We study the Bethe-Salpeter equations for spin zero diquark composites in the color superconducting phase of \( N_f = 2 \) cold dense QCD. The explicit form of the spectrum of the diquarks, containing an infinite tower of narrow (at high density) resonances, is derived. It is argued that there are five pseudo-Nambu-Goldstone bosons (pseudoscalars) that remain almost massless at large chemical potential. These five pseudoscalars, along with the massless quarks of the third color should play an important role in the infrared dynamics of \( N_f = 2 \) dense QCD.

Only a few years ago, not much was known about the properties of different phases in dense quark matter (see, however, Refs. [1, 2]). The situation drastically changed after the ground breaking estimates of the color superconducting order parameter were obtained in Refs. [3, 4]. Within the framework of a phenomenological model, it was shown that the order parameter could be as large as 100 MeV. Afterwards, the same estimates were also obtained within the microscopic theory, quantum chromodynamics [5–12]. The further progress in the field was mostly motivated by the hope that the color superconducting phase could be produced either in heavy ion experiment, or in the interior of neutron (or rather quark) stars.

Despite many advances [13–16] in study of the color superconducting phase of dense quark matter, the detailed spectrum of the diquark bound states (mesons) is still poorly known. In fact, most of the studies deal with the Nambu-Goldstone (NG) bosons of the three flavor QCD. At best, the indirect methods of Refs. [13–16] could probe the properties of the pseudo-NG bosons. It was argued in Ref. [17], however, that, because of long-range interactions mediated by the gluons of the magnetic type [5, 6], the presence of an infinite tower of massive diquark states could be the key signature of the color superconducting phase of dense quark matter.

In this Letter, we consider the problem of spin zero bound states in the two flavor color superconductor using the Bethe-Salpeter (BS) equations. We find that the spectrum contains an infinite tower of bound states in the doublet, antidoublet and singlet representations of the unbroken \( \text{SU}(2)_c \) subgroup. Furthermore, the estimates of masses of the doublets and antidoublets are given by

\[
M_n^2 \simeq |\Delta|^2 \left[ 1 - \exp \left( \frac{-3\pi^3/2n}{\sqrt{2}\alpha} \right) \right], \quad n = 0, 1, \ldots, \quad (1)
\]

while, in the case of singlets, the estimates are

\[
M_n^2 = 0, \quad (2a)
\]
\[
M_n^2 = 4|\Delta|^2 \left( 1 - \frac{\alpha^2\kappa}{(2n-1)^4} \right), \quad n = 1, 2, \ldots, \quad (2b)
\]

where \( \kappa \) is a constant of order 1, \( |\Delta| \) is the dynamical Majorana mass of quarks in the color superconducting phase, and \( \alpha \) is the value of the running coupling constant related to the scale of the chemical potential \( \mu \).

At large chemical potential, we also notice an approximate degeneracy between scalar and pseudoscalar channels. As a result of this parity doubling, all massive diquark states come in pairs. In addition, there also exist five massless scalars and five (nearly) massless pseudoscalars [a doublet, an antidoublet and a singlet under \( \text{SU}(2)_c \)]. While the scalars are removed from the spectrum of physical particles by the Higgs mechanism, the pseudoscalars remain in the spectrum, and they are the relevant degrees of freedom of the infrared dynamics. At last, at high density, all the massive and (nearly) massless resonances are narrow.

In the case of two flavor dense QCD, the original gauge symmetry \( \text{SU}(3)_c \) breaks down to the \( \text{SU}(2)_c \) by Higgs mechanism. The flavor \( \text{SU}(2)_L \times \text{SU}(2)_R \) group remains intact at the vacuum. The appropriate order parameter is an antitriplet in color and a singlet in flavor. Without loss of generality, we assume that the order parameter points in the third direction of the color space. In order to have a convenient description of the bound states at the true vacuum, we introduce the following Majorana spinors,

\[
\Psi^i_a = \psi^i_a + \varepsilon_{3ab}\varepsilon^{ij}(\psi^C)_{b}^j, \quad a = 1, 2, \quad (3)
\]
\[
\Phi^i_a = \phi^i_a - \varepsilon_{3ab}\varepsilon^{ij}(\phi^C)_{b}^j, \quad a = 1, 2, \quad (4)
\]

