Towards softer scales in hot QCD*

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Abstract

An effective theory for the soft field modes in hot QCD has been obtained recently by integrating out the field modes of momenta of order $T$ and $gT$. The mean hard particle distribution obeys a transport equation with a collision term. The scale that comes out for the soft field is $g^2 T \ln 1/g$. Explicit solutions have been written for the linearized mean field distribution and for the polarization tensor.

The $n$-gluon soft amplitudes are shown to be similar to the hard thermal loop ones as they are effectively one-loop diagrams. We constrain the three-gluon soft vertex to obey Ward identities. Then we compute the one-loop contribution to the gluon self-energy when the loop momentum is of order $g^2 T \ln 1/g$ with effective vertices and propagators. It allows to identify a new collision term which should be included in the transport equation in order to integrate out the momenta of order $g^2 T \ln 1/g$ and to be able to reach an effective theory for the soft field modes with momenta $g^2 T$.

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1 Introduction

The framework of this paper is pure gauge theory in thermal equilibrium with $g << 1$

It has been realized that the near-equilibrium dynamics of high temperature QCD involves a hierarchy of length scales:

- scale $(T)^{-1}$ inverse of the typical momentum of the plasma particles
- scale $(gT)^{-1}$ electric screening length and first scale of collective excitations
- scale $(g^2T \ln 1/g)^{-1}$ damping length of color excitations
- scale $(g^2T)^{-1}$ non-perturbative magnetic fluctuations.

Effective theories can be constructed at a given scale that integrate out the smaller scales. The prototype is the hard thermal loop (HTL) effective theory that integrate out the scale $(T)^{-1}$ \cite{1, 2, 3}. To integrate out the hard modes $p \sim T$ means to calculate loop diagrams with external momenta $<< T$ and internal momenta of order $T$. It generates effective propagators and effective vertices for the field modes with momenta $<< T$. To leading order, only one-loop diagrams survive and the result is the hard thermal loop $n$-point functions. Remarkably, the generating functional of the hard thermal loops is gauge invariant.

Bödeker’s break-through \cite{4} was to realize that one may integrate out the scale $(gT)^{-1}$. The summation now involves an infinite set of multiloop diagrams with loop momenta $gT$ and effective HTL propagators and vertices.

One way to construct effective theories for the soft field modes of the plasma is to use classical transport equations. The soft field modes behave classically, this is due to the fact that the momentum scales of interest are small compared to the temperature. Transport equations are able to take into account the successive phenomena that are controlling the scales. The soft degree of freedom are represented by mean fields, while the hard ones are integrated out using perturbation theory. For the collective dynamics at the scale $(gT)^{-1}$, a collisionless kinetic equation for the mean field $W$ generates the hard thermal loop effective theory \cite{5}.

For the collective dynamics at larger wavelength, involving color fluctuations, the mean field $W$ satisfies a transport equation with a collision term \cite{4, 6, 7, 8, 9}. The collision term is made of two parts. One part describes the total interaction rate for hard particles of momentum $T$, it is dominated by small angle scattering i.e. soft momentum transfer $q \sim gT$ in the magnetic sector. Indeed color exchange can take place in the forward direction. The scale $g^2T \ln 1/g$ appears in the collision term with

$$\ln 1/g = \int_{\mu_1}^{\mu_2} \frac{dq}{q}$$

where $\mu_1$ separates the scale $T$ from the scale $gT$, and $\mu_2$ the scale $gT$ from the next scale. The second part of the collision term interprets $g^2T \ln 1/g$ as the hard gluon damping rate $\gamma$. This transport equation generates Bödeker effective theory which integrates out the field modes with momentum of order $T$ and $gT$.

No mention has yet been made of the time scales in the theory and of the noise term that appears in this effective theory. Indeed, as long as one considers time-independent correlation functions, no noise term is needed in the transport equation, the fluctuation-dissipation theorem operates at each scale.

Up to now, explicit amplitudes have only been obtained for quantities that depends on the linearized form of the transport equation, i.e. for the linearized mean field $W$ and for the polarization tensor \cite{10, 11}. $n$-point amplitudes have not yet been studied.

One quantity that turns out to be particularly sensitive to the change of scale is the polarization tensor in the magnetic sector. Precisely that quantity enters the collision term of the transport equation in a strategic way. Indeed the interaction rate between two hard particles is dominated...
by soft momentum transfer $q \sim gT$ where the squared modulus of the resummed gluon propagator enters

$$I \simeq m_D^2 \int_{q_0 << q} dq_0 dq \frac{1}{q^4 + (\text{Im} \Pi')^2 q^2}$$

where $1/q^2$ comes from energy-momentum conservation at each vertex and $m_D^2 = g^2 NT/3$. When the scale $T$ has been integrated out, $\Pi'$ has the well known HTL expression $\text{Im} \Pi' = -\pi m_D^2 q_0/q$ for $q_0 << q$ so that $I \sim \int dq/q$ leading to the $\ln 1/g$ factor in the scale of the collision term. That expression for $\Pi'$ turns out to be valid at the scale $gT$ only. Indeed, when the scale $gT$ is integrated out the resulting $\Pi'$ is $[10, 11] \text{Im} \Pi' = -m_D^2 q_0/3\gamma$ for $q_0 << q << \gamma$ so that the integral $I$ now behaves as $I \sim \int dq/q^2$. Again this $\Pi'$ is valid at the scale $g^2 T \ln 1/g$. If one wants to reach the scale $g^2 T \ln 1/g$, one has to integrate out the scale $g^2 T \ln 1/g$.

Arnold and Yaffe [11] have recently attacked the problem of integrating out this scale. As they are interested in the color conductivity, they concentrate on static quantities. They have computed the one-loop contribution to $\Pi(q_0 = 0, q)$ making use of effective theories.

In this paper, the one-loop contribution to the polarization tensor $\Pi^{\mu\nu}(q_0, q)$ will be computed for loop momenta $\sim g^2 T \ln 1/g$ and external momenta $\sim g^2 T$ assuming $\ln 1/g >> 1$. It is only the first term of an infinite series of multiloop diagrams. However, following a remark made by Blaizot and Iancu [10] for the case of momenta $\sim gT$, it will be possible to identify in the one-loop term, the new collision term that should be added in the transport equation in order to integrate out the scale $g^2 T \ln 1/g$.

Sections 2 and 3 are devoted to the effective amplitudes at the scale $g^2 T \ln 1/g$. In Section 2 the polarization tensor $\Pi^{\mu\nu}$ is written in terms of the mean W field propagator. It is interpreted as an effective one-loop diagram. $\Pi^{\mu\nu}$ appears under a form that allows to interpolate between the scale $gT$ and the scale $g^2 T \ln 1/g$. In Section 3 the three-gluon vertex is constructed as an effective one-loop diagram. It is required to satisfy HTL-like Ward identities. The Retarded/Advanced formalism is used, this constraint is due to the W propagator’s retarded type. Section 4 details how the one-loop contribution to the gluon self-energy with loop momenta $g^2 T \ln 1/g$ allows to identify the new collision operator, under which assumptions. In Section 5 the one-loop self-energy diagram with 3-gluon effective vertices is evaluated in the Retarded/Advanced formalism. Loop momenta $gT$ and $g^2 T \ln 1/g$ are successively considered. For the case $g^2 T \ln 1/g$ one part of the new collision term is identified, its physical interpretation, its scale, its infrared behaviour are examined. In Section 6, the one-loop self-energy diagram with a 4-gluon vertex is treated in a similar way. The total one-loop $\Pi^{\mu\nu}$ is transverse, the total collision operator has a zero mode. The similarities between the two collision operators appear. Section 7 contains a summary of the results and conclusions. In the Appendix, the explicit expression of $\Pi(q_0, q)$ at the scale $g^2 T \ln 1/g$ is written down and its analytical properties in the complex $q_0$ plane are studied.

