Hausdorff space. Every locally trivial vector bundle admits non-trivial sections. For every vector bundle (E, p, X) the set $\Gamma(E)$ has the algebraic structure of a $C(X)$ -module.*

Any isomorphism of vector bundles induces an isomorphism of the corresponding modules of sections. Whitney sums of vector bundles correspond to direct $C(X)$ -module sums of the related modules of sections, tensor products of vector bundles correspond to bimodule tensor products.

For compact X the $C(X)$ -module $\Gamma(E)$ is projective and finitely generated, in particular, $\Gamma(X \times \mathbb{C}^n) \cong C(X)^n$ for every $n \in \mathbb{N}$.

Proof. The existence of continuous sections can be proved applying Uryson's Lemma to constant sections in the (trivial) part of the vector bundle over one chart U , getting continuous sections of the whole vector bundle supported in one chart U over which the vector bundle is trivial.

Any bundle homomorphism $\phi : (E, p, X) \rightarrow (F, q, X)$ maps sections in E to sections in F . If ϕ is a bundle isomorphism, then $\phi_* : \Gamma(E) \rightarrow \Gamma(F)$ is a $C(X)$ -module isomorphism. We observe that $\Gamma(X \times \mathbb{C}^n) \cong C(X)^n$. These $C(X)$ -modules are free and finitely generated. Since $C(X)^n \cong \Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$ for a given vector bundle (E, p, X) , some vector bundle (F, q, X) and $n < \infty$ by Swan's theorem, $\Gamma(E)$ is projective and finitely generated. $\square$

Theorem 1.7. (J.-P. Serre, 1957/58, R. G. Swan, 1962)

Let X be a compact Hausdorff space and $\mathcal{E}$ be a finitely generated projective $C(X)$ -module. If $\mathcal{E} \oplus \mathcal{G} \cong C(X)^n$ for some $n < \infty$, then let P be the projection of $C(X)^n$ onto $\mathcal{E}$ along $\mathcal{G}$. Interpreting P as an element of $M_n(C(X)) \cong C(X, M_n(\mathbb{C}))$ define

$$\Xi(\mathcal{E}) := \{(x, e) \in X \times \mathbb{C}^n : e \in \text{ran}(P)\}.$$

Then $\Xi(\mathcal{E})$ is a locally trivial vector bundle over X , $\Gamma(\Xi(\mathcal{E})) \cong \mathcal{E}$. Moreover, if $\mathcal{E} = \Gamma(E)$ for some vector bundle E , then $\Xi(\Gamma(E)) \cong E$.

Proof. $\Xi(\Gamma(E)) \cong E$: Assume $E \oplus F \cong X \times \mathbb{C}^n$ by Swan's theorem. Let $\pi_x : \mathbb{C}^n \rightarrow E_x$ be the fibrewise projection, $x \in X$. Define $\pi : X \times \mathbb{C}^n \rightarrow E$ by $\pi(x, e) = (x, \pi_x(e))$ for $x \in X$, $e \in \mathbb{C}^n$. Then π is a correctly defined surjective bundle homomorphism.

Let $P = \pi_* : \Gamma(X \times \mathbb{C}^n) \rightarrow \Gamma(E) \oplus \Gamma(F)$ be the induced $C(X)$ -module map, a projection onto $\Gamma(E)$. Note, that $P(x) = \pi_x$ for every $x \in X$. Therefore, $E = \Xi(\Gamma(E))$ by construction.

$\Gamma(\Xi(\mathcal{E})) = \mathcal{E}$: Note, that $(\Xi(\mathcal{E}))_x = \{x\} \times \{e \in \mathbb{C}^n : e \in \text{ran}(P(x))\}$ are the fibres of $\Xi(\mathcal{E})$. The family of projections $\{P(x)\}$ is continuous, and $\Xi(\mathcal{E})$ becomes a locally trivial vector bundle. Thus, $\Gamma(\Xi(\mathcal{E})) = \{f \in C(X, \mathbb{C}^n) : f \in \text{ran}(P) = \mathcal{E}\} \cong \mathcal{E}$. $\square$

Formulating the result in a categorical language we obtain a categorical equivalence between an algebraic and a geometric category if suitable sets of $C(X)$ -module and bundle homomorphisms are chosen:

$$\begin{array}{ccc} \text{projective, finitely generated } C(X)\text{-modules} & \Leftrightarrow & \text{locally trivial vector bundles } (E, p, X) \\ \text{proper } C(X)\text{-module maps} & & \text{proper bundle homomorphisms} \end{array}$$

We would like to point out that this categorical equivalence can be extended to the situation of infinite-dimensional fibres, however we will lose local triviality of the Banach bundles if we try to preserve a suitable category of $C(X)$ -modules like Banach or Hilbert $C(X)$ -modules on the left side. Moreover, most locally trivial bundles over compact Hausdorff spaces X with fibre l_2 turn out to be automatically globally trivial.

Now, we specify the compact Hausdorff space X to be a compact smooth manifold M . The observation to be made is that every locally trivial vector bundle over M with continuous transition functions in some atlas is in fact equipped with an atlas containing smooth transition functions, i.e. there is no reason to distinguish between 'continuous' and 'smooth' vector bundles over smooth compact manifolds M , cf. [17] for a proof.

Lemma 1.8. *For every vector bundle (E, p, M) there exists an atlas on M such that E is trivial over every chart U_α and the transition functions $\overline{f_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{C})$ are smooth functions.*

The Fréchet algebra $C^\infty(M)$ and the C^* -algebra $C(M)$ have the same set of characters: every character on $C^\infty(M)$ is automatically continuous and a measure on M and hence, a character of $C(M)$. Consequently, $C^\infty(M)^n \cong \Gamma^\infty(M \times \mathbb{C}^n)$, and the categorical equivalence between projective $C^\infty(M)$ -modules and vector bundles over M is a reduction of Serre-Swan's categorical equivalence. For the Whitney sum and the bundle tensor product

we get the following corresponding module operations on the $C^\infty(M)$ -modules of smooth sections:

$$\begin{aligned}\Gamma^\infty(E \oplus F) &= \Gamma^\infty(E) \oplus_{C^\infty(M)} \Gamma^\infty(F), \\ \Gamma^\infty(E \otimes F) &= \Gamma^\infty(E) \otimes_{C^\infty(M)} \Gamma^\infty(F).\end{aligned}$$

2. HERMITEAN STRUCTURES AND FRAMES FOR SETS OF SECTIONS

As an essential tool we need the existence and the properties of a continuous field of scalar products on the fibres of vector bundles. This structure is not needed to prove the Serre-Swan' theorem, it arises additionally.

