Non-Perturbative Mass Renormalization in Quenched QED from the Worldline Variational Approach

C. Alexandrou 1, R. Rosenfelder 2 and A. W. Schreiber 3
1 Department of Physics, University of Cyprus, CY-1678 Nicosia, Cyprus
2 Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland
3 Department of Physics and Mathematical Physics and Research Centre for the Subatomic Structure of Matter, University of Adelaide, Adelaide, S. A. 5005, Australia
(March 22, 2000)

Abstract

Following Feynman’s successful treatment of the polaron problem we apply the same variational principle to quenched QED in the worldline formulation. New features arise from the description of fermions by Grassmann trajectories, the supersymmetry between bosonic and fermionic variables and the much more singular structure of a renormalizable gauge theory like QED in 3 + 1 dimensions. We take as trial action a general retarded quadratic action both for the bosonic and fermionic degrees of freedom and derive the variational equations for the corresponding retardation functions. We find a simple analytic, non-perturbative, solution for the anomalous mass dimension $\gamma_m(\alpha)$ in any mass-independent, dimensionally regularized, renormalization scheme. For small couplings we compare our result with recent four-loop perturbative calculations while at large couplings we find that $\gamma_m(\alpha)$ becomes proportional to $\sqrt{\alpha}$. The anomalous mass dimension shows no obvious sign of the chiral symmetry breaking observed in calculations based on the use of Dyson-Schwinger equations, however we find that a perturbative expansion of $\gamma_m(\alpha)$ diverges for $\alpha > 0.7934$. Finally, we investigate the behaviour of $\gamma_m(\alpha)$ at large orders in perturbation theory.
I. INTRODUCTION

Variational methods are widely used in many areas of physics but are not very prominent in field theory [1]. This is due to the infinite number of degrees of freedom and the singular short-distance behaviour of relativistic field theories. A very successful application of variational methods in a non-relativistic field theory is provided by Feynman’s treatment of the polaron [2]: after integrating out the phonon degrees of freedom and approximating variationally the remaining effective action by a retarded quadratic trial action one obtains the best approximation scheme which works for both small and large coupling constants. Detailed numerical investigations [3] have shown that Feynman’s approximate solution deviates at most 2.2% from the true ground state energy for all coupling constants. It is therefore very attractive to apply similar techniques to problems in relativistic quantum field theory where there is much need for non-perturbative methods. In previous publications we have done that in the context of a scalar, super-renormalizable model theory [4].

In this paper we present the first results obtained by applying polaron variational methods to a realistic theory, namely Quantum Electrodynamics (QED) in the quenched approximation where electron-positron loops are neglected. While the actual coupling constant between electrons and photons $\alpha = e^2/(4\pi) \simeq 1/137$ is small enough to apply perturbation theory in most cases, there is enough interest to study the theory at larger coupling: first, the strong coupling behaviour of any physical theory is of interest in itself, second, the possibility of chiral symmetry breaking [5] demands an investigation at large $\alpha$ and, finally, bound state problems are inherently non-perturbative and involve powers of $\ln 1/\alpha \simeq 4.92$ in radiative corrections.

The extension of our methods to QED requires a formalism to include fermions and a treatment of the more severe singularities encountered in a renormalizable field theory rather than a super-renormalizable or non-relativistic one. We do this within the worldline technique which has recently experienced a revival [6]. In this formulation, the degrees of freedom describing the electron are its bosonic worldline $x^\mu (t)$, which is the four-dimensional analogue to the polaron trajectory, as well as a Grassmannian path $\zeta (t)$ needed to describe the electron’s spin [7]. Here $t$ is the proper time which parametrizes the paths and runs from 0 to $T$. The dynamics of the electron in an external vector field $A_\mu (x)$ with field strength $F_{\mu\nu} (x)$ are then described by the following worldline Lagrangian

$$L = -\frac{\kappa_0}{2} \dot{x}^2 + i \zeta \cdot \dot{\zeta} + \frac{1}{T} \dot{x} \cdot \zeta \chi - e \dot{x} \cdot A(x) - \frac{i e}{\kappa_0} F_{\mu\nu} (x) \zeta^\mu \zeta^\nu. \quad (1)$$

Here $\kappa_0$ is an arbitrary parameter which may be used to reparametrize the proper time without changing the physics and $\chi$ is a Grassmannian (super-)partner of the proper time $T$. Note that the above action exhibits a well-known supersymmetry between bosonic and fermionic degrees of freedom [8]. For further details about the application of the worldline formalism to QED we refer the reader to Ref. [9].