2 Polarization tensor and W-propagator

Our notations are $K(k_0, k)$

2.1 The collision-resummed W propagator

The transport equation for the mean W field is $[4, 11, 10]$

$$(v.D_x + \hat{C})W(x, v) = v.E(x)$$ (1)
where $D_x$ is the covariant derivative and $\hat{C}$ the collision operator. The linearized equation is in Fourier space
\[(−iv.K + \hat{C}) W(K, v) = v.E(K)\] (2)
where $v.K = v_0 k_0 - v.k$ and $v(v_0 = 1, v)$ with $v^2 = 1$, and $\hat{C}$ is an operator in $v$ space
\[
\hat{C} W = \int \frac{dΩ_{v'}}{4π} C(v, v')W(K, v')
\] (3)
$C(v, v')$ is a real function, symmetric in $v$ and $v'$
\[
C(v, v') = \gamma S_2(v - v') - m^2 g^2 NT \frac{2}{2} \Phi(v, v')
\] (4)
and the scale of $\hat{C}$ is set by
\[
\gamma = \frac{g^2 NT}{4π} \ln 1/g
\] (6)
The explicit form of $(v:K)$ is in Sec.5.2
The linearized $W$ propagator is the Green function associated to Eq. (2) i.e. the inverse operator
\[
\hat{G}_R(K, v) = (v.K + i\hat{C})^{-1}
\] (7)
When $\hat{C}$ is replaced by $\epsilon > 0$, $G_R$ is the retarded Green function associated with the drift equation. Note that this operator in $v$ space should properly be written
\[
\hat{G}_R(K, v) = (v.K I + iv_0 \hat{C})^{-1}
\] (8)
where $I$ is the identity operator in $v$ space, so that
\[
\hat{G}_R(K, v_0, v) = -\hat{G}_R(K, -v_0, -v)
\] (9)
The advanced propagator is
\[
\hat{G}_A(K, v) = (v.K - i\hat{C})^{-1}
\] (10)
and one has
\[
\hat{G}_A(K, v) = \hat{G}_R(K, v)^* \quad \hat{G}_A(K, v) = -\hat{G}_R(-K, v)
\] (11)
Operations in $v$ space
We adopt the notations of Arnold and Yaffe [11]. The measure is $\int dΩ_{v'}/(4π)$,
$< >_v$ denotes averaging over the direction $v$
\[
< δS_2(v - v') >_v = 1 = < I >_v
\] (12)
\[
\hat{C}W = < C(v, v')W(v') >_v
\] (13)
\[
< v_i \hat{C} v_j > = < v_i C(v, v') v_j' >_{v, v'}
\] (14)
so that $>$ represent any function independent of $v$. As stressed in [11] an important property of $\hat{C}$ is
\[
\hat{C} >= 0 \quad \text{or} \quad < \hat{C} = 0
\] (15)
from (5) with the useful consequence
\[
(v.K + i\hat{C})^{-1}v.K >= (v.K + i\hat{C})^{-1}(v.K + i\hat{C}) >= I >= >
\] (16)
and a similar one for $(v.K - i\hat{C})^{-1}$.
2.2 The collision-resummed polarisation tensor

Blaizot and Iancu [10] have extracted from the linearized color current, the form of the polarisation tensor that takes into account the full effect of collisions. Defining

\[ W(K, v) = i W^i(K, v) E^i(K) \] (17)

the new functions \( W^i(K, v) \) satisfy, from (2)

\[ (v.K + i\hat{C}) W^i(K, v) = v^i \] (18)

and they obtained (see Eqs. (4.6-4.7) in [10])

\[ \Pi^{\mu i} = q_0 m_D^2 \int \frac{d\Omega_v}{4\pi} v^\mu W^i(Q, v) \] (19)

\[ \Pi^{\mu 0} = q^i m_D^2 \int \frac{d\Omega_v}{4\pi} v^\mu W^i(Q, v) \] (20)

with \( m_D^2 = g^2 NT^2 / 3 \). The solution of (18) may be written in terms of the retarded inverse operator defined in Eq. (7)

\[ W_R^i(K, v) = (v.K + i\hat{C})^{-1} v^i \] (21)

Eqs. (20) become

\[ \Pi_R^{\mu i}(Q) = m_D^2 q_0 < v^\mu(v.Q + i\hat{C})^{-1} v^i >_{v,v'} \] (22)

\[ \Pi_R^{\mu 0}(Q) = m_D^2 < v^\mu(v.Q + i\hat{C})^{-1} v.q >_{v,v'} \] (23)

With \( v.q = q_0 - v.Q \) and Eq. (16), (23) becomes

\[ \Pi^{\mu 0} = m_D^2 [ q_0 < v^\mu(v.Q + i\hat{C})^{-1} >_{v,v'} - < v^\mu T >_{v,v'} ] \] (24)

so that \( \Pi^{\mu\nu} \) may be written in the compact form

\[ \Pi_R^{\mu\nu}(Q) = m_D^2 [ q_0 < v^\mu(v.Q + i\hat{C})^{-1} v^\nu >_{v,v'} - g^{\mu 0} g^{\nu 0} ] \] (25)

(25) explicits many properties of \( \Pi^{\mu\nu} \). Indeed \( \Pi^{\mu\nu}(Q) = \Pi^{\nu\mu}(Q) \) since \( \hat{C} \) is symmetric in \( v \) space, and one readily verifies that \( Q_\mu \Pi^{\mu\nu} = Q_\nu \Pi^{\mu\nu} = 0 \) since, again with Eq. (16)

\[ Q_\mu \Pi^{\mu\nu} = m_D^2 [ q_0 < v^\nu >_{v,v'} - q_0 g^{\nu 0} g^{\nu 0} ] = 0 \] (26)

Moreover, for momenta \( q_0, q >> g^2 \ln 1/g \), one may replace the operator \( \hat{C} \) by \( \epsilon > 0 \), the \( W \) propagator is now diagonal in \( v \) space and one recovers the well-known HTL form for \( \Pi^{\mu\nu} \)

\[ \Pi_R^{\mu\nu}(Q) = m_D^2 [ q_0 < v^\mu(v.Q + i\epsilon)^{-1} v^\nu >_{v,v'} - g^{\mu 0} g^{\nu 0} ] \] (27)

(25) interpolates between the collisionless scale \( (T)^{-1} \) where the mean field \( W \) propagates on a straight line and the scale \( (gT)^{-1} \) where collisions cause fluctuations of the direction \( v \) of the \( W \) field.