Definition 2.1. Let X be a compact Hausdorff space and (E, p, X) be a vector bundle with base space X . A $C(X)$ -valued inner product on (E, p, X) is a bilinear mapping $\langle \cdot, \cdot \rangle : \Gamma(E) \times \Gamma(E) \rightarrow C(X)$ that is continuous in both the arguments, acts fibrewise (i.e. is $C(X)$ -linear in the first argument) and its restriction to any fibre E_x generates a scalar product on it. (Some authors refer to this structure as to a *Hermitean structure* on the vector bundle.)

Theorem 2.2. Let X be a compact Hausdorff space and (E, p, X) be a vector bundle with base space X . Then (E, p, X) admits $C(X)$ -valued inner products $\langle \cdot, \cdot \rangle$ on the $C(X)$ -module $\Gamma(E)$ such that $\Gamma(E)$ is complete with respect to the resulting norm $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$. Any two $C(X)$ -valued inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are related by a positive invertible $C(X)$ -linear operator S on $\Gamma(E)$ via the formula $\langle \cdot, \cdot \rangle_1 \equiv \langle S(\cdot), \cdot \rangle_2$.

If X is a smooth manifold then $\langle \cdot, \cdot \rangle$ can be chosen in such a way that its restriction to $\Gamma^\infty(E) \times \Gamma^\infty(E)$ takes values in $C^\infty(X)$.

Proof. Because of the categorical equivalence by J.-P. Serre and R. G. Swan it is sufficient to indicate the existence and the properties of $C(X)$ -valued inner products on finitely generated projective $C(X)$ - or $C^\infty(X)$ -modules. For $C(X)^n$ the $C(X)$ -valued inner product is defined as $\langle (f_1, \dots, f_n), (g_1, \dots, g_n) \rangle = \sum_{i=1}^n f_i \bar{g}_i$. For direct summands $P(C(X)^n)$ of $C(X)^n$ we reduce this $C(X)$ -valued inner product to elements of them.

The relation between two $C(X)$ -valued inner products follows from an analogue of Riesz' representation theorem for $C(X)$ -linear bounded module maps from $C(X)^n$ into $C(X)$. (Attention: This may fail for more general $C(X)$ -modules with $C(X)$ -valued inner products.)

If X is a smooth manifold, then the $C(X)$ -valued inner product defined above maps elements with smooth entries to smooth functions on X . A perturbation of the $C(X)$ -valued inner product by a positive invertible operator S that preserves the range $C^\infty(X)$ of it or the restriction to a direct summand of $C^\infty(X)^n$ do not change this fact. $\square$

We would like to remark that for more general $\ast$ -algebras $\mathcal{A} \subset C^\infty(X)$ that are closed under holomorphic calculus and contain the identity the property of $\mathcal{A}$ -valued inner products on the correspondingly reduced set of sections $\Gamma^\mathcal{A}(E) \subset \Gamma^\infty(E)$ to possess an analogue of the Riesz' property has to be axiomatically supposed, in general.

Now, we indicate the existence of finite sets of generators of $\Gamma^\infty(E)$ as a $C^\infty(M)$ -module for vector bundles (E, p, M) over smooth manifolds M . Consider the free $C^\infty(M)$ -module

$\Gamma^\infty(M \times \mathbb{C}^n) = C^\infty(M)^n$ for $n \in \mathbb{N}$ and a $C^\infty(M)$ -valued inner product $\langle \cdot, \cdot \rangle_0$ on it. Then there exists an orthonormal with respect to $\langle \cdot, \cdot \rangle_0$ basis consisting of n elements of this module. Indeed, on free $C(M)$ -modules $C(M)^n$ every $C(M)$ -valued inner product is related to the canonical $C(M)$ -valued inner product by a bounded invertible positive module operator S that fulfills the identity $\langle \cdot, \cdot \rangle_{can.} \equiv \langle S(\cdot), \cdot \rangle$. The restriction of $\langle \cdot, \cdot \rangle_{can.}$ to $C^\infty(M)^n$ is $C^\infty(M)$ -valued, and $\langle \cdot, \cdot \rangle_0$ can be extended to $C(M)^n$. So the linking operator S exists on $C(M)^n$, and its restriction to $C^\infty(M)^n$ maps smooth elements to smooth elements. However, the canonical $C^\infty(M)$ -valued inner product on $C^\infty(M)^n$ admits an orthonormal basis consisting of smooth elements:

$$\{e_1, \dots, e_n : e_i = (0, \dots, 0, 1_{(i)}, 0, \dots, 0)\}.$$

Consequently, $\{S^{-1/2}(e_i) : i = 1, \dots, n\}$ is an orthonormal basis of $C^\infty(M)^n$ with respect to the given $C^\infty(M)$ -valued inner product $\langle \cdot, \cdot \rangle_0$.

Let $\mathcal{E}$ be a projective finitely generated $C^\infty(M)$ -module, i.e. $\mathcal{E} \oplus \mathcal{F} = C^\infty(M)^n$ for a finite integer n . Denote by P the $C^\infty(M)$ -linear projection onto $\mathcal{E}$ along $\mathcal{F}$. Then the set $\{P(e_i) : i = 1, \dots, n\}$ of elements of $\mathcal{E}$ has the remarkable property that

$$\xi = \sum_{i=1}^n \langle \xi, P(e_i) \rangle_0 P(e_i)$$

for every $\xi \in \mathcal{E}$. The engineering literature on wavelets calls such sets of generators of Hilbert spaces (*normalized tight*) *frames*, whereas the literature on conditional expectations calls them *quasi-bases* or (*module*) *bases*. The notion 'basis' is, however, misleading since the elements of the generator sequence $\{P(e_i) : i = 1, \dots, n\}$ may allow a non-trivial $C^\infty(M)$ -linear decomposition of the zero element of $\mathcal{E}$. To see that let $\mathcal{E}$ be simply the subset of all elements admitting only all over equal entries in their n -tuple representation. For more details we refer the reader to [9]. To summarize the arguments we formulate

Theorem 2.3. *Let M be a smooth compact manifold and (E, p, M) be a vector bundle with base space M . Let $\langle \cdot, \cdot \rangle$ be a Hermitean structure on it. Then the projective finitely generated $C^\infty(M)$ -module $\Gamma^\infty(E)$ possesses a finite subset $\{\eta_i : i \in \mathbb{N}\}$ such that $\Gamma^\infty(E)$ is generated as a $C^\infty(M)$ -module by this set and the equality*

$$\xi = \sum_{i=1}^n \langle \xi, \eta_i \rangle \eta_i$$

is satisfied for every $\xi \in \Gamma^\infty(E)$.