The photon field $A_\mu$ may be integrated out exactly in complete analogy to the phonons in the polaron case, resulting in an effective action for the electron only

$$S_{\text{eff}} = S_0 - \frac{e^2}{2} \int_0^T dt_1 dt_2 \int \frac{d^4k}{(2\pi)^4} G^{\mu\nu} (k) \left[ \dot{x}_\mu (t_1) + \frac{2}{\kappa_0} \zeta_\mu (t_1) k \cdot \zeta (t_1) \right]$$

2
Here $S_0$ denotes the free action and $G^\mu\nu(k)$ the gauge-fixed photon propagator. The electron propagator is obtained from this action by carrying out a path integral over $x_\mu(t)$ and $\zeta_\mu(t)$, which is approximated variationally below, as well as a final weighted integral over the proper times $T$ and $\chi$.

II. VARIATIONAL APPROACH

Feynman’s variational principle has its root in Jensen’s inequality for convex functions applied to $\exp(-S_E)$, where $S_E$ is a Euclidean action. In Minkowski space and/or for complex actions the variational principle remains valid, however it becomes a stationary principle rather than a minimum principle. To be more precise, the path integral over bosonic and fermionic paths obeys

$$\left\langle \exp\left[i(S - S_t)\right]\right\rangle_t \simeq \exp\left[i \langle S - S_t \rangle_t\right]$$

(3)

where $\langle \ldots \rangle_t$ indicates an average involving the weight function $e^{iS}$ in the relevant functional integral and $S_t$ is a suitable trial action. Note that corrections to this variational approximation may be calculated in a systematic way and that, furthermore, to first order in the interaction (i.e. to order $\alpha$) the relation is in fact an equality if $S_t$ reduces to the free action for small couplings.

For the trial action required in eq. (3) we choose a general retarded quadratic action

$$\tilde{S}_t = S_0 + i\kappa_0^2 \int_0^T dt_1 dt_2 \left[ -g_B(\sigma) \dot{x}(t_1) \cdot \dot{x}(t_2) + \frac{2i}{\kappa_0} g_F(\sigma) \zeta(t_1) \cdot \dot{\zeta}(t_2) \right. \left. + 2 \frac{\sigma}{\kappa_0 T} g_{SO}^r(\sigma) \dot{x}(t_1) \cdot \zeta(t_2) \chi \right] + \int_0^T dt \left[ \lambda_1 p \cdot \dot{x}(t) - \frac{i}{2} \lambda_2 \Gamma \cdot \dot{\zeta}(t) \right].$$

(4)

Here the variational parameters are contained in the retardation functions $g_i(\sigma)$ for bosonic, fermionic and spin-orbit interactions; these are even functions of $\sigma = t_1 - t_2$ and they become identical for a supersymmetric trial action [10]. The variational principle ‘adjusts’ these functions in order to compensate for the fact that the true effective action (2) is not quadratic in the variables $x(t)$, $\zeta(t)$. Feynman’s polaron result was obtained by taking a specific Ansatz for the retardation functions but here we leave their functional form free. This is because one expects that the correct short-time behaviour of these functions is much more important for a renormalizable theory like QED than for the polaron problem which does not exhibit any ultraviolet divergences. Indeed one finds that for small $\sigma$ the “best” $g_B(\sigma)$ behaves like $\sqrt{\sigma}$, $\ln \sigma$ and $1/\sigma$ in the polaron, super-renormalizable and QED case, respectively. The “tilde” over the trial action indicates that boundary terms (the last terms in the above equation) are added; we are using “momentum averaging” [4] where one includes the Fourier transform over the endpoint $x = x(T)$ in the path integral in order to obtain the propagator in momentum space. These boundary terms, involving the external momentum $p$ and the Grassmann variable $\Gamma = \zeta(0) + \zeta(T)$ (which represents the Dirac
matrix $\gamma$ in the worldline formalism) are multiplied by $\lambda_1$ and $\lambda_2$, which are additional variational parameters providing the freedom to modify the strength of the boundary terms. We have allowed this freedom because of our experience in scalar relativistic field theory [4], where the variational parameter $\lambda_1$ turned out to be essential for describing the instability of the Wick-Cutkosky model.