Another aspect of (25) follows. An interpretation in terms of tree diagrams may be attached to the first term, i.e. to the Landau damping term. The propagator along the tree is the collision-remummed \( W \) propagator. It carries the momentum of the incoming gluon, \( v^\mu T^a \) is the vertex for a gluon (polarisation \( \mu \), color \( a \)) attached to the \( W \) line (the \( W \) propagator is blind to color). The retarded prescription of the incoming gluon determines the retarded prescription for the \( W \) propagator.
This interpretation of $\Pi^{\mu\nu}$ in terms of a tree diagram will generalize to the $n$-point functions. Indeed, it is worth remembering how the HTL form is obtained from the exact $n$-point one-loop diagram. If $k_0, k$ is the momentum running around the loop, one first integrate over $k_0$ with the weight $(n(k_0) + 1/2)$ in the complex $k_0$ plane, i.e. one picks up successively the residue of each $k_0$ pole. To pick up the contribution of a pole associated with one propagator is to put that line on shell, i.e. the one-loop diagram is cut and becomes a tree diagram, the cut line does not carry any external momentum. For the case of the two-point function $\Pi^{\mu\nu}(Q)$, there are two trees corresponding respectively to momenta $K$ or $K + Q$ on shell, they have the same propagators $(2K.Q + Q^2)^{-1}$. In the HTL approximation, the external momenta are much smaller that the loop momentum $K$, one selects the terms associated with Landau damping, the weight are combined $n(k_0 + q_0) - n(k_0) \approx q_0(dn/dk_0)$ the momentum integral decouples from the angular integral and with $k_0^2 = k^2$ and $k = kv$,

$$(2K.Q + Q^2)^{-1} \approx (2k)^{-1}(Q.v)^{-1}$$

In the tree diagram $(Q.v)^{-1}$ is now interpreted as the straightline propagator of the mean $W$ field. In (27) $q_0$ shows up, the integration over $k$ brings $m^2_L$ and $v^\mu$ is the factor that survives from the 3-gluon vertex when one leg is soft.

If one looks at the collision-resummed case, i.e. Eq. (25), one sees that the interpretation of $\Pi^{\mu\nu}$ as a tree diagram still holds, one just has to substitute the collision-resummed $W$ propagator to the straightline $W$ propagator. The $W$ propagator’s fluctuations in $v$ is what is left from the sum of multiloop diagrams with loop momenta $gT$.

To summarize, when going from collisionless HTL amplitudes to collisionfull amplitudes, one just has to change the propagator of the mean field $W$, it is now an operator in $v$ space, and the vertices have to be ordered consistently.

In the Appendix, an explicit form of $\Pi^{\mu\nu}$ is given, as a continuous fraction, and its analytic properties in the complex $q_0$ plane are examined.

3 The three-gluon collision-resummed vertex

We shall require that

i) it gives back the HTL expression when the effect of collisions is neglected

ii) it obeys HTL-like Ward Identities for any of the legs

$$p_{3\mu}V^{\mu\nu}(P_1, P_2, P_3) = \Pi^{\mu\nu}(P_1) - \Pi^{\nu\mu}(P_1 + P_3)$$

(all momenta are incoming) Going from the HTL case to the collisionfull case will amount to a substitution of propagators in tree diagrams. As the $W$ propagator is of the retarded-advanced type, the natural framework is the Retarded/Advanced one.

3.1 The Retarded/Advanced formalism

It is a real-time formalism. General properties have been established [12, 13, 14]. An external leg $l$ may be of type $R$, or of type $A$, i.e. its incoming energy is $p_0 + i\epsilon_l, \epsilon_l > 0$ type $R$, $\epsilon_l < 0$ type $A$. For an $n$-point amplitude, with all momenta incoming, $\sum_{l=1}^n \epsilon_l = 0$. Then the only non-zero three-point functions have two legs of type $R$ and one leg of type $A$, or two legs of type $A$ and one leg of type $R$. The three-point function is defined with all momenta incoming, and the general properties are, for bosons

$$V(P_{1R}, P_{2A}, P_{3A}) = V^*(P_{1A}, P_{2R}, P_{3R})$$

6
and for massless bosonic fields in the QED/QCD case [13]

\[ V((−P_1)_A, (−P_2)_R, (−P_3)_R) = −V(P_{1R}, P_{2A}, P_{3A}) \]  

(30)

Those relations are valid in the complex \( p_0 \) planes [15]. With the same notation, the two-point function, should be written

\[ \Pi_R(P) \equiv \Pi(P_R) = \Pi(P_R, (−P)_A) \]  

(31)

and the relations analogous to (29,30) are

\[ \Pi(P_A) = \Pi^*(P_R) \]  

(32)

\[ \Pi((−P)_R) = \Pi(P_A) \]  

(33)

Our results will obey those general properties. See (25) for \( \Pi^{\mu\nu} \).

3.2 Diagrammatic Rules for tree diagrams

In order to generalize to the collision-resummed case, one needs simple diagrammatic rules to write down the basic HTL \( n \)-point amplitude (i.e. before any symmetry and color structure are taken into account) [1, 2]. For the rules’ origin, see the discussion about tree diagrams in Sec. 2.2.

1. all external momenta are incoming
2. choose one orientation around the loop for the \( v \) vector
3. to an \( n \)-point one-loop diagram, there are \( n \) associated trees that are obtained by cutting one propagator of the loop. Choose any \( n − 1 \) of those trees (i.e. one propagator will never be cut). The amplitude is the sum of \( n − 1 \) terms, each linked to a tree.
4. to each propagator \( l \) of a tree, associate \((v.K_l)^{-1}\) where \( K_l \) is the sum of the external momenta going through the propagator whose orientation is the same as the \( v \) vector’s one (or \(-\) the sum of those whose orientation is opposite)

\[ K_l = \sum_l P_l = −\sum'_v P_v \]

5. the numerator associated with the tree is the energy \( k'_0 \) carried by the propagator which is never cut, with the same rule for the sign

\[ k'_0 = \sum_l p_{0l} = −\sum'_v p_{0v} \]

6. with each vertex associate \( v^\mu T_a \), where \( \mu \) is the polarization of the external leg, \( a \) its color
7. sum over the directions of \( v \) with the measure \( d\Omega_v/4\pi \), trace over the color, and multiply by \( g^{n-2}m_D^2 \).

From rule 3. there are \( n \) equivalent forms for the \( n \)-point amplitude. Indeed, in order to exhibit a difference of statistical factors (Landau damping) one has to use a partial fraction expansion to express the denominator associated with one tree as \((-\) the sum over the \( n − 1 \) other trees [2].

The \( R/A \) formalism adds another information on the external legs, they are of the \( R \) or \( A \) type, and one can define a causality flow (or \( \epsilon \)-flow) along the tree diagram, the \( R \)-flow is incoming the \( R \) legs and outgoing the \( A \) legs [16]. That flow obeys the rule of an electric current as there are no source or sink at a vertex (since \( \sum_{j=1}^{3} \epsilon_j = 0 \)). The \( R \)-flow along a tree diagram will be indicated with a specific arrow. Rule 4 is completed with

4’. When the \( \epsilon \)-flow has the same orientation as the \( v \) vector, the propagator is

\[ (v.K_l + i\epsilon)^{-1} = v.(\sum_l P_l + i\epsilon)^{-1} \]

\[ = (v.K_lR)^{-1} \]
When the $\epsilon$-flow has the opposite orientation, the propagator is

$$ (v.(-\sum_{l} P'_{l}) - i\epsilon)^{-1} = (v.K_{l} - i\epsilon)^{-1} = (v.K_{lA})^{-1} $$

### 3.3 The three-gluon vertex

With the above rules, one chooses the sum of diagrams (a) and (b) in Fig. 1. Leaving apart a factor $gf_{abc}$, the expression for an HTL three-gluon RAA vertex is

$$ V^{\mu\nu\rho}(P_{1R},P_{2A},P_{3A}) = m_{D}^{2} $$

$$ [ (p_{1}^{0} + p_{3}^{0}) < v^{\mu}(P_{1}.v + i\epsilon)^{-1} v^{\rho}((P_{1} + P_{3}).v + i\epsilon)^{-1} v^{\nu} >_{v} $$

$$ -(p_{1}^{0} + p_{2}^{0}) < v^{\rho}(-(P_{1} + P_{2}).v - i\epsilon)^{-1} v^{\nu}(-(P_{1}.v - i\epsilon)^{-1} v^{\mu} >_{v} ] $$

(34)

The expression exhibits the signs as they follow from the rules; the order follows the $v$ vector around the loop. The vertex $V(P_{1A},P_{2R},P_{3R})$ is obtained by reversing all arrows of the causality flow; it is easily checked that the general relations (29,30) are satisfied.