3. CLIFFORD AND SPINOR BUNDLES, SPIN MANIFOLDS

Let (M, g) be a smooth Riemannian manifold, where the Riemannian metric g_x induces a scalar product on $T_x M$ for any $x \in M$. Note, that the tangent space $T_x M$ and the cotangent space $T_x^* M$ are isomorphic via the scalar product on $T_x M$ for any $x \in M$. If $(T_x M, g_x)$ denotes the Hilbert tangent space then let $(T_x^* M, g_x^{-1})$ denote the resulting Hilbert cotangent space.

Let $Cl(T_x M, g_x)$ be the real Clifford algebra of the tangent space $T_x M$ with respect to the scalar product induced by the Riemannian metric g_x , $x \in X$ arbitrarily fixed. This

algebra is defined to be a quotient of the tensor algebra $\mathcal{T}(T_x M)$ generated by the linear space $T_x M$, i.e. of

$$\mathcal{T}(T_x M) = \mathbb{C} \oplus T_x M \oplus (T_x M \otimes T_x M) \oplus \dots \oplus (T_x M \otimes \dots \otimes T_x M) \oplus \dots$$

More precisely,

$$Cl(T_x M, g_x) := \mathcal{T}(T_x M) / \text{Ideal}(e \otimes e - g_x(e, e) : e \in T_x M).$$

The real Clifford algebra $Cl(T_x M, g_x)$ possesses a $\mathbb{Z}_2$ -grading induced by the map $\chi_x : (x, e) \in T_x M \rightarrow (x, -e) \in T_x M$, i.e. by the linear operator χ on $Cl(T_x M, g_x)$ with the property $\chi^2 = \text{id}$, with eigen-values $\{1, -1\}$ and isomorphic eigenspaces $Cl^{\text{even}}(T_x M, g_x)$, $Cl^{\text{odd}}(T_x M, g_x)$ summing up to the algebra itself. The words 'even' and 'odd' refer to the highest degree of the element under consideration and its property to be an even or odd number. If $n = 2m + 1$ then χ is realized as a multiplication by a central element. Extending the isomorphism between tangent space and cotangent space via the scalar product g_x on the first space we obtain a canonical algebraic isomorphism of the Clifford algebras $Cl(T_x M, g_x)$ and $Cl(T_x^* M, g_x)$ for any fixed $x \in X$.

The *Clifford algebra bundle* $\mathcal{Cl}(M)$ is defined fibrewise using the atlas on M induced by the tangent bundle atlas of TM (or the cotangent bundle atlas of T^*M):

$$\mathcal{Cl}_x(M) := Cl(T_x M, g_x) \otimes_{\mathbb{R}} \mathbb{C} \cong \begin{cases} M_{2^m}(\mathbb{C}) & : n = 2m \\ M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) & : n = 2m + 1 \end{cases}.$$

Note that the isomorphism τ is quite complicated, and in case $n = 2m + 1$ it maps both the even and the odd part of the Clifford algebra to both the blocks of the matrix sum at the right (see [10, p. 15] for details). The Clifford bundle possesses a $\mathbb{Z}_2$ -grading induced from that one on its fibres. The C^* -algebra structure of $\Gamma(\mathcal{Cl}(M))$ comes from the algebra structure of the Clifford algebra fibres and from the involution induced by $\otimes_{\mathbb{R}} \mathbb{C}$ from $\mathbb{C}$. The C^* -norm exists and is uniquely defined since the multiplication and the involution are given and every fibre is finite-dimensional.

Some authors (cf. [24]) prefer to restrict the Clifford algebra bundle to the even part in case the dimension of the manifold is $n = 2m + 1$. The loss of that alternative definition is the $\mathbb{Z}_2$ -grading. The advantage of that approach is the structure of $\mathcal{Cl}(M)$ as a continuous field of *simple* C^* -algebras allowing the attempt to interpret this bundle as a homomorphism bundle derived from some other vector bundle with base space M . We prefer to postpone this reduction until the spinor bundle has to be built up.

Consider either the Clifford algebra bundle $\mathcal{Cl}(M)$ over M for $n = 2m$ or the first matrix block part $\mathcal{Cl}(M)^\dagger$ of the Clifford algebra bundle $\mathcal{Cl}(M)$ over M for $n = 2m + 1$ (in its matrix representation) locally: for every $x \in X$ we find vector spaces S_x such that the (first part of the) Clifford algebra bundle is locally isomorphic to the trivial homomorphism bundle $\text{Hom}(S_x)$ of the trivial bundle $(U \times S_x, p_U, U)$. The C^* -algebra structure on $\Gamma(\mathcal{Cl}_x(M))$ (resp., $\Gamma(\mathcal{Cl}_x(M)^\dagger)$) induces a unique scalar product on S_x compatible with it. The dimension of the linear spaces S_x is constant and equals $\dim(S_x) = 2^m$ for any manifold dimensions $n, m := [n/2]$.

Whether we can glue these trivial pieces together to obtain a vector bundle S over the compact Riemannian manifold M carrying an irreducible left action of the Clifford bundle (resp., the first part of it) that acts locally in the manner described, or not? Unfortunately,

not always. If $n = 2m$ the Clifford bundle $\text{Cl}(M)$ serves as a homomorphism bundle for some other vector bundle with the same compact base space M if and only if the Dixmier-Douady class $\delta(\text{Cl}(M)) \in H^3(M, \mathbb{Z})$ equals zero, where $\delta(\text{Cl}(M))$ also equals the third integral Stiefel-Whitney class $w_3(TM) \in H^3(M, \mathbb{Z})$. If $n = 2m + 1$ the first part $\text{Cl}(M)^\dagger$ of the Clifford bundle is a homomorphism bundle of some other vector bundle if and only if the second part of it does so, if and only if the Dixmier-Douady class $\delta(\text{Cl}(M)^\dagger) \in H^3(M, \mathbb{Z})$ equals zero, where $\delta(\text{Cl}(M)^\dagger)$ also equals the third integral Stiefel-Whitney class $w_3(TM) \in H^3(M, \mathbb{Z})$. The first fact was observed by J. Dixmier in [7, Th. 10.9.3] and again investigated in connection with spinor bundles by R. J. Plymen [19].