made of the Weyl spinors of the first two colors,

\[
\psi^i_a = \mathcal{P}_+ (\Psi_D)^i_a, \quad (\psi^C)_{b}^j = \mathcal{P}_- (\Psi_D)^C_{b}^j, \quad (5)
\]
\[
\phi^i_a = \mathcal{P}_- (\Psi_D)^i_a, \quad (\phi^C)_{b}^j = \mathcal{P}_+ (\Psi_D)^C_{b}^j, \quad (6)
\]

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Here $\mathcal{P}_\pm = (1 \pm \gamma^5)/2$ are the left- and right-handed projectors, $\Psi_D$ is the Dirac spinor, and $\Psi_C^T = C \Psi_D^T$ is its charge conjugate. Notice that the Majorana spinors in Eqs. (3) and (4) satisfy the following generalized Majorana condition:

\begin{align*}
\langle \Psi_C^T \rangle_i^a &= \varepsilon^{a b} \varepsilon_{i j} \Psi_b^j, \\
\langle \Phi_C^T \rangle_i^a &= -\varepsilon^{a b} \varepsilon_{i j} \Phi_b^j.
\end{align*}

Regarding the quark of the third color, we use the Weyl spinors, $\psi^\alpha$ and $\phi^\alpha$, for left and right components, respectively (notice that the color index is omitted).

The BS wave functions of the bound diquark states in the channels of interest are given by

\begin{align*}
\chi_{a}^{(b)}(p, P) &= \langle 0 | T \bar{\Psi}_{a}^{(p + P/2)} \bar{\psi}_{b} (p - P/2) | P; \bar{b} \rangle_L, \\
\lambda_{a}^{(b)}(p, P) &= \langle 0 | T \bar{\psi}_{a}^{(p + P/2)} \bar{\bar{\Psi}_{b}} (p - P/2) | P; \bar{a} \rangle_L, \\
\eta(p, P) &= \langle 0 | T \bar{\Psi}_{a}^{(p + P/2)} \bar{\psi}_{b} (p - P/2) | P \rangle_L, \\
\sigma(p, P) &= \langle 0 | T \bar{\psi}_{a}^{(p + P/2)} \bar{\bar{\psi}_{b}} (p - P/2) | P \rangle_L,
\end{align*}

plus the BS wave functions constructed out of the right handed fields $\Phi_a^T$ and $\phi^T$. One might notice that there is another diquark channel, a triplet under $SU(2)_c$, that we do not consider here. The reason is that the repulsion dominates in such a channel, and no bound states are expected (the triplet comes from the $SU(3)_c$ sextet of the symmetric phase). It might look that the same applies to the $\sigma$-singlet channel. We keep it in the analysis, however, because the equations for the BS wave functions of two singlets, $\sigma$ and $\eta$, may not decouple.

In order to derive the BS equations, we use the method developed in Ref. [18] for the case of zero chemical potential. To this end, we need to know the quark propagators and the quark-gluons interactions.

By introducing the multicomponent spinor that combines the Majorana spinors of the first two colors and the Weyl spinors of the third color, \( \langle \Psi_b^T, \psi^j, \psi_C^T \rangle \), we find that the inverse propagator takes the following block-diagonal form:

\[
G_p^{-1} = \operatorname{diag} \left( S_p^{-1} \delta_3^{b, i}, \ s_p^{-1} \delta_3^{i, j}, \ s_p^{-1} \delta_3^{j, i} \right),
\]

where, upon neglecting the wave functions renormalization of quarks [5–12],

\begin{align*}
S_p^{-1} &= -i \left( \not{p} - \mu \gamma^5 + \Delta_p \mathcal{P}_- + \bar{\Delta}_p \mathcal{P}_+ \right), \\
s_p^{-1} &= -i \left( \not{p} - \mu \gamma^\mu \right) \mathcal{P}_+, \\
\bar{s}_p^{-1} &= -i \left( \not{p} - \mu \gamma^\mu \right) \mathcal{P}_-.
\end{align*}

Here the notation, $\Delta_p = \mathcal{D}_\gamma^{\mu} \Lambda^{\gamma \mu}_p + \bar{\Delta}_p \Lambda^{\mu 0}_p$, $\bar{\Delta}_p = \gamma^0 \Delta^{0 \mu}_p$, and $\Lambda^{\gamma \mu}_p = (1 \pm \gamma^5 \not{p}/|p|)/2$ are the same as in Ref. [7].