One now considers the collision-resummed case. The propagator along a tree is in fact the propagator, which is now an operator in $v$ space, one obtains

$$ V^{\mu\nu\rho}(P_{1R},P_{2A},P_{3A}) = m_{D}^{2} $$

$$ [ (p_{1}^{0} + p_{3}^{0}) < v^{\mu}(P_{1}.v + i\tilde{C})^{-1} v^{\rho}((P_{1} + P_{3}).v + i\tilde{C})^{-1} v^{\nu} >_{v,v',v''} $$

$$ -(p_{1}^{0} + p_{2}^{0}) < v^{\rho}(-(P_{1} + P_{2}).v - i\tilde{C})^{-1} v^{\nu}(-(P_{1}.v - i\tilde{C})^{-1} v^{\mu} >_{v,v',v''} ] $$

(35)

In each $v$ average, one may reverse the whole string as the $W$ propagator is symmetric in $v$ space. The vertex is antisymmetric in the exchange $P_{2} \leftrightarrow P_{3}$.

Turning to the Ward identities, one considers $p_{2v} V^{\mu\nu\rho}$. When $v^{\nu}$ sits at the end of a string, one makes use of relation (16); when $v^{\nu}$ sits in the middle, one writes

$$ v.P_{2} = (v.(P_{1} + P_{2}) + i\tilde{C}) - (v.P_{1} + i\tilde{C}) $$

(36)

and one obtains

$$ p_{2v} V^{\mu\nu\rho}(P_{1R},P_{2A},P_{3A}) = m_{D}^{2} [ -(p_{1}^{0} + p_{3}^{0}) < v^{\mu}(P_{1}.v + i\tilde{C})^{-1} v^{\rho} >_{v,v'} $$

$$ +p_{3}^{0} < v^{\rho}(P_{1}.v + i\tilde{C})^{-1} v^{\mu} >_{v,v'} -p_{3}^{0} < v^{\rho}((P_{1} + P_{2}).v + i\tilde{C})^{-1} v^{\mu} >_{v,v'} ] $$

(37)

$$ p_{2v} V^{\mu\nu\rho} = \Pi^{\mu\nu\rho}(P_{1} + P_{2})_{R} - \Pi^{\mu\nu\rho}(P_{1}R) $$

(38)

with $\Pi^{\mu\nu\rho}(P_{R})$ given by (25), and remembering that $\sum_{i=1}^{3} \epsilon_{i} = 0$ at the three-point function.

For $p_{1\mu} V^{\mu\nu\rho}$, $v^{\mu}$ only enters at the end of a string in (35) so that

$$ p_{1\mu} V^{\mu\nu\rho}(P_{1R},P_{2A},P_{3A}) = \Pi^{\nu\rho}(P_{1} + P_{3})_{R} - \Pi^{\nu\rho}(P_{1} + P_{2})_{R} $$

(39)

To conclude, (35) is a good candidate for the three-gluon collision-resummed vertex as it satisfies the three HTL-like Ward identities.
3.4 Other constructions

One may wish to have more antisymmetry between the three legs of the effective vertex, i.e. to average over the three possible forms as a sum of two trees. Then the vertex is written as a sum of three trees, and from the rules, one gets that the tree’s numerator is now \((1/3)\) the sum of the energies carried by the propagators with the appropriate sign (see Fig. 1).

\[
V^{\mu\nu\rho}(P_{1R}, P_{2A}, P_{3A}) = m_D^2 \frac{1}{3} \left[ \begin{array}{c}
(p_1^0 + p_3^0 + p_1^0) < v^\mu(P_1, v + i\tilde{C})^{-1}v^\rho((P_1 + P_3), v + i\tilde{C})^{-1}v^\nu > v, v, v > \\
-(p_1^0 + p_2^0 + p_1^0) < v^\rho(P_1 + P_2, v + i\tilde{C})^{-1}v^\nu(P_1, v + i\tilde{C})^{-1}v^\mu > v, v', v > \\
+(p_2^0 - p_3^0) < v^\nu(-(P_1 + P_3), v - i\tilde{C})v^\mu((P_1 + P_2), v + i\tilde{C})^{-1}v^\rho > v, v', v' > \end{array} \right] \quad (40)
\]

Comparing (40) and (35) one has

\[
V^{\mu\nu\rho} = V^{\mu\nu\rho} + \frac{1}{3}(p_2^0 - p_3^0) I^{\mu\nu\rho} \quad (41)
\]

with

\[
I^{\mu\nu\rho}(P_{1R}, P_{2A}, P_{3A}) = -< v^\rho(P_3, v - i\tilde{C})^{-1}v^\mu(P_1, v + i\tilde{C})^{-1}v^\nu > \\
+ < v^\mu(P_1, v + i\tilde{C})^{-1}v^\nu((P_1 + P_3), v + i\tilde{C})^{-1}v^\rho > \\
+ < v^\nu((P_1 + P_3), v + i\tilde{C})^{-1}v^\mu(P_3, v - i\tilde{C})^{-1}v^\rho > \quad (42)
\]

As \(I^{\mu\nu\rho}\) is zero in the HTL case, it is proportional to \(\tilde{C}\).

The Ward Identity (38) is satisfied for \(p_{2\nu}V^{\mu\nu\rho}\) and a similar one for \(p_{3\rho}V^{\mu\nu\rho}\). However for \(p_{1\mu}V^{\mu\nu\rho}\), the \(v^\mu\) factor in the last term of (40) sits in the middle of a string and the identity analogous to (36) does not allow to kill the inverse operators. It happens because in diagram(c) of Fig. 1 a reversal of the causality flow occurs around the loop, at the vertex with incoming \(P_1\). In other words, in (42)

\[
p_{1\mu}I^{\mu\nu\rho} \neq 0 \quad p_{2\nu}I^{\mu\nu\rho} = 0 = p_{3\rho}I^{\mu\nu\rho} \quad (43)
\]

One may try to cure the problem by modifying the vertex where the leg \(P_1\) is attached \([17]\), however we have not found a symmetric modification that does not affect the two other terms.

Turning now to the 4-point function, the rules of Sec 3.2 gives the HTL amplitude as a sum of three tree diagrams. Simple Ward identities are obeyed for the Imaginary Time amplitudes, or equivalently for the functions of the type \(RAAA\) or \(ARRR\) \([13]\). For the collision-resummed case, the amplitude is easily written down and one encounters a new problem. For example, if the uncut propagator is chosen between the legs \(P_1\) and \(P_4\)

\[
p_{4\nu}V^{\mu\nu\rho}(P_{1R}, P_{2A}, P_{3A}, P_{4A}) = \]

\[
V^{\mu\nu\rho}(P_{1R}, P_{2A}, (P_3 + P_4), A) - V^{\mu\nu\rho}((P_1 + P_4), R, P_{2A}, P_{3A}) \\
+ (p_1^0 + p_2^0)(I^{\mu\nu\rho}(P_{1R}, P_{2A}, (P_3 + P_4), A) - I^{\mu\nu\rho}((P_1 + P_4), R, P_{2A}, P_{3A})) \quad (44)
\]

with \(V^{\mu\nu\rho}\) as in (35) and \(I^{\mu\nu\rho}\) similar to (42) with a somewhat different distribution of the signs in front of \(i\tilde{C}\).