To formulate the definition of a spinor bundle on a given compact smooth Riemannian manifold M or, equivalently, the definition of the property of M to be a Spin^c -manifold we have to introduce the notion of Morita equivalence of certain unital $*$ -algebras. We will do that only for the two $*$ -algebras of interest, for more general cases we refer to [20, 19]. Let us fix the unital $*$ -algebra

$$B = \begin{cases} C^\infty(M, \text{Cl}(M)) = \Gamma^\infty(\text{Cl}(M)) & : n = 2m \\ C^\infty(M, \text{Cl}(M)^\dagger) = \Gamma^\infty(\text{Cl}(M)^\dagger) & : n = 2m + 1 \end{cases}.$$

Definition 3.1. Let M be a compact smooth Riemannian manifold. Consider the unital $*$ -algebras $A = C^\infty(M)$ and B . They are *Morita-equivalent as algebras* if there exists a B - A bimodule $\mathcal{E}$ and an A - B bimodule $\mathcal{F}$ such that $\mathcal{E} \otimes_A \mathcal{F} \cong B$ and $\mathcal{F} \otimes_B \mathcal{E} \cong A$ as B - and A -bimodules, respectively.

In our case $\mathcal{F}$ can be chosen to be a projective and finitely generated (left) module over the unital $*$ -algebra $A = C^\infty(M)$ denoted by $\tilde{S}$. As a projective finitely generated $C^\infty(M)$ -module $\tilde{S}$ admits a $C^\infty(M)$ -valued inner product $\langle \cdot, \cdot \rangle_{C^\infty(M)}$. Then B can be realized as the $*$ -algebra of bounded module operators over $\tilde{S}$ generated as $\text{Lin}\{\langle \xi, \eta \rangle_{C^\infty(M, \text{Cl}(M))} : \xi, \eta \in \tilde{S}\}$, where

$$\langle \xi, \eta \rangle_{C^\infty(M, \text{Cl}(M))}(\nu) := \langle \nu, \xi \rangle_{C^\infty(M)} \eta \quad \text{for } \nu \in \tilde{S}.$$

So the counterpart $\mathcal{E}$ of $\mathcal{F}$ can be described as the set $\{\bar{\xi} : \xi \in \mathcal{F}, \bar{\xi}a := \overline{a^*\xi}, a \in A\}$. Obviously, the right action of B on $\mathcal{F}$ is simultaneously swept to a left B -action on $\mathcal{E}$. The C^∞ -module $\mathcal{F}$ together with the $C^\infty(M)$ -valued inner product $\langle \cdot, \cdot \rangle_{C^\infty(M)}$ is said to be a *B - A imprimitivity bimodule*.

Definition 3.2. (R. J. Plymen, 1982)

Let (M, g) be a compact smooth Riemannian manifold, let $A = C^\infty(M)$ and B as defined above in dependency on the dimension of M . Both A and B are unital $*$ -algebras of smooth mappings.

We say that *the tangent bundle TM of M admits a Spin^c -structure* if TM is orientable as a vector bundle and the Dixmier-Douady class $\delta(\text{Cl}(M))$ equals zero for $n = 2m$ or, respectively, $\delta(\text{Cl}(M)^\dagger) = 0$ for $n = 2m + 1$.

If this condition is fulfilled then *the Spin^c -structure on TM* is a pair $(\epsilon, \tilde{S})$ consisting of an orientation ϵ of TM and a B - A imprimitivity bimodule $\tilde{S}$.

The compact smooth Riemannian manifold M is a $\text{Spin}^{\mathbb{C}}$ -manifold if the tangent bundle TM of M admits a $\text{Spin}^{\mathbb{C}}$ -structure.

Questions like existence or uniqueness of $\text{Spin}^{\mathbb{C}}$ -structures are complicated and depend on several properties of the manifold M . For an accessible and detailed geometrical account see [10].

By the Serre-Swan' theorem the B - A imprimitivity bimodule $\tilde{S}$ can be realized as the $C^\infty(M)$ -module of smooth sections $\Gamma^\infty(S)$ of a uniquely determined vector bundle (S, p_S, M) with base space M . The vector bundle (S, p_S, M) is called *the spinor bundle*.

If $n = 2m$ then the spinor bundle admits a non-trivial $\mathbb{Z}_2$ -grading arising from the grading of the Clifford bundle $\mathcal{C}\ell(M)$: $S = S^+ \oplus S^-$, where $\dim(S_x^+) = \dim(S_x^-) = 2^{m-1}$. Furthermore, the set of smooth sections of S always admits a $C^\infty(M)$ -valued inner product $\langle \cdot, \cdot \rangle_{C^\infty(M)}$. The smooth sections of the spinor bundles are called *spinors*, or *chiral vector fields* in physics.

Definition 3.3. Let H be the Hilbert space

$$H := \overline{\left\{ \xi \in \Gamma^\infty(S) : \int_M \langle \xi, \xi \rangle_{C^\infty(M)} dg < +\infty \right\}},$$

Sometimes H is referred to as *the spinor Hilbert space*. The Hilbert space H consists of all square-integrable sections of the spinor bundle S , i.e. $H = L_2(M, S)$.

To obtain more well-behaved structure we have to assume additionally that the manifold M is compact. The spinor Hilbert space H inherits the non-trivial $\mathbb{Z}_2$ -grading arising from the grading of the spinor bundle in case $n = 2m$: $H = H^+ \oplus H^-$, where $H^\pm := L_2(M, S^\pm)$.

The sections of the Clifford bundle $\mathcal{C}\ell(M)$ act naturally on H . To look for details recall that $\Gamma(\mathcal{C}\ell(M)) = C(M) + \Gamma(T^*M) + \dots$. Then the elements of $C(M)$, i.e. of the zeroth component of $\Gamma(\mathcal{C}\ell(M))$, act as multiplication operators on $\Gamma(S)$ and, hence, on H by continuity. Identifying $\Gamma^\infty(T^*M)$ by the $C^\infty(M)$ -module $A^1(M)$ of 1-forms on M , the images of 1-forms under γ fulfill the rule

$$\gamma(\alpha)\gamma(\beta) + \gamma(\beta)\gamma(\alpha) = 2g^{ij}\alpha_i\beta_j \quad \text{for } \alpha, \beta \in A^1(M).$$

Consequently, $\gamma(dx^k)^2 > 0$ and non-trivial 1-forms are faithfully represented. The representation $\gamma : \Gamma(\mathcal{C}\ell(M)) \rightarrow B(H)$ is called *the spin representation*. We will use it again in the last part of the present survey.

4. SPIN CONNECTION AND DIRAC OPERATOR

Let (M, g) be a compact smooth Riemannian $\text{Spin}^{\mathbb{C}}$ -manifold, where the Riemannian metric g_x induces a scalar product in the cotangent spaces T_x^*M for any $x \in M$. The Riemannian metric g on M gives rise to a unique *Levi-Civita connection* ∇^g (or *Riemannian connection*). It is defined on (contra-/covariant) C^∞ -tensor fields over M of arbitrary order, ∇^g is symmetric, $\nabla^g(g^{ij}) = 0$ and the torsion of ∇^g vanishes.