Since the trial action (4) is at most quadratic in $x(t)$ and $\zeta(t)$ it is possible to evaluate the various averages required in eq. (3) analytically. Details of the calculation will be given elsewhere [11]. The results are greatly simplified on the electron’s mass shell as this corresponds to taking the proper time $T$ to infinity. Furthermore, this point defines the electron’s physical mass $M$ in terms of its bare mass $M_0$, i.e.

$$M_0^2 = M^2(2\lambda - \lambda^2) - 2\left(\Omega[A_B] - \Omega[A_F] + V[\mu_B^2, \mu_F^2]\right).$$  \hfill (5)

(We have labeled this relationship Mano’s equation as K. Mano first applied polaron techniques to a scalar relativistic field theory [12].) Finally, on mass shell the variational equations resulting from eq. (3) are equivalent to demanding stationarity of Mano’s equation.

The nomenclature in Mano’s equation corresponds to that introduced in Ref. [4]: $\Omega[A_B]$ and $\Omega[A_F]$ originate from contributions (bosonic and fermionic, respectively) of the terms in eq. (3) involving $S_0$ and $S_1$ only. They are the analogue to the kinetic term in variational quantum mechanical calculations, while the analogue of the contribution from a potential term (explicitly proportional to the strength of the coupling) resides in $V$.

More explicitly, similarly to Ref. [4], the retardation functions have been expressed in terms of the variational “profile functions” $A_i(E)$ and the “pseudotimes” $\mu_i^2(\sigma), i = B, F$ defined by

$$A_i(E) = 1 + i\kappa_0 \int_0^\infty d\sigma \, g_i(\sigma) \cos(E\sigma)$$  \hfill (6)

$$\mu_i^2(\sigma) = \frac{4}{\pi} \int_0^\infty dE \frac{1}{E^2 A_i(E)} \sin^2\left(\frac{E\sigma}{2}\right).$$  \hfill (7)

Furthermore it is convenient to define $\lambda = \lambda_1/A_B(0)$ [13]. The specific properties of QED are encoded in the “interaction” terms $V_1, V_2$, with $V = V_1 + V_2$, which in $d = 4 - 2\epsilon$ dimensions read

$$V_1[\mu_B^2, \mu_F^2] = -(d - 1)\pi\alpha \frac{\nu^2e}{\kappa_0} \int_0^\infty d\sigma \int \frac{dk}{(2\pi)^d} \left[(\mu_F^2(\sigma))^2 - (\mu_B^2(\sigma))^2\right] E(k, \sigma)$$  \hfill (8)

$$V_2[\mu_B^2] = -\frac{4\pi\alpha \nu^2e^2}{\kappa_0} \int_0^\infty d\sigma \int \frac{dk}{(2\pi)^d} \frac{1}{k^2} \left[M^2 + (d - 2)(k \cdot p)^2/k^2\right] E(k, \sigma)$$  \hfill (9)

where $E(k, \sigma) = \exp\{i[k^2\mu_B^2(\sigma) - 2\lambda k \cdot p\sigma]/(2\kappa_0)\}$ and $p^2 = M^2$. Note that the fermionic contributions, both in the ‘kinetic term’ $\Omega_F$ as well as in $V_1$, appear with an opposite sign to the bosonic contributions.