If one average over the four possible “sum of three trees” for the 4-point function, one just make the problem symmetric. The Ward identities do affect the symmetry of the \(V^{\mu\nu\rho}\) \([2]\), hence the final expression of the 4-gluon vertex when symmetry and color are taken into account; starting from QED-like identities may not be the optimal way to find out the full amplitude.
In further sections, one will consider the soft one-loop diagrams that contribute to the gluon self-energy. There will be needed the 4-point function with two legs of type $A$ and two legs of type $R$. The causality flow is not completely determined along the tree and one has to sum over the possible cases with an appropriate thermal weight [16]. As a consequence thermal weights enter the Ward identities in the HTL case [13]. We shall not pursue the general construction of this 4-point amplitude as it is needed in a very specific kinematical case, the forward direction.

4 The collision operator and the one-loop soft exchange in the self-energy

In his quest for an effective theory for the soft field modes $Q \sim g^2 T$, Bödeker’s first step was to compute the contribution from the one-loop soft $K$ momentum exchange to the polarization tensor $\Pi^{\mu\nu}(Q)$ for $K \sim gT$ and $Q \sim g^2 T$ [4, 6, 18]. Those one-loop diagrams involve hard thermal loop effective vertices and propagators.

Then higher loop diagrams with momentum $K \sim gT$ were resummed via a transport equation, so that both scales $T$ and $gT$ were integrated out [4, 6]. This transport equation is in terms of the field $W(K, v)$ that describes the mean distribution of hard particles. Fluctuations in $v$ are caused by collisions and taken into account via an operator $\hat{C}(v, v')$. The linearized transport equation may be written (see Eq. (18))

\begin{equation}
(v.K + i\hat{C}) W^i(K, v) = v^i \tag{45}
\end{equation}

while the integration over the scale $T$ only, lead to the linearized transport equation [5]

\begin{equation}
v.K W^i = v^i \tag{46}
\end{equation}

The complete solution to the transport equation (45) involves the inverse operator $(v.K + i\hat{C})^{-1}$

\begin{equation}
W^i_R(K, v) = (v.K + i\hat{C})^{-1} v^i \tag{47}
\end{equation}

As recently stressed by Blaizot and Iancu [10], alternatively one may solve the transport equation (45) by iteration. One obtains a series expansion in powers of $\hat{C}$ whose first terms reproduce the expansion in the number of loops with momentum $K \sim gT$. Indeed, writing Eq. (45)

\begin{equation}
(v.K W^i = v^i - i\hat{C} W^i \tag{48}
\end{equation}

one obtains

\begin{equation}
W^{i(0)}_R = \frac{v^i}{v.K + i\epsilon} , \quad \epsilon > 0 \tag{49}
\end{equation}

\begin{equation}
W^{i(1)}_R = \frac{1}{v.K + i\epsilon} (-i)\hat{C} \frac{v^i}{v.K + i\epsilon} = \frac{1}{v.K + i\epsilon} (-i)C(v, v') \frac{v^i}{v'.K + i\epsilon} > v' \tag{50}
\end{equation}

Since the polarization tensor is simply related to the $W^i$ field (see Eq. (19))

\begin{equation}
\Pi^{ji}(Q) = q_0 m_D^2 < v^j W^i(Q, v) > v \tag{51}
\end{equation}

one obtains

\begin{equation}
\Pi^{j(0)i}(Q) = q_0 m_D^2 < v^j \frac{1}{v.Q + i\epsilon} v^i > v \tag{52}
\end{equation}
\[ \Pi^{ji}(Q) = q_0 m_D^2 < \frac{v^j}{v.Q + i\epsilon} (-i) \hat{C} \frac{v^i}{v.Q + i\epsilon} >_{v,v'} \]
\[ = q_0 m_D^2 < \frac{v^j}{v.Q + i\epsilon} (-i) C(v,v') \frac{v^i}{v'.Q + i\epsilon} >_{v,v'} \]  

(53)

\( \Pi^{ji(0)} \) is the hard thermal loop contribution to the polarization tensor, \( \Pi^{ji(1)} \) is the one-loop soft momentum contribution for \( K \sim gT \) with \( C(v,v') \) as in (4) [10, 18]. The conclusion is, one can identify the collision operator \( \hat{C} \) if \( \Pi^{ji(1)} \) is known and appears under the form (53).

We shall follow this strategy. We want to integrate out the momenta \( K \sim g^2T \ln 1/g \) to see how they affect the dynamics of the soft fields \( Q \sim g^2T \). We shall assume \( \ln 1/g >> 1 \) so that the scales are well separated. If the summation over the scale \( g^2T \ln 1/g \) can be made via a transport equation which involves a collision operator \( \hat{C}' \), i.e.

\[ (v.K + i\hat{C} + i\hat{C}')W_i(K,v) = v^i \]  

(54)

the full solution will be in terms of an inverse operator

\[ W_i = (v.K + i\hat{C} + i\hat{C}')^{-1}v^i \]  

(55)

and the iterative solution will be

\[ W^{(0)} = (v.K + i\hat{C})^{-1}v^i \]  

(56)

\[ W^{(1)} = (v.K + i\hat{C})^{-1} (-i) \hat{C}' (v.K + i\hat{C})^{-1}v^i \]  

(57)

so that

\[ \Pi^{ji(0)}(Q) = q_0 m_D^2 < v^j (v.Q + i\hat{C})^{-1}v^i >_{v,v'} \]  

(58)

\[ \Pi^{ji(1)}(Q) = q_0 m_D^2 < v^j (v.Q + i\hat{C})^{-1} (-i) \hat{C}' (v.Q + i\hat{C})^{-1}v^i >_{\text{all } v} \]  

(59)

\( \Pi^{ji(0)} \) is the form found in Eq. (25) for the polarization tensor, when the scale \( T \) and \( gT \) have been integrated out. \( \Pi^{ji(1)} \) will correspond to the one-loop soft momentum exchange \( K \sim g^2T \ln 1/g \), where the one-loop diagrams involve effective vertices and resummed propagators with the scales \( T \) and \( gT \) integrated out.

The effective vertices has been constructed in Sec. 3.3. The one-loop soft momentum exchange \( K \sim g^2T \ln 1/g \) contribution to the polarization tensor \( \Pi^{ji(1)} \) will be computed in Sec. 5.2, it will indeed show up as in (59). This fact will allow to identify the operator \( \hat{C}' \).

The expression for \( C(v,v') \) involves by construction an integration over the momentum \( K \sim gT \). So does \( C'(v,v') \) with momentum \( K \sim g^2T \ln 1/g \). There wont be any over-counting in \( \hat{C} + \hat{C}' \) in Eq. (54,55) if one limits the integration range over the space momentum \( k \)

\[ \mu_2 < k < \mu_1 \text{ for } \hat{C} \]
\[ \mu_3 < k < \mu_2 \text{ for } \hat{C}' \]
\[ gT < \mu_1 < T, \quad g^2T \ln 1/g < \mu_2 < gT, \quad g^2T < \mu_3 < g^2T \ln 1/g \]  

(60)

5 The one-loop self-energy diagram with 3-gluon effective vertices

5.1 A general expression in the Retarded/Advanced formalism

At the HTL level, the Retarded/Advanced formalism is an alternative to the Imaginary Time formalism. It avoids the analytic continuations that are necessary in the Imaginary Time formalism.
It has been used for explicit calculations of the photon self-energy within the HTL framework at
the two-loop level by Aurencche and collaborators [19]. We adopt their conventions.