In particular, $\nabla^g : A^1(M) \rightarrow A^1(M) \otimes_A A^1(M)$ obeying a Leibniz rule:

$$\nabla^g(\omega a) = \nabla^g(\omega)a + \omega \otimes da$$

for $a \in C^\infty(M)$ and arbitrary tensor fields ω on M . Lifting this Levi-Civita connection to the spinor bundle S (where $\Gamma(S)$ is equipped with the $C(M)$ -valued inner product arising from $\hat{S}$) we obtain another Levi-Civita connection there, the spin connection.

Definition 4.1. The *spin connection* is an operator $\nabla^S : \Gamma^\infty(S) \rightarrow \Gamma^\infty(S) \otimes_A A^1(M)$ that is linear and satisfies the two Leibniz rules

$$\nabla^S(\psi a) = \nabla^S(\psi)a + \psi \otimes da,$$

$$\nabla^S(\gamma(\omega)\psi) = \gamma(\nabla^g(\omega))\psi + \gamma(\omega)\nabla^S(\psi)$$

for $a \in A = C^\infty(M)$, $\omega \in A^1(M) = \Gamma^\infty(T^*M)$, $\psi \in \Gamma^\infty(S)$.

The spin connection on the spinor bundle (S, p_S, M) gives rise to the Dirac operator acting on the spinor Hilbert space H .

Definition 4.2. Let $m : \Gamma^\infty(S) \otimes_A A^1(M) \rightarrow \Gamma^\infty(S)$ be the mapping defined by the rule $m(\psi \otimes \omega) = \gamma(\omega)(\psi)$ for $\omega \in A^1(M) = \Gamma^\infty(T^*M) \subset \Gamma^\infty(\text{Cl}(M))$, $\psi \in \Gamma^\infty(S)$. The *Dirac operator on S* is the mapping $\mathcal{D} := m \circ \nabla^S$ that acts on the domain $\Gamma^\infty(S) \subset H$ of the spinor Hilbert space H as an unbounded operator.

The Dirac operator $\mathcal{D}$ has a number of remarkable properties. We list them without proof. For references see [5, 10, 17, 15, 23]:

- If $n = 2m$ then $\mathcal{D} : \Gamma^\infty(S^\pm) \rightarrow \Gamma^\infty(S^\mp)$. Moreover, with respect to this decomposition of $\Gamma^\infty(S)$ the Dirac operator can be represented as

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^+ \\ \mathcal{D}^- & 0 \end{pmatrix}, \quad \langle \mathcal{D}^+(h^+), h^- \rangle = \langle h^+, \mathcal{D}^-(h^-) \rangle$$

for $h^\pm \in \Gamma^\infty(S^\pm)$.

- If $n = 2m$ and $\chi : (h^+, h^-) \in H^+ \oplus H^- \rightarrow (h^+, -h^-) \in H^+ \oplus H^-$ is the grading operator on the spinor Hilbert space H then $\chi\mathcal{D} + \mathcal{D}\chi = 0$.
- $\mathcal{D}$ is symmetric and extends to an unbounded self-adjoint operator on H . (Same denotation.)
- $[\mathcal{D}, a]$ is compact and $[[\mathcal{D}, a], b] = 0$ for every $a, b \in C^\infty(M)$.
- $\mathcal{D}$ is a Fredholm operator, i.e. $\ker(\mathcal{D})$ is finite-dimensional.
- The operator $\mathcal{D}^{-1}$ defined on the orthogonal complement of $\ker(\mathcal{D})$ is compact. The eigenvalues $\{\lambda_k\}$ of $\mathcal{D}^{-1}$ counted with multiplicity fulfil the relation $\lambda_k \leq C \cdot k^{-1/n}$ for some constant C and $n = \dim(M)$.
- The spectrum of $\mathcal{D}$ is discrete and consists of eigenvalues of finite multiplicity.
- $\mathcal{D}$ is an elliptic first order differential operator.
- The algebra $A = C^\infty(M)$ is represented on the spinor Hilbert space H by multiplication operators (via γ). We obtain

$$[\mathcal{D}, a] = \mathcal{D}(a\psi) - a\mathcal{D}(\psi) = \gamma(da)\psi$$

for $a \in A = C^\infty(M)$, $\psi \in H$. In particular, since a is smooth and M is compact, the operator $[\mathcal{D}, a]$ is bounded with the sup-norm $\|\gamma(da)\|_\infty$ of the multiplication operator by $\gamma(da)$.

- For the geodesic distance of two points $p, q \in M$ we have

$$d(p, q) = \sup\{|\hat{p}(a) - \hat{q}(a)| : a \in C^\infty(M), \|[\mathcal{D}, a]\| \leq 1\},$$

where $\hat{p}$ is the character on $C^\infty(M)$ induced by evaluation in $p \in M$ and $\|\gamma(da)\|_\infty = \|a\|_{Lip} = \|[\mathcal{D}, a]\|$.

- The Lichnrowicz formula is valid:

$$\mathcal{D}^2 = \Delta^S + \frac{1}{4}R,$$

where R is the scalar curvature of the metric and Δ^S is the Laplacian operator lifted to the spinor bundle that can be described in local coordinates by $\Delta^S = -g^{ij}(\nabla_i^S \nabla_j^S - \Gamma_{ij}^k \nabla_k^S)$ with Γ_{ij}^k the Christoffel symbols of the connection.

- For any $f \in C^\infty(M)$ one has the formula

$$\int_M f dg = \frac{(-1)^n n \Gamma(n/2)}{2^{[n/2]+1-n} \pi^{-n/2}} \cdot \text{Tr}_\omega(f |\mathcal{D}|^{-n}),$$

where Tr_ω denotes the Dixmier trace.

5. THE UNIVERSAL DIFFERENTIAL ALGEBRA $\Omega C^\infty(M)$ AND CONNES' DIFFERENTIAL ALGEBRA $\Omega_{\mathcal{D}} C^\infty(M)$

As a good source for the commutative approach to differential algebras we can refer to the monograph of G. Landi [13]. Complementary information can be found in [16].

Definition 5.1. Let M be a compact smooth manifold. Identify a suitable completion of the algebraic tensor product $C^\infty(M) \odot \dots \odot C^\infty(M)$ with $C^\infty(M \times \dots \times M)$, the same number of $\odot/\times$ operations supposed.