By construction Mano’s equation is stationary under variation of the parameters. It is important to note that we have not demanded the various retardation functions $g_{B,F,SO}$ to be identical (before variation). Had we done so, the resulting profile functions $A_B$ and $A_F$ would have also been identical, the pseudotimes $\mu_{B,F}^2$ would have been one and the same and hence

$$\mu_{B,F}^2 = \mu_{B,F,S}^2.$$
\[ \Omega[A_B] - \Omega[A_F] \] as well as \( V_1 \) would have vanished. The absence of a ‘kinetic’ contribution would have been fatal to the variational principle as this contribution provides the restoring ‘force’ to the potential \( V \). On the other hand, closer examination of \( V_1 \) reveals that \( \mu_B^2 \neq \mu_F^2 \) is also dangerous: The contribution of each of these terms is quadratically divergent, while the combination of the two should only display the usual logarithmic UV divergence of QED. Although at leading order in the coupling we are guaranteed to reproduce the correct perturbative result (see eq. (3)), at higher orders the cancellation of these quadratic divergences is ensured by the supersymmetry. To summarize, on the one hand the trial action cannot be restricted to contain only supersymmetric terms but on the other hand allowing non-supersymmetric terms may destroy the renormalizability of the theory.

The way out of this predicament is provided by the variational principle itself: although it is unavoidable that the trial action breaks supersymmetry, the actual solutions to the variational equations may in fact be nearly supersymmetric. That this indeed turns out to be the case may be seen by recognizing that \( V_1 \) is the most singular part of the interaction whereas \( V_2 \), which involves only bosonic contributions and is the only source of supersymmetry breaking, is similar in structure to the scalar super-renormalizable model studied before [14]. Divergent contributions in the limit \( \epsilon \to 0 \) to the variational equations are solely determined by \( V_1 \) and hence the divergent contributions to \( A_B(E) \) and \( A_F(E) \) are identical. The corresponding variational equation for \( A_B(E) = A_F(E) \equiv A(E) \) becomes (after performing the \( k \)-integration in eq. (8))

\[
A(E) = 1 + (1 - \epsilon) c_{\epsilon} \int_0^\infty d\sigma \frac{\sin E\sigma}{E} \frac{\mu^2(\sigma)}{[\mu^2(\sigma)]^2} \exp \left[ -i \frac{\lambda^2 M^2 \sigma^2}{2\kappa_0 \mu^2(\sigma)} \right]
\]

where \( c_{\epsilon} = (\alpha/\pi)(2\pi i \nu^2/\kappa_0)^\epsilon (3 - 2\epsilon)/((1 - \epsilon)(2 - \epsilon)) \). Since \( \mu^2(\sigma) \to \sigma \) for small \( \sigma \) one sees that the \( \sigma \)-integral in eq. (10) would diverge for \( \epsilon = 0 \); this justs reflects the \( 1/\sigma \) behaviour of the retardation function in eq. (6) as was discussed before. The crucial difference between super-renormalizable and renormalizable theories therefore is that for the latter ones the variational equations themselves are UV-divergent. In this way the divergent structure of higher-order diagrams is effectively summed up.

### III. MASS RENORMALIZATION

Renormalizability of (quenched) QED means that these divergences can all be collected in the mass and wave function renormalization constants. In the present investigation we concentrate on the mass renormalization constant \( Z_M \) defined via \( M_0 = Z_M M_\nu \) where \( M_\nu \) is an intermediate mass scale. In the MS scheme it should have the perturbative expansion

\[
Z_M = 1 + \frac{b_{11}}{\epsilon} \frac{\alpha}{\pi} + \left[ \frac{b_{12}}{\epsilon^2} + \frac{b_{12}}{\epsilon} \right] \left( \frac{\alpha}{\pi} \right)^2 + \ldots,
\]

where it is known from perturbation theory [15] that the expansion coefficients \( b_{ij} \) are pure, i.e. mass independent, numbers. Furthermore, the renormalization group provides relations between many of these coefficients; at order \( n \) in perturbation theory only the coefficient \( b_{1n} \) contains new information. This is encapsulated in the solution of the renormalization group equation for \( Z_M \), namely...
\[ Z_M = \exp \left[ -\frac{1}{2\epsilon} \int_0^\alpha dx \frac{\gamma_m(x)}{x} \right] = \exp \left[ -\frac{1}{2\epsilon} \sum_{n=1}^\infty \frac{\gamma_{n-1}}{n} \left( \frac{\alpha}{\pi} \right)^n \right]. \] (12)

where \( \gamma_m(\alpha) \) is the anomalous mass dimension of the electron [16]. In perturbation theory, \( \gamma_m(\alpha) \) can be extracted from perturbative QCD calculations, which have been performed up to 4-loop order. One obtains \( \gamma_0 = 3/2, \gamma_1 = 3/16 \) [17], \( \gamma_2 = \frac{129}{64} = 2.0156 \) and \( \gamma_3 = -\frac{1}{128} \left[ \frac{1261}{8} + 336 \zeta(3) \right] = -4.3868 \) [18].