The propagators are the retarded and advanced propagators, i.e. with $\epsilon > 0$

$$
\Delta(p_R) = \Delta(p_0 + i\epsilon, p) = \frac{i}{(p_0 + i\epsilon)^2 - p^2 - \Pi(p_0 + i\epsilon, p)} 
$$  \hspace{1cm} (61)

$$
\Delta(p_A) = \Delta(p_0 - i\epsilon, p) = -\Delta(p_R)^* \hspace{1cm} \Delta(p_R) = \Delta((-P)_A) 
$$  \hspace{1cm} (62)

from Eqs (32,33) for $\Pi(P)$. The effective gluon propagator is

$$
D_{\mu\nu}(P) = \mathcal{P}_t^{\mu\nu} \Delta^t(P) + \mathcal{P}_l^{\mu\nu} \Delta^l(P) + i\xi \frac{p^\mu p^\nu}{P^4} 
$$  \hspace{1cm} (63)

$$
\mathcal{P}_t^{ij} = \delta^{ij} - \frac{p^i p^j}{P^2} \hspace{1cm} \mathcal{P}_t^{00} = 0 
$$  \hspace{1cm} (64)

$$
\mathcal{P}_l^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{P^2} - \mathcal{P}_t^{\mu\nu} 
$$  \hspace{1cm} (65)

$$
\Pi^{\mu\nu}(P) = \mathcal{P}_t^{\mu\nu}(P)\Pi^t + \mathcal{P}_l^{\mu\nu}(P)\Pi^l 
$$  \hspace{1cm} (66)

In the R/A formalism the thermal weights $n(p_0)$ that are associated with the loop momenta are
carried by the vertices in a very specific way [12, 14]. For the 3-point vertices, they are attached to
the $A$ legs in the vertices with two legs of type $A$, i.e. with all incoming momenta, the vertices are

$$
\Gamma^{\mu\nu\sigma}(p_R, q_A, r_R) = V^{\mu\nu\sigma}(p_R, q_A, r_R) 
$$  \hspace{1cm} (67)

$$
\Gamma^{\mu\nu\sigma}(p_A, q_R, r_A) = - V^{\mu\nu\sigma}(p_A, q_R, r_A)(n(p_0) + n(r_0) + 1) 
$$  \hspace{1cm} (68)

where the general properties of the 3-gluon vertices $V^{\mu\nu\sigma}$ were given in Sec. 3.1. Two relations will
be useful

$$
V^{\mu\nu\sigma}((-p)_R, (-q)_A, (-r)_R) = - V^{\mu\nu\sigma}(p_A, q_R, r_A) 
$$  \hspace{1cm} (69)

$$
n(p_0) + 1/2 = - (n(-p_0) + 1/2) 
$$  \hspace{1cm} (70)

When one considers the self-energy $\Pi(Q_R)$, the one-loop diagram with 3-gluon vertices is, in
the R/A formalism, the sum of the three diagrams drawn on Fig.2 (Note that because any vertex
is defined with all incoming momenta, a propagator joins an $R$ to an $A$, since a $P_R(p_0 + i\epsilon, p)$
incoming a vertex comes from another vertex with incoming $(-p_0 - i\epsilon, -p)$ i.e. $(-P)_A$ ). With the
definition of momenta of Fig.2(d), leaving out a color factor $\delta_{ab}$

$$
\Pi^{\mu\nu}_{(3g)}(Q_R) = \frac{Ng^2}{2} \int \frac{dp}{i(2\pi)^4} \left[ \left( n(p_0) + 1/2 \right) [D_{\rho\rho'}(P_R) V^{\rho\mu\sigma}(P_R, Q_R, (-S)_A) D_{\sigma\sigma'}(S_R) V^{\sigma\nu\rho'}((-P)_A, (-Q)_A, S_R)] 

- (n(s_0) + 1/2) [D_{\rho\rho'}(P_A) V^{\rho\mu\sigma}(P_A, Q_R, (-S)_R) D_{\sigma\sigma'}(S_A) V^{\sigma\nu\rho'}((-P)_R, (-Q)_A, S_A)] 

- (n(p_0) - n(s_0)) [D_{\rho\rho'}(P_A) V^{\rho\mu\sigma}(P_A, Q_R, (-S)_A) D_{\sigma\sigma'}(S_R) V^{\sigma\nu\rho'}((-P)_R, (-Q)_A, S_R)] \right] 
$$  \hspace{1cm} (71)

Several comments have to be made
- From (69) the two vertices entering each term are identical, up to a sign.
- The prescription $R$ or $A$ on each leg fixes the way one should approach the discontinuity in that
  variable. Then one writes $S = P + Q$ and one may integrate over $p_0$ either on the real axis, or in
  the complex $p_0$ plane. Although $s_0 = p_0 + q_0$, it make sense to speak of discontinuities in the $p_0$
or \( q_0 \) or \( s_0 \) variable, just as at \( T = 0 \) one considers singularities in the \( s \) or \( t \) or \( u \) channel of the 4-point function.

- The third term of the right side of Eq.(71) corresponds to diagram(c) of Fig. 3. In the first and second term, a term \( (n(q_0) + 1/2) \) has been dropped in the thermal weight. The reason is: all factors in the bracket of the first (or second) term have singularities in the \( p_0 \) complex plane only on one side of the real axis; by closing the integration contour on the other half plane, they are seen to give a vanishing contribution when they are multiplied by \( (n(q_0) + 1/2) \). By the same token \( 1/2 \) could be dropped, it has been kept because of the parity property (70). Conversely, \( n(p_0) \) has poles at \( p_0 = \pm i n 2\pi T \) all along the imaginary axis.

Thermal weights depending on the external momenta do appear in the R/A formalism. They give a vanishing contribution at any loop order because they are associated with terms whose singularities all lie one one side of the real axis. It has to be so in order to agree with the Imaginary Time formalism, where thermal weights only depend on loop momenta (see the coth method [20]). This feature will turn out to be an useful one when one considers soft momentum exchange \( p \).

- One may make the change of variable \( s = -p' \) in the two terms whose statistical weight is \( (n(s_0) + 1/2) \). They can be folded into the terms whose statistical weight is \( (n(p_0) + 1/2) \) provided that the vertex is antisymmetric in the exchange of \( P_A \) and \((-S)_R = S_A \)

- Eq. (71) is a completely general formula and appears in many cases [19, 13].

5.2 The case of a soft exchange around the loop

We want to use Eq.(71) to compute soft momentum exchange around the loop, i.e. \( p_0 << T \). Now comes Bödeker’s argument [18] (translated from Imaginary Time to R/A formalism). For loop momenta \( p_0 << T \) one may drop the first and second term in Eq.(71). Indeed, as said above, the terms in brackets have singularities only on one side of the real \( p_0 \) axis, and if one closes the integration contour on the other side, the first contribution comes from the pole \( p_0 = \pm i 2\pi T \), i.e. a hard energy, and it will lead to terms of order \( q_0/T \). As a consequence, one is left with the third term, and for \( p_0 << T \) and \( q_0 << T \), one may approximate the thermal weights i.e. keep only the pole close to the origin and neglect the other ones’ contribution. Changing the loop variable from \( P \) to \( K \)

\[
P = K - Q/2 \quad , \quad S = K + Q/2
\]

\[
n(p_0) - n(s_0) \approx T\left(\frac{1}{p_0} - \frac{1}{s_0}\right) = \frac{Tq_0}{(k_0 - q_0/2)(k_0 + q_0/2)}
\]

so that a factor \( q_0 \) is extracted from the thermal weight. Note that in terms of the set of diagrams drawn on Fig. 2, this argument amounts to keep only the diagram where the vertex with two legs of type \( A \) is not the one with a \((-Q)A\) leg. It will be a useful feature when one turns to the other case, i.e. one soft loop diagram with a 4-point vertex, as it allows from the start to select a very limited number of diagrams.