The *universal differential algebra* $\Omega C^\infty(M) = \oplus_p \Omega^p C^\infty(M)$ is defined by the linear spaces:

$$\begin{aligned} \Omega^0 C^\infty(M) &:= C^\infty(M) \\ \Omega^p C^\infty(M) &:= \overline{\{f \in \odot_1^{p+1} C^\infty(M) : \\ &\quad f(x_1, \dots, x_{k-1}, x, x, x_{k+2}, \dots, x_{p+1}) = 0, \forall k\}} \end{aligned}$$

The *exterior differential* $\delta : \Omega^p \rightarrow \Omega^{p+1}$ is defined by

$$\begin{aligned} (\delta f)(x_1, x_2) &:= f(x_2) - f(x_1) \\ (\delta f)(x_1, \dots, x_{p+1}) &:= \sum_{k=1}^{p+1} (-1)^{k-1} f(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{p+1}) \end{aligned}$$

The $C^\infty(M)$ -bimodule structure on $\Omega C^\infty(M) := \oplus_p \Omega^p C^\infty(M)$ is given by:

$$\begin{aligned} (gf)(x_1, \dots, x_{p+1}) &:= g(x_1) f(x_1, \dots, x_{p+1}) \\ (fg)(x_1, \dots, x_{p+1}) &:= f(x_1, \dots, x_{p+1}) g(x_{p+1}) \end{aligned}$$

It extends to a general multiplication by the formula

$$(fh)(x_1, \dots, x_{(p+q)+1}) := f(x_1, \dots, x_{p+1}) h(x_{p+1}, \dots, x_{(p+q)+1})$$

for $f \in \Omega^p C^\infty(M)$, $h \in \Omega^q C^\infty(M)$.

Key properties of the exterior differential are linearity, the Leibniz rule and the vanishing of its square:

$$\begin{aligned}\delta(ab) &= (\delta a)b + (-1)^p a(\delta b) \quad , \quad \delta^2 = 0, \\ \delta(\alpha a + \beta b) &= \alpha(\delta a) + \beta(\delta b)\end{aligned}$$

for $a \in \Omega^p C^\infty(M)$, $b \in \Omega C^\infty(M)$, $\alpha, \beta \in \mathbb{C}$. These three properties give rise to another representation of the differential algebra as a linear hull of standard elements as it is used in the noncommutative case:

$$\Omega^p C^\infty(M) = \text{Lin}\{a_0 \delta a_1 \dots \delta a_p : a_i \in C^\infty(M)\},$$

$$\delta(a_0 \delta a_1 \dots \delta a_p) = \delta a_0 \delta a_1 \dots \delta a_p.$$

We take the parity of the degree p as a grading for the differential algebra $\Omega C^\infty(M) = \bigoplus_p \Omega^p C^\infty(M)$.

To go further and to construct Connes' differential algebra we need another property of our compact smooth manifold M – it has to be Riemannian and $\text{Spin}^{\mathbb{C}}$. Then we have a spectral triple $(C^\infty(M), H = L_2(M, S), \not{D})$ by construction, and we consider an algebraic representation of $\Omega C^\infty(M)$ on $B(H)$:

$$\pi : \Omega C^\infty \rightarrow B(H) \quad , \quad \pi(a_0 \delta a_1 \dots \delta a_p) := a_0 [\not{D}, a_1] \dots [\not{D}, a_p]$$

where $a_i \in C^\infty(M)$, and $C^\infty(M)$ acts on $\tilde{S} \subseteq H$ by the usual module action. (If one introduces an involution on the differential algebra then π becomes a $*$ -representation, however we do not need this additional structure for our purposes.) If we want π to be a representation commuting with the action of the differential in some way we run into difficulty since $\pi(\omega) = 0$ does not imply $\pi(\delta\omega) = 0$, in general. Fortunately, there exists a differential ideal of $\Omega C^\infty(M)$, the 'junk ideal' J .

Lemma 5.2. *Let $J_0 := \bigoplus_p J_0^p$ be the graded two-sided ideal of $\Omega C^\infty(M)$ given by*

$$J_0^p := \{\omega \in \Omega^p C^\infty(M) : \pi(\omega) = 0\}.$$

Then $J := J_0 + \delta J_0$ is a graded differential two-sided ideal of $\Omega C^\infty(M)$.

Definition 5.3. (A. Connes)

The *graded differential algebra of Connes' forms* over the algebra $C^\infty(M)$ is defined by

$$\Omega_p C^\infty(M) := \Omega C^\infty(M) / J \cong \pi(\Omega C^\infty(M)) / \pi(\delta J_0).$$

The space of Connes' p -forms is $\Omega_p^p C^\infty(M) = \Omega^p C^\infty(M) / J^p$. On $\Omega_p C^\infty(M)$ there exists a differential induced by δ with the usual properties:

$$d : \Omega_p^p C^\infty(M) \rightarrow \Omega_{p+1}^p C^\infty(M) \quad , \quad d([\omega]) := [\delta\omega] \cong \pi([\delta\omega]).$$

6. THE EXTERIOR ALGEBRA BUNDLE $\Lambda(M)$ AND THE DE RHAM COMPLEX

Let M be a compact smooth manifold equipped with an atlas inherited from the cotangent bundle T^*M . Denote by $\Lambda(T_x^*M)$ the real exterior algebra of the cotangent space T_x^*M , $x \in X$. Recall that

$$\Lambda(T_x^*M) := \mathcal{T}(T_x^*M) / \text{Ideal}(e \otimes e : e \in T_x^*M).$$

The real exterior algebra $\Lambda(T_x^*M)$ possesses a $\mathbb{Z}_2$ -grading, i.e. a linear operator χ on it with $\chi^2 = \text{id}$, eigenvalues $\{1, -1\}$ and isomorphic eigen-spaces $\Lambda^+(T_x^*M)$, $\Lambda^-(T_x^*M)$ summing up to the algebra itself. The signs $\pm$ stand for the parity of the degree p of the exterior form. An exterior p -form on M is locally given by

$$\omega = \sum_{i_1, \dots, i_p} a_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

with smooth functions $a_{i_1, \dots, i_p}(x)$ defined on a chart U .