The bare vertex, $\gamma^{A \mu}$, is also a $3 \times 3$ matrix.

By making use of this vertex and the propagator in Eq. (13), it is straightforward to derive the BS equations in the (hard dense loop improved) ladder approximation. The details of the derivation, as well as the explicit form of equations will be given elsewhere [19]. Here we just note that the most transparent form of the equations appears for the amputated BS wave functions, defined by

\[
\chi(p, P) = S^{-1} (p + P/2) \chi(p, P) s^{-1} (p - P/2),
\]

\[
\lambda(p, P) = S^{-1} (p + P/2) \lambda(p, P) S^{-1} (p - P/2),
\]

\[
\sigma(p, P) = s^{-1} (p + P/2) \sigma(p, P) s^{-1} (p - P/2).
\]

In order to get a feeling for the problem at hand, let us briefly discuss the analysis of the BS equation for the $\chi$-doublet. In general, the BS wave function contains eight different Dirac structures [20]. It is of great advantage to notice that only four of them survive in the center of mass frame, $P = (M_b, 0)$.

\[
\chi_{a}^{(b)}(p, 0) = \delta_a^b \left[ \chi_1 \Lambda^+_p + (p_0 - \epsilon_p + M_b/2) \chi_2 \gamma_0 \Lambda^+_p \right.
\]

\[
\left. + \chi_1 \Lambda^-_p + (p_0 + \epsilon_p + M_b/2) \chi_2 \gamma_0 \Lambda^-_p \right] \mathcal{P}_+.
\]

This is the most general structure that is allowed by the space-time symmetries of the model.

Now, in the particular case of the NG bosons, $M_b = 0$, we will show that the BS wave function is fixed by the Ward identities. Indeed, let us consider the following non-amputated vertex:

\[
\Gamma^{A, i \mu}_{a j \mu}(x, y) = \langle 0 | T j^A_{\mu}(0) \bar{\Psi}_a(x) \bar{\psi}_j(y) | 0 \rangle,
\]

where, for our purposes, it is sufficient to consider $A = 4, \ldots, 8$ (that correspond to the five broken generators). The vertex satisfies the following Ward identity [19]:

\[
P^{\mu} \Gamma^{A, i \mu}_{a j \mu}(k + P, k) = i T_a^{A3} \delta^i_j \left[ s_k - S_{k + P} \right] \mathcal{P}_+.
\]

As in the case of the BS wave functions, it is more convenient to deal with the corresponding amputated quantity,

\[
\Gamma_{a j \mu}(k + P, k) = S_k^{-1} \Gamma^{A, i \mu}_{a j \mu}(k + P, k) s_k^{-1}.
\]

This latter satisfies the following identity:

\[
P^{\mu} \Gamma_{a j \mu}(k + P, k) = i T_a^{A3} \delta^i_j \left[ S_k^{-1} - s_k^{-1} \right] \mathcal{P}_+.
\]

By making use of the explicit form of the quark propagators in Eqs. (14) and (15), we could check that the right hand side of Eq. (25) is non-zero in the limit $P \to 0$. This is possible only if the vertex on the left hand side develops a pole as $P \to 0$. After a simple calculation, we obtain
where, we introduced \( \bar{P}^\mu = (P_0, c_\chi^2 \vec{P}) \) with \( c_\chi \) being the velocity of the NG boson in the \( \chi \)-doublet channel.