As the variational calculation is applicable for arbitrary values of the coupling, comparison to perturbation theory provides a useful guide to its utility. As mentioned before, to first order in the coupling the calculation is guaranteed to be exact as long as one has used a trial action which can reduce to the free action in the limit \( \alpha \to 0 \). A genuine test of the variational scheme is only obtained by comparing the coefficients in higher order. It should be noted that this test is much more demanding than in the polaron case where one can only compare the numerical value of the second-order coefficient for the energy: here, in addition, one tests the \( \epsilon \)-dependence of this coefficient and also whether it is mass-independent as it should be in the exact theory. In second order one has to insert the lowest order result \( \mu^2(\sigma) = \sigma \) into the variational equation (10) for the profile function and then to substitute the solution into eq. (9) for the interaction part \( V_2 \). Finally the \( \lambda \)-variation of Mano’s equation has to be performed. This gives \( b_{22}^{\text{var}} = 9/32 \), which is correct, and \( b_{12}^{\text{var}} = 0 \), which should be compared to the exact value of \( b_{12} = -9/32 \). As in the Wick-Cutkosky model, the \( \lambda \)-variation is of crucial importance: for example, fixing \( \lambda = 1 \) would give a wrong result for \( b_{22} \) and a logarithmic mass-dependence for \( b_{12} \).

It is possible to develop the perturbative expansion of the variational result further, with the result that no mass dependence in the coefficients appears even at higher order. Indeed, it turns out that it is in fact possible to obtain the \textit{full} analytic expression for the anomalous mass dimension in the worldline variational approximation. To derive this we first drop the mass term in the variational equation (10) since it only affects long-distance physics and not the ultra-violet behaviour contained in \( Z_M \). Then we change variables from \( \sigma, E \) to \( y = c_s(v^2\sigma)^\epsilon, z = c_s(v^2/E)^\epsilon \). This has the effect of making the system of integral equations (7–10) independent of the coupling. By making the \textit{Ansatz} \( \mu^2(\sigma)/\sigma = \exp[-\omega(y)/\epsilon + \mathcal{O}(\epsilon^0)] \) the variational equation for the pseudotime can be transformed in this way into

\[ \omega'(y) = \left[ 1 - v(y) \right] \frac{\pi}{2} v(y) \cot \left[ \frac{\pi}{2} v(y) \right] e^{-\omega(y)}. \] (13)

where \( v(y) := y \omega'(y) \). For the calculation of the anomalous dimension from Mano’s equation we use the same technique to evaluate \( V_2 \) in leading order of \( \epsilon \) and the fact that the coupling constant and the combination \( 4\pi v^2/(\lambda^2 M^2) \) always occur together. This allows us to use the variational equation for \( \lambda \) and to obtain, after some algebra [11]

\[ \gamma_{m}^{\text{var}}(\alpha) = \frac{v_0}{1 - v_0} \] (14)

where \( v_0 = v(y_0) \) is determined implicitly from \( 3\alpha/(2\pi) = y_0 \exp[-\omega(y_0)] \). An implicit equation naturally occurs in the present approach when integrals of the type

\[ \int_0^\infty dy G(y) \exp \left[ ia(F(y)/z)^{1/\epsilon} \right] \overset{\epsilon \to 0}{\longrightarrow} \int_0^{y_0} dy G(y), \quad \text{with} \quad F(y_0) = z \] (15)
(F(y) being a monotonous function) are evaluated in leading order in \( \epsilon \). The differential equation (13) can now be used to eliminate the function \( \omega(y_0) \) completely. After solving eq. (14) for \( r_0 \) as function of \( \gamma^\text{var}_m \) one is left with an implicit algebraic equation for the anomalous dimension

\[
\frac{3}{4} \alpha = (1 + \gamma^\text{var}_m) \tan \left( \frac{\pi/2 \gamma^\text{var}_m}{1 + \gamma^\text{var}_m} \right).
\]

Eq. (16) is the main result of this paper.