Definition 6.1. The *exterior algebra bundle* $\Lambda(M)$ is fibrewise defined using the atlas on M induced by the cotangent bundle atlas of T^*M :

$$\Lambda_x(M) := \Lambda(T_x^*M) \otimes_{\mathbb{R}} \mathbb{C}, \quad x \in X.$$

Consequently, $\Lambda^0(M)$ is a trivial line bundle over M and $\Lambda^1(M) = T^*M$ and $\Lambda^k(M) = 0$ for $k > n = \dim(M)$. The set $\Lambda^p(M)$ is said to be the set of all p -forms, and $\Lambda(M) := \bigoplus_p \Lambda^p(M)$ is a linear space by definition. The multiplication is fibrewise defined by the $\wedge$ -multiplication of $\Lambda_x(M)$, i.e. $\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1$ for $\omega_1 \in \Lambda^p$, $\omega_2 \in \Lambda^q$. The exterior differential $d : \Gamma^\infty(\Lambda^p(M)) \rightarrow \Gamma^\infty(\Lambda^{p+1}(M))$ induced by the local differential d_x on $\Lambda(T_x^*M)$ is linear and obeys the rules

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d\omega_1 \wedge \omega_2 + (-1)^p \cdot \omega_1 \wedge d\omega_2, \\ d(d\omega) &\equiv 0 \end{aligned}$$

for $\omega_1 \in \Lambda^p$, $\omega_2 \in \Lambda^q$. Moreover, in local coordinates we have

$$\begin{aligned} df &= \sum_i \frac{\partial f}{\partial x_i} dx^i \quad \text{for } f \in C^\infty(M), \\ d\omega &= \sum_{i_1, \dots, i_p} da_{i_1, \dots, i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}, \quad \omega \in \Lambda^p. \end{aligned}$$

As a result we obtain a complex, the de Rham' complex

$$0 \rightarrow \Gamma^\infty(\Lambda^0(M)) \xrightarrow{d} \Gamma^\infty(\Lambda^1(M)) \xrightarrow{d} \dots \xrightarrow{d} \Gamma^\infty(\Lambda^n(M)) \rightarrow 0$$

that gives rise to cohomology groups that are isomorphic to the cohomology groups $H^*(M, \mathbb{R})$. The name of the complex comes from the application of de Rham's theorem to this particular situation, cf. [17, 15].

7. $\Omega_p C^\infty(M)$ VERSUS $\Lambda(M)$

The final goal of the present notes is another theorem relating a structure formally depending on the Riemannian metric on the compact smooth manifold M to another structure that does not depend even on its existence or absence. This observation by A. Connes was the starting point for his subsequent noncommutative generalizations, cf. [5].

Theorem 7.1. (*A. Connes*)

Comparing the components of Connes' differential algebra $\Omega_p C^\infty(M)$ and the smooth sections of components of the exterior algebra bundle $\Lambda(M)$ we obtain an isomorphism $\Omega_p C^\infty(M) \cong \Gamma^\infty(\Lambda^p(M))$ for every $p \geq 0$. Moreover, it extends to the commutative diagrams

$$\begin{array}{ccc} \Omega_p C^\infty(M) & \xrightarrow{d} & \Omega_{p+1} C^\infty(M) \\ \downarrow \cong & & \downarrow \cong \\ \Gamma^\infty(\Lambda^p(M)) & \xrightarrow{d} & \Gamma^\infty(\Lambda^{p+1}(M)) \end{array}$$

showing an equivalence of differential algebras.

Harald Upmeyer kindly communicated a new approach to a proof of this theorem. We present here the variant that arose after some discussions and that preserves his basic ideas.

Proof. For every $p \geq 0$ consider the subbundle $\mathcal{C}l(M)^{p-ev}$ that consists of the intersection of the subbundle of all elements of $\mathcal{C}l(M)$ of degree at most p with either the subbundle $\mathcal{C}l(M)^{even}$ or $\mathcal{C}l(M)^{odd}$ in accordance with the parity of p . In the same manner we define $\Omega^{p-ev} C^\infty(M) = \oplus_{k=p-ev} \Omega^k C^\infty(M)$, where k runs over all indices between 0 and p differing from p by zero or an even number.

Claim 1: $\pi(\Omega^{p-ev} C^\infty(M)) \cong \gamma(\Gamma^\infty(\mathcal{C}l^{p-ev}(M)))$ for every $p \in \mathbb{N}$.

We prove the claim by induction. For $p = 0$ a comparison of the definitions shows that $\pi(f) = \gamma(f) = f \cdot \text{id}_H$ for every $f \in C^\infty(M)$. In case $p = 1$ we obtain $\pi(df) = [\not{D}, \pi(f)] = [\not{D}, \gamma(f)] = \gamma(df)$ for every $f \in C^\infty(M)$.

To show the general argument recall that the complexified Clifford algebra of a real vector space V and the complexified exterior algebra of V are related by the isomorphisms $Cl_{\mathbb{C}}^{p-ev}(V)/Cl_{\mathbb{C}}^{(p-2)-ev}(V) \cong \Lambda_{\mathbb{C}}^p(V)$ for every $p \in \mathbb{N}$. So there exist induced *symbol maps* between the components of the Clifford bundle and the exterior algebra bundle over M ,

$$\sigma^p : \Gamma^\infty(\mathcal{C}l^{p-ev}(M)) \rightarrow \Gamma^\infty(\Lambda^p(M)),$$

$p \in \mathbb{N}$, with kernels $\ker(\sigma^p) = \Gamma^\infty(\mathcal{C}l^{(p-2)-ev}(M))$. Consequently, every smooth section of the p -th component $\mathcal{C}l^p(M)$ of the Clifford bundle can be represented as a finite linear combination of elementary elements of the form $\{f_0 df_1 \dots df_p : f_i \in C^\infty(M)\}$, where the central dot denotes the Clifford multiplication. To see this apply Swan's theorem and take the projection P of the canonical orthonormal basis $\{e_i\}$ of the trivial bundle that houses $\Lambda^p(M)$ as a direct summand. By Theorem 2.3 the set $\{P(e_i)\}$ generates $\Gamma^\infty(\Lambda^p(M))$ as a $C^\infty(M)$ -module. There exists a finite atlas of M and a partition of unity $\{u_\alpha\}$ corresponding to it such that every component $u_\alpha P(e_i)$ of a certain generator $P(e_i)$ can be written as a finite sum of elements of the set $\{f_{0,\alpha} df_{1,\alpha} \wedge \dots \wedge df_{p,\alpha} : f_{i,\alpha} \in C^\infty(M), \text{supp}(f_{i,\alpha}) \subseteq \text{supp}(u_\alpha)\}$. Since all sums are finite we get the desired decomposition property for smooth sections of $\Lambda^p(M)$. Pulling this system of generators back via σ^p we obtain it for smooth sections of $\mathcal{C}l^p(M)$, too.