By making use of the definition in Eqs. (9) and (17), it is also not difficult to show that the pole contribution to the vertex function (26) is directly related to the BS wave function. By omitting the details, the velocity of the NG boson in the \( \chi \)-doublet channel.

\[
\chi^{(a)}_\lambda(p, 0) \equiv \delta^{(a)}_\lambda \chi(p, 0) = \delta^{(a)}_\lambda \frac{\Delta p}{F(\chi)} \mathcal{P}^+, \tag{27}
\]

where \( F(\chi) \) is the decay constant of the corresponding doublet whose formal definition is given by

\[
\langle 0 | \sum_{A = 4}^7 T^A_{a j} j^A_\mu(x) | P, \vec{b} \rangle_L = i \delta^{\vec{b} \hat{P} \chi}_a F(\chi). \tag{28}
\]

By comparing the Dirac structures in Eqs. (21) and (27), we see that no components of the \( \chi^A \) type appear in Eq. (27) which follows from the Ward identities. The analysis of the BS equations, on the other hand, shows that, while being small, the component functions \( \chi^A \) cannot be exactly zero. The origin of this discrepancy is clear. Indeed, in our approximation, we completely neglected the wave function renormalization effects of quarks. Upon taking them into account, the Ward identity (25) would lead to a modified structure of the BS wave function, and all allowed Dirac structures of Eq. (21) would be non-zero.

In the leading order, the corrections due to the wave function renormalization, as well as the non-zero component functions \( \chi^A \) are small. So, we neglect both of them here. After enforcing \( \chi^A = 0 \), we check that the expression in Eq. (27) is indeed the solution to the BS equation if \( \Delta p \) is the solution to the gap equation [5–12]. This proves that the approximation used here is consistent with the Ward identities.

In passing, we would like to indicate one important point regarding the \( \eta \)-singlet. The Ward identity in the appropriate channel is given in terms of a single propagator, \( S_\chi \). As a result, the corresponding BS wave function does not depend on the wave function renormalization at all. It was very rewarding, therefore, to establish that the structure of the BS wave function, required by the Ward identity, is the exact solution to the BS equation for the \( \eta \)-singlet.

In connection with the Ward identities, it is appropriate to mention here the complementary analysis of Ref. [21]. The authors of this paper consider the contribution to the Ward identity that is directly related to the wave function renormalization of quarks.

Now let us discuss the fate of the massless states that we obtain. Altogether, there are five scalars and five pseudoscalars (a doublet, an antiduallelt and a singlet). Because of the Higgs mechanism, the scalars are removed from the spectrum. Nevertheless, these scalar bound states exist in the theory as “ghosts” [22], and one cannot get rid of them completely, unless a unitary gauge is found. Actually, these ghosts play a very important role in getting rid of unphysical poles from the on-shell scattering amplitudes [22].

As for the pseudoscalars, they remain in the spectrum as pseudo-NG bosons. In the (hard dense loop improved) ladder approximation, they look like NG bosons because the left and right sectors of quarks decouple. One could think of this as an effective enlargement of the original color symmetry from \( SU(3)_c \) to an approximate \( SU(3)_{c, L} \times SU(3)_{c, R} \). Then, since the approximate symmetry of the ground state is \( SU(2)_{c, L} \times SU(2)_{c, R} \), five scalar NG bosons (which are removed by the Higgs mechanism) and five pseudoscalar NG bosons (which remain in the spectrum) should appear. Of course, in the full theory, the pseudoscalars are only pseudo-NG bosons. Indeed, they should get non-zero masses due to higher orders corrections that are beyond the improved ladder approximation. At the same time, since the theory is weakly coupled at large chemical potential, it is natural to expect that the masses of the pseudo-NG bosons are small even compared to the value of the dynamical quark mass.

To complete our discussion of the massless diquark states, let us calculate their decay constants. Here we outline only the calculation for the \( \chi \)-doublet (see Ref. [19] for others). From the definition of the conserved current and Eq. (28), one could derive the following exact relation:

\[
\bar{P}_\mu F(\chi) = - \int \frac{d^4q}{(2\pi)^4} \text{tr} \left[ \gamma_\mu P + S q + P/2 \chi(q, P) S_p - P/2 \right]. \tag{29}
\]

Unfortunately, it is very hard to obtain the solution for \( \chi(q, P) \) when \( P \neq 0 \). In order to get an estimate, we use \( \chi(q, 0) \) instead, see Eq. (27). A simple calculation gives

\[
F(\chi) = \frac{\mu}{2\pi} \text{ and } c_\chi = \frac{1}{\sqrt{3}}. \tag{30}
\]

Notice that, as in the case of NG bosons in three flavor QCD [13–16], the velocity of the scalar and pseudoscalar NG bosons is equal to \( 1/\sqrt{3} \), while the decay constants themselves are of order \( \mu \).