IV. DISCUSSION AND SUMMARY

When expanded in powers of \( \alpha \), eq. 16 immediately yields

\[
\gamma^\text{var}_m(\alpha) = \frac{3 \alpha}{2 \pi} - \frac{9}{32} \pi^2 \left( \frac{\alpha}{\pi} \right)^3 + \frac{27}{32} \pi^2 \left( \frac{\alpha}{\pi} \right)^4 - \frac{243}{128} \pi^2 \left( 1 - \frac{\pi^2}{20} \right) \left( \frac{\alpha}{\pi} \right)^5 + O(\alpha^6),
\]

which may be compared to perturbation theory. Numerically the values of the coefficients are different but of the same order of magnitude as the exact perturbative results. Note, however, that this comparison is not particularly meaningful: the variational result is an approximation which is valid at all \( \alpha \). It need not have the same, or even approximately the same, perturbative expansion in \( \alpha \) as the exact result. It should, however, be numerically similar. In Fig. 1 we plot the variational result as a function of the coupling and compare it to perturbation theory up to 4-loop order. For \( \alpha \gtrsim 1 \) the 3- and 4-loop anomalous dimensions start to deviate so much from each other that one cannot trust either of them. Also shown is the result up to 5 loops, where the 5-loop coefficient has been estimated from Padé approximations to the perturbation theory (see eq. (2.12) of Ref. [19], which needs to be adapted to QED with \( n_f = 0 \) flavours; one finds \( \gamma^\text{Pade}_4 = 3.848 \)). Clearly this does not significantly extend the numerical validity of the perturbative result. In short, the variational estimate for \( \gamma_m \) is roughly in agreement with (albeit apparently a little below) the perturbative result in the region where the perturbative result can be trusted.

Also shown in Fig.1 is the only other easily available non-perturbative result for \( \gamma_m \) based on the use of dimensionally regularized Dyson-Schwinger (DS) equations in “rainbow approximation” within the Landau gauge in a mass independent regularization scheme such as \( MS \) or \( \overline{MS} \). This may be obtained by adapting the discussion in Ref. [20] with the result that \( \gamma^\text{DS}_m = 1 - \sqrt{1 - 3 \alpha/\pi} \) (which is the same as derived by Miransky [21] using a hard momentum cutoff). We see that this result deviates from perturbation theory in a region where, at least numerically, perturbation theory still appears to converge. Above \( \alpha = \pi/3 = 1.047 \) the DS result becomes complex, this value of the coupling constant coinciding with the coupling \( \alpha_{cr} \) at which the onset of chiral symmetry breaking takes place in those calculations. This is in contrast to the variational result which remains real for all values of the coupling and in fact has the strong coupling limit

\[
\gamma^\text{var}_m(\alpha) \xrightarrow{\alpha \to \infty} \frac{1}{4} \sqrt{6\pi \alpha} - \frac{1}{2} + O\left( \frac{1}{\sqrt{\alpha}} \right).
\]

7
FIG. 1. Anomalous mass dimension $\gamma_m$ as function of the coupling constant $\alpha$ in quenched QED. The variational result (16) is shown as a solid curve while the solution from the Dyson-Schwinger equations in rainbow approximation is indicated as a dot-dashed curve. The curves labeled “n-loop” show the result up to n-loop perturbation theory. Finally, the Padé estimation of the 5-loop result is also shown.

Further investigations are necessary to clarify the absence of any obvious sign of chiral symmetry breaking in the variational result for $\gamma_m(\alpha)$. Indeed, it even needs to be clarified in what way this symmetry breaking should show up in an anomalous mass dimension obtained in a dimensionally regulated calculation (The reader should note that the issue of dynamical chiral symmetry breaking in a dimensionally regulated theory is a notoriously subtle problem; see ref. [20]).