To show the inclusion $\pi(\Omega^{p-ev} C^\infty(M)) \subseteq \gamma(\Gamma^\infty(\mathcal{C}l^{p-ev}(M)))$ we have only to check the canonical elements $f_0 df_1 df_2 \dots df_p \in \Omega^p C^\infty(M)$ since the inclusion is supposed to be already

established for lower degrees by induction. We have

$$\begin{aligned}\pi(f_0 df_1 df_2 \dots df_p) &= \pi(f_0)[\mathcal{D}, \pi(f_1)] \dots [\mathcal{D}, \pi(f_p)] \\ &= \gamma(f_0)\gamma(df_1) \dots \gamma(df_p) \\ &= \gamma(f_0 df_1 \dots df_p).\end{aligned}$$

Conversely, we have to show that $\pi(\Omega^{p-ev}C^\infty(M)) \supseteq \gamma(\Gamma^\infty(\mathcal{C}^{p-ev}(M)))$ for every $p \in \mathbb{N}$. By the results of our considerations on the symbol maps and by induction we have to verify the inclusion only for finite sums $c = \sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p,l} \in \mathcal{C}^p(M)$. We get

$$\gamma(c) = \sum_{fin.,l} \gamma(f_{0,l})[\mathcal{D}, f_{1,l}] \dots [\mathcal{D}, f_{p,l}] = \pi\left(\sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p,l}\right) \in \pi(\Omega^p C^\infty(M)).$$

This establishes the statement of the first claim.

Claim 2: $\pi(d \ker(\pi^{p-1})) \equiv \gamma(\ker(\sigma^p))$ for any $p \in \mathbb{N}$ with $p \geq 2$.

Suppose, $\omega = \sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p-1,l} \in \ker(\pi^{p-1}) \subset \Omega_{\mathcal{D}}^{p-1}$. By the first step

$$\gamma\left(\sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p-1,l}\right) = \pi(\omega) = 0$$

and $\sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p-1,l} = 0$ since γ is injective on such elementary elements. Consider $d\omega = \sum_{fin.,l} df_{0,l} \cdot df_{1,l} \dots df_{p-1,l} \in \mathcal{C}^p(M)$:

$$\begin{aligned}\pi(d\omega) &= \gamma\left(\sum_{fin.,l} df_{0,l} \cdot df_{1,l} \dots df_{p-1,l}\right) \\ \sigma^p\left(\sum_{fin.,l} df_{0,l} \cdot df_{1,l} \dots df_{p-1,l}\right) &= \sum_{fin.,l} df_{0,l} \wedge df_{1,l} \wedge \dots \wedge df_{p-1,l} \\ &= d\left(\sum_{fin.,l} f_{0,l} df_{1,l} \wedge \dots \wedge df_{p-1,l}\right) \\ &= d\sigma^{p-1}\left(\sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p-1,l}\right) \\ &= 0.\end{aligned}$$

Consequently, $\pi(d \ker(\pi^{p-1})) \subseteq \gamma(\ker(\sigma^p))$ for every $p \in \mathbb{N}$ with $p \geq 2$.

To show the reverse inclusion, let $c \in \Gamma^\infty(\mathcal{C}^{(p-2)}(M))$. If $\{u_\alpha\}$ is a partition of unity corresponding to the selected atlas then we can assume $\text{supp}(c) \subset U$ since $\gamma(c) = \gamma(\sum_\alpha u_\alpha c) = \sum_\alpha \gamma(u_\alpha c)$. Furthermore, by the discussions in the first part of this proof

$$c = \sum_{fin.,l} f_{0,l} df_{1,l} \dots df_{p-2,l}$$

for some functions $f_i \in C^\infty(M)$.

Let $h \in C^\infty(M)$ with $h(y) \geq \lambda > 0$ for any $y \in M$ and $\langle d_x h, d_x h \rangle_{g^{-1}}(x) \geq \mu > 0$ for every $x \in U$. This forces $h, h^{-1} \in C^\infty(M)$ and

$$\tilde{f}_{0,l}(x) := \frac{f_{0,l}(x)}{2\langle d_x h, d_x h^{-1} \rangle_{g^{-1}}} = -\frac{h(x)^2}{2\langle d_x h, d_x h \rangle_{g^{-1}}} \cdot f_{0,l}(x) \in C^\infty(M)$$

for every l . Furthermore,

$$(1) \quad \tilde{f}_{0,l}(dh \cdot dh^{-1} + dh^{-1} \cdot dh) = 2\tilde{f}_{0,l}\langle dh, dh^{-1} \rangle_{g^{-1}} = f_{0,l}$$

for every l , and by the first step we obtain

$$\pi(h dh^{-1} + h^{-1} dh) = \gamma(h dh^{-1} + h^{-1} dh) = \gamma(d(hh^{-1})) = 0.$$

Therefore, for any l

$$\omega_l := (h dh^{-1} + h^{-1} dh) df_{1,l} df_{2,l} \dots df_{p-2,l} \in \ker(\pi^{p-1}),$$

$$\tilde{f}_{0,l} d\omega_l \in (\ker(\pi^p) + d\ker(\pi^{p-1})),$$

since the latter is an ideal in Ω^p . Finally, by (1) and the first step:

$$\begin{aligned} \gamma(c) &= \gamma \left(\sum_{fin.,l} \tilde{f}_{0,l} (dh \cdot dh^{-1} + dh^{-1} \cdot dh) \cdot df_{1,l} \cdot df_{2,l} \cdot \dots \cdot df_{p-2,l} \right) \\ &= \pi \left(\sum_{fin.,l} \tilde{f}_{0,l} (dh dh^{-1} + dh^{-1} dh) df_{1,l} df_{2,l} \dots df_{p-2,l} \right) \\ &= \pi \left(\sum_{fin.,l} \tilde{f}_{0,l} d\omega_l \right) \\ &\in \pi(\ker(\pi^p) + d\ker(\pi^{p-1})) = \pi(d\ker(\pi^{p-1})). \end{aligned}$$

We arrive at $\pi(d\ker(\pi^{p-1})) \supseteq \gamma(\ker(\sigma^p))$ for every $p \in \mathbb{N}$ with $p \geq 2$, and claim 2 is proved.

As the final step we list the following chain of identifications and isomorphisms:

$$\begin{aligned} \Omega_p^p C^\infty(M) &= \pi(\Omega^p C^\infty(M)) / \pi(d\ker(\pi^{p-1})) \\ &\cong \Gamma^\infty(\mathbb{C}^{p-ev}(M)) / \ker(\sigma^p) \\ &\cong \text{im}(\sigma^p) \\ &= \Gamma^\infty(\Lambda^p(M)). \end{aligned}$$

□

Corollary 7.2. $\Omega_p^p C^\infty(M) = 0$ for every $p > \dim(M) = n$.

Finishing we point out that there are new ideas and results in noncommutative geometry that are closely related to Theorem 7.1 and invent quantum de Rham cohomology on Poisson manifolds in a sense different from A. Connes' work. Pioneering results have been obtained by M. Kontsevich [12], Huai-Dong Cao and Jian Zhou [3, 4], among others. We refer to these sources for details.

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