First, momenta \( K \sim gT \) around the loop will be considered and the result obtained in [18] will be reproduced, then momenta \( K \sim g^2 T \ln 1/g \) will be studied. The vertex \( V \) is the effective vertex, bare + one-loop, however the leading term only comes from the one-loop part, see the discussion in [18]. We shall not consider the case \( \mu = \nu = 0 \) in \( \Pi^{\mu\nu} \).

For momenta \( K \sim gT \) the effective vertex is the HTL one, Eq. (34) with \( P_1 = Q, ~ P_2 = -S = -(K + Q/2), ~ P_3 = P = K - Q/2 \)

\[
V^{\mu\nu\rho}(Q_R, (-S)_A, P_A) = m_D^2 < v_1^{\mu} v_1^{\nu} v_1^{\rho} (k_0 + q_0/2) v_1.(K + Q/2)_R - (k_0 - q_0/2) v_1.(K - Q/2)_A > v_1
\]

13
and the resummed propagators (63,66) are those with the HTL $\Pi^{\mu
u}$, as in Eq. (27), so that keeping only the third term in Eq. (71). and with (72,73)

$$\Pi_{(3g)}^{\mu\nu}(Q_R) = q_0 m_D^4 g^{2NT} \frac{1}{2} \int \frac{d_4 k}{i(2\pi)^4} \frac{1}{(k_0 - q_0/2)(k_0 + q_0/2)} D_{\rho\rho'}((K - Q/2)_A) D_{\sigma\sigma'}((K + Q/2)_R) \nu^{\mu\rho'}(K, Q)$$

(76)

$\nu^{\mu\rho'}\sigma'$ is obtained from $\nu^{\mu\sigma}$ with the substitutions $\mu \rightarrow \nu, \rho \rightarrow \rho', \sigma \rightarrow \sigma', v_1 \rightarrow v_2$

If one now drops $Q \sim g^2 T$ in front of $K \sim g T$ in (76,74)

$$k_0 \left( \frac{1}{v_1.K_R} - \frac{1}{v_1.K_A} \right) = k_0 \text{disc} \frac{1}{v_1.K} = -k_0 2\pi i \delta(v_1.K)$$

(77)

and

$$\Pi_{(3g)}^{\mu\nu}(Q_R) = q_0 m_D^4 g^{2NT} \frac{1}{2} \int \frac{d_4 k}{i(2\pi)^4} \frac{1}{v_1.K} \frac{1}{v_1.K} \frac{1}{v_2.K} > v_1 < v_2$$

(78)

The result does not depend on the gauge parameter $\xi$ owing to the $\delta$ functions. $\Pi_{(3g)}^{\mu\nu}$ was found via this perturbative approach in [18] in the Imaginary Time formalism.

Comparing (78) with (53) for the case $\mu = j, \nu = i$, one identifies the contribution from this diagram to the collision operator $\hat{C}$, or equivalently to $\Phi$ defined in Eq. (4)

$$\Phi(v_1,v_2) = \int \frac{d_4 k}{(2\pi)^4} |v_1.D(K).v_2|^2 (2\pi)^2 \delta(v_1.K)\delta(v_2.K)$$

(79)

$\Phi$ was also found by other methods [4, 7, 8, 9]. $\Phi(v_1,v_2)$ describes the collision cross section of two fast particles with soft momentum exchange $K$; the $\delta$ functions enforce the conservation of energy-momentum at each vertex in the near-forward direction. The integration range for the space momentum $k$ is limited $\mu_2 < k < \mu_1$ (see (60)).

With the explicit form of the HTL vertex, one may write down the two first terms of Eq. (71). and with (72,73)

$$v^{\mu\rho'}(K, Q) = < v^\mu_1(v_1.Q + i\hat{C})^{-1}$$

$$((k_0 + q_0/2)v^\rho_1(v_1.(K + Q/2) + i\hat{C})^{-1}v^\sigma_1$$

$$- (q_0/2 - k_0)v^\sigma_1(v_1.(Q/2 - K) + i\hat{C})^{-1}v^\rho_1 > v_1, v'_1, v''_1$$

(80)

and $\nu^{\mu\rho'}\sigma'$ is obtained by the substitution $\mu \rightarrow \nu, \rho \rightarrow \rho', \sigma \rightarrow \sigma'$ and $v_1, v'_1, v''_1 \rightarrow v_2, v'_2, v''_2$

The Ward identities (39,38) imposed on this 3-gluon vertex read

$$Q_\mu \nu^{\mu\rho\sigma} = m_D^2 [\Pi^{\rho\sigma}((K + Q/2)_R) - \Pi^{\rho\sigma}((Q/2 - K)_R)]$$

(81)

14
\[-(K + Q/2) \mathcal{V}^{\mu\rho\sigma} = m_D^{-2} [\Pi^{\mu\rho}((Q/2 - K)_R) - \Pi^{\mu\rho}(Q_R)]\]  
(82)

\(\Pi^{\mu\nu}_{(3g)}\) is in the desired form (59). If one now drops \(Q \sim g^2 T\) in front of \(K \sim g^2 T \ln 1/g\), which is only valid for \(\ln 1/g \gg 1\)

\[\mathcal{V}^{\mu\rho\sigma}(K, Q) = k_0 < v_1^p (v_1, Q + i \hat{C})^{-1} v_1^q (v_1, K - i \hat{C})^{-1} v_1^\rho \rangle >_{v_1, v_1', v_1''} \]  
(83)

In particular, for \(\rho\) and \(\sigma\) spacelike

\[\mathcal{V}^{\mu\rho\sigma} = k_0 < v_1^p (v_1, Q + i \hat{C})^{-1} \{v_1^q (v_1, K + i \hat{C})^{-1} v_1^r - v_1^r (v_1, K - i \hat{C})^{-1} v_1^q \} >_{v_1, v_1', v_1''} \]  
(84)

where the mean field \(W^j_R(K, v_1)\) has been defined in (21) and \(W^i_A(K, v_1)\) in a similar way (see (10)). The resulting contribution to the polarization tensor is

\[\Pi^{\mu\nu}_{(3g)}(Q_R) = q_0 m_D^4 [\frac{g^2 NT}{2} \int \frac{d k}{(2\pi)^4} D_{\rho\sigma'}(K_A) D_{\sigma\sigma'}(K_R) \frac{1}{k_0^2} \mathcal{V}^{\mu\rho\sigma}(K, Q) \mathcal{V}^{\nu\rho\sigma'}(K, Q) \]  
(85)

with \(\mathcal{V}\) as in (83) or (84). A consequence of (85) is, from (81)

\[Q_v \Pi^{\mu\nu}_{(3g)} = q_0 m_D^2 \frac{g^2 NT}{2} \int \frac{d k}{(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \frac{1}{k_0^2} \mathcal{V}^{\mu\rho\sigma'}(K, Q)(\Pi^{\rho\sigma'}(K_R) - \Pi^{\rho\sigma'}(K_A)) \]  
(86)

In Sec. 6 the other one-loop diagram with a 4-gluon vertex will be computed and one will obtain

\[Q_v (\Pi^{\mu\nu}_{(3g)} + \Pi^{\mu\nu}_{(4g)}) = 0 \]  
(87)