On the other hand, it may be shown that a perturbative inversion of eq. (16), i.e. the expansion $\gamma_m^{\text{var}}(\alpha) = \sum_{n=1}^{\infty} c_n \alpha^n$, has a finite radius of convergence, given by $\alpha_{\text{con}} = 0.7934$ [11], which is not too different from the radius of convergence of the DS result [22]. It is not clear whether this similarity between $\alpha_{\text{cr}}$ and $\alpha_{\text{con}}$ is accidental or not. In connection with this, it is interesting to note that for large $n$ the behaviour of the expansion coefficients $c_n$ in both the variational result as well as the DS result are rather similar:

$$c_n \approx \alpha_{\text{con}}^{n} \frac{e^{-\beta}}{n^{3/2}}$$

where for the variational case we have $\beta \approx 1.4$ while for the DS result one obtains $\beta = \log(2\sqrt{\pi}) = 1.27$. In addition, the variational result is actually multiplied by a sinusoidal function of $n$, which places the branchpoint responsible for the finite radius of convergence (which, for the DS result, is on the positive real axis) into the complex plane. Furthermore, it is remarkable that the large-$\alpha$ limit of $|\gamma_m(\alpha)|$ obtained in eq. (18) is almost the same as for the DS result: $|\gamma_m^{\text{var}}(\alpha)| \to 1.09\sqrt{\alpha}$ vs. $|\gamma_m^{\text{DS}}(\alpha)| \to 0.98\sqrt{\alpha}$.
To summarize, we have applied polaron variational techniques to quenched QED in 3+1 dimensions and obtained, using dimensional regularization as well as a mass independent renormalization scheme, a remarkably simple expression for the anomalous mass dimension valid for arbitrary couplings. The approach has considerable advantages over other techniques in that it automatically maintains gauge invariance, as well as the requirements of the renormalization group, and corrections can be systematically calculated (as has been done in the polaron case [23]). Furthermore, we have shown that the numerical results for $\gamma_m$ are rather reasonable at small coupling and that at large couplings the perturbative expansion of this quantity fails in a similar way to rainbow DS results. It would be interesting to compare to DS calculations which go beyond the ladder approximation, thus decreasing the strong gauge dependence inherent in that approximation. Furthermore, it should be possible to obtain the behaviour of $\gamma_m$ at large orders in perturbation theory using the methods of Lipatov and others [24], which could then be compared with the predictions made above. Finally, we note that the calculation of physical observables or application to bound state problems also seem feasible within the variational worldline approach developed here.

ACKNOWLEDGMENTS

We would like to thank Reinhard Alkofer for helpful discussions. One of us (AWS) is supported by the Australian Research Council through an Australian Research Fellowship. C.A. would like to thank PSI for its hospitality on several visits during which parts of this work were done.
REFERENCES


[10] In supersymmetric formulation the interaction part of our trial action is given by

\[ i \int_0^T dt_1 dt_2 \int d\theta_1 d\theta_2 g(T_{12}) DX_1 \cdot DX_2 \]


[13] The “kinetic” term \( \Omega[A_i] \) is given explicitly in Ref. [4]. In the free case one has \( A_i(E) = 1, \mu^2(\sigma) = \sigma, \lambda = 1, \Omega_i = 0 \). The other variational parameters are linked to the spin structure of the propagator and therefore do not show up in Mano’s equation.

[14] If the ultraviolet divergences are regulated by a formfactor or a hard cutoff, dimensional arguments show that \( V_2 \) is suppressed by a factor \( (M/\text{cutoff})^2 \) compared to \( V_1 \).


[16] We use the conventions and definitions of R. Coquereaux, Ann. Phys. 125, 401 (1980) except that \( \epsilon_{\text{Coquereaux}} = 2\epsilon \).


The reader should note that DS calculations which go beyond the rainbow approximation tend to reduce the value of the critical coupling to about $\alpha_{cr} \approx 0.9$, which is even closer to the radius of convergence of the variational result. However, it is not guaranteed that $\alpha_{cr} = \alpha_{con}$ in these calculations.