We conclude our discussion of the massless diquarks by emphasizing that the low-energy dynamics of the two flavor QCD is dominated by massless quarks of the third color and by the five pseudoscalars that remain almost massless in the dense quark matter. Of course, the three gluons of the unbroken \( SU(2)_c \) may also be of some relevance but we do not study this question here.

Now, let us consider massive diquarks. The structure of the BS equations becomes even more complicated in
we know that the most important region of momenta in the experience of solving the gap equation (which coincides with the BS equation for the massless states), we closely follow the approximation used for infrared region where \( p < \mu \). In this region, the kernel of the BS equations for massive states, \( M \ll |\Delta| \), is almost the same. The deviations appear only in the infrared region where \( p \ll |\Delta| \).

Therefore, in our analysis of the BS equations for massive states, we closely follow the approximation used for the massless diquarks. By dropping all the component functions like \( \chi^\pm_3 \) in the anzatz (21), and assuming that other functions depend only on the time component of momentum (compare with the analysis of the gap equation in Refs. [5–12]), we arrive at the following equation:

\[
\chi_1^{-}(p) = \frac{2\alpha}{9\pi} \int_0^\Lambda dq K^{(x)}(q) \chi_1^{-}(q) \ln \frac{\Lambda}{|q-p|},
\]  
(31)

where \( \Lambda = (4\pi)^{3/2}\mu/\alpha^{5/2} \), and the kernel reads

\[
K^{(x)}(q) = \frac{M_b^2 \left( q^2 + \sqrt{q^2 + |\Delta|^2} - |\Delta|^2 \right)^2 - |\Delta|^4}{\sqrt{q^2 + |\Delta|^2} \sqrt{4M_b^2q^2 - (|\Delta|^2 - M_b^2)^2}}
\]  
(32)

This kernel is a smooth function that could be approximated by

\[
K^{(x)}(q) \simeq \begin{cases} 
\frac{|\Delta|^2 - M_b^2}{|\Delta|^2}, & \text{as } q \ll |\Delta|^2 - M_b^2; \\
\frac{1}{q}, & \text{as } q \gg |\Delta|^2 - M_b^2.
\end{cases}
\]  
(33)

In order to obtain the solution to the BS equation (31), we use the same method as in the case of the gap equation [6–12]. The result for the spectrum is given in Eq. (1). Note that the Meissner effect was neglected in this analysis [23].

The analysis for the massive singlet is similar. The difference appears in the kernel which is given by

\[
K^{(n)}(q) = \frac{\sqrt{q^2 + |\Delta|^2}}{q^2 + \sqrt{q^2 + |\Delta|^2} - (M_n/2)^2}.
\]  
(34)

In this case, the best approximation is achieved by considering three regions, \( 0 < q < \sqrt{|\Delta|^2 - (M_n/2)^2} \), \( \sqrt{|\Delta|^2 - (M_n/2)^2} < q < |\Delta| \) and \( |\Delta| < q < \Lambda \), and then matching the corresponding solutions. By omitting the details, the spectrum is presented in Eq. (2).

We note that the bound diquark states may truly be just resonances in the full theory, since they could decay into gluons of the unbroken \( SU(2) \). At high density, however, both the running coupling \( \alpha(\mu) \) and the effective Yukawa coupling \( g_r = |\Delta|/F \sim |\Delta|/\mu \) are small, and, therefore, these massive resonances are narrow.

[20] Note that in the case of zero chemical potential, see Ref. [18], there are only four different Dirac structures.


[23] Taking the Meissner effect into account should decrease the binding energy of the doublets. We stress that, while in the case of singlets the binding interaction is partially mediated by the unscreened gluons of the unbroken $SU(2)_c$, the binding of doublets is exclusively due to five massive gluons.