Comparing (85)(83) with the expression (59) for the case \(\mu = j, \nu = i\), one obtains the collision term arising from the diagram of Fig. 2. Defining

\[C^i(v_1, v_2) = -m_D^2 \frac{g^2 NT}{2} [(\Phi^{i}_{(3g)}(v_1, v_2) + \Phi^i_{(4g)}(v_1, v_2)] \]  
(88)

\[\Phi^{i}_{(3g)}(v_1, v_2) = \int \frac{d k}{(2\pi)^4} D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) \]  
(89)

\[< v_1^p (v_1, K + i \hat{C})^{-1} v_1^q - v_1^q (v_1, K - i \hat{C})^{-1} v_1^p >_{v_1} \]

\[< v_2^p (v_2, K + i \hat{C})^{-1} v_2^q - v_2^q (v_2, K - i \hat{C})^{-1} v_2^p >_{v_2} \]

where the range of integration over the space momentum is \(\mu_3 < k < \mu_2\) (see(60)). Note that one may go back to the expression describing the momentum \(K \sim g T\) by replacing the operator \(\hat{C}\) by \(\epsilon > 0\), then the inverse operator becomes diagonal in \(v\) space, and one gets back \(\Phi(v_1, v_2)\) as in (79). The dependence on the gauge parameter \(\xi\) needs further study, one has

\[k_\rho < v_1^p (v_1, K + i \hat{C})^{-1} v_1^q - v_1^q (v_1, K - i \hat{C})^{-1} v_1^p >_{v_1} \]

\[= - < i \hat{C}(v_1, K + i \hat{C})^{-1} v_1^q - v_1^q >_{v_1} \]  
(90)

which is zero when integrated over \(v_1\) or contracted with \(k_\sigma\).
An intuitive picture for the collision term $\Phi'_3(v_1, v_2)$ is obtained if one limits oneself to the dominant exchange of transverse gluons. From (63)

$$D_{\rho\rho'}(K) \approx \Delta^i(K)\mathcal{P}_{i}^{\rho\rho'} = \Delta^i(K)\mathcal{P}_{i}^{i'}$$

(91)

then from (84), the mean field $W^i(K, \nu)$ appears in $\Phi'_3$, its transverse part is parallel to $v^i_
u$

$$W^i(K, \nu) = \hat{k}_i W^t + v^i_
u W^t$$

(92)

$$\mathcal{P}_i^{ii'}(k) W^{ii'}(K, \nu) = \mathcal{P}_i^{ii'} v^{ii'} W^{ii'}(K, \nu)$$

(93)

so that

$$\Phi'_3(v_1, v_2) = \int \frac{d^4k}{(2\pi)^4} |\Delta^i(K_R)(v_1, \nu_1, v_2)|^2$$

$$(-)[W^t_R(K, v_1) - W^t_A(K, v_1)][W^t_R(K, v_2) - W^t_A(K, v_2)]$$

(94)

$\Phi'_3(v_1, v_2)$ may be interpreted as the collision cross section of two mean distributions of the fast particles, in the near forward direction. The constraint at each vertex is no more a strict particle-like conservation $2\pi \delta(v_1.K)$ as in (79), but a looser one involving the “width” of the $W$ field.

A consequence of (89) is

$$\int \frac{d\Omega_{v_2}}{4\pi} \Phi'_3(v_1, v_2) = \int \frac{d^4k}{(2\pi)^4} D_{\rho\rho'}(K_A)D_{\sigma\sigma'}(K_R) \frac{1}{k_0 m^2_D}$$

$$[\Pi^{i'\sigma'}(K_R) - \Pi^{i'\sigma'}(K_A)]$$

(95)

One notices that the contribution to the damping rate of a hard gluon arising from the range $K \sim g^2 T \ln 1/g$ is obtained from (95) if one substitutes to the $v^i_1$ average the quantity $v^i_1 v^q_2 (-2\pi i) \delta(v_1.K)$. In Sec. 6, one will find that

$$\int \frac{d\Omega_{v_2}}{4\pi} [\Phi'_3(v_1, v_2) + \Phi'_4(v_1, v_2)] = 0$$

(96)

i.e. a relation similar to (5) for the case of the $\hat{C}$ operator.

5.3 A sum rule for the collision operators $\hat{C}$ and $\hat{C}'$

In order to know the scale of $\Phi'_3$, and of $\hat{C}'$ defined in (88), one may average over $v_1$ and $v_2$

$$\Phi = \int \frac{d\Omega_{v_1}}{4\pi} \frac{d\Omega_{v_2}}{4\pi} \Phi'_3(v_1, v_2)$$

$$= \int \frac{d^4k}{(2\pi)^4} D^{*}_{\rho\rho'}(K_R)D_{\sigma\sigma'}(K_R)$$

$$[\Pi^{i'\sigma'}(K_R) - \Pi^{i'\sigma'}(K_A)] [\Pi^{i\sigma}(K_R) - \Pi^{i\sigma}(K_A)](-1)$$

(97)

This relation, found in the case $K \sim g^2 T \ln 1/g$, also holds in the case $K \sim g T$, i.e. when $\Phi'_3(v_1, v_2)$ is replaced by $\Phi(v_1, v_2)$ as in (79). Indeed, for $K \sim g T$, $\Pi^{i\sigma}(K)$ is as in (27) i.e.

$$\frac{i}{k_0 m^2_D} [\Pi^{i\sigma}(K_R) - \Pi^{i\sigma}(K_A)] = \int \frac{d\Omega_{v_1}}{4\pi} v^\rho_i v^\sigma_1 2\pi \delta(v_1.K)$$

(98)
Relation (97) allows an easy comparison between the infrared behaviour of the cases $K^g T$ and $K^g T \ln 1/g$. Restricting oneself to the dominant transverse gluon exchange, and coming back to the notation $\Pi(K^g) \equiv \Pi_R(K)$, (97) is

$$\Phi = \int \frac{d_4 k}{(2\pi)^4} |\Delta^i(K_R)|^2 \left[ (\Pi'_R(K) - \Pi'_A(K)) \frac{i}{k_0 m_D^2} \right]^2$$

(99)

with $\Pi' = \frac{1}{2} D_t^{ij} \Pi^{ij}$. In the infrared sector $k_0 << k$

$$|\Delta^i(K_R)|^2 = \frac{1}{(k_0^2 - k^2 - \text{Re} \Pi')^2 + (\text{Im} \Pi')^2} \approx \frac{1}{k^4 + (\text{Im} \Pi')^2}$$

(100)

For the case $k \sim g T$, from (98)

$$\text{Im} \Pi' = -k_0 m^2 \frac{1}{4k^2} \frac{\pi}{k} (1 - \frac{k_0^2}{k^2}) \theta(k^2 - k_0^2) \approx -m^2 \frac{k_0 \pi}{k}$$

for $k_0 << k$

(101)

and

$$I_0 = \int \frac{dk_0}{k^4 + k_0^2} \frac{1}{m_D^2 k^2} \approx \frac{4}{km_D^2}$$

(102)

so that

$$\Phi \sim \int dk \frac{k^2}{m_D^2} \frac{1}{k^2} \approx \frac{1}{m_D^2} \int_{\mu_2} \frac{d\mu_1}{k}$$

(103)

the integral has an infrared divergence and

$$\gamma = m_D^2 \frac{g^2 N T}{2} \Phi = \frac{g^2 N T}{4\pi} \ln \frac{\mu_1}{\mu_2}$$

(104)

For the case $k \sim g^2 T \ln 1/g$ and for $k << \gamma$ see in the appendix Eqs (140,138)

$$\Pi'_R(K) = \frac{m_D^2}{3} \frac{k_0}{k_0 + i(\gamma - \delta_1)} \approx -i\frac{m_D^2}{3} \frac{k_0}{\gamma - \delta_1}$$

for $k_0 << k << \gamma$

(105)

Now

$$I_0 = \int \frac{dk_0}{k^4 + k_0^2} \frac{1}{3(\gamma - \delta_1)} \approx \frac{1}{m_D^2} \int_{\mu_3} \frac{d\mu_3}{\gamma - \delta_1}$$

(106)

so that

$$\Phi \sim \int dk \frac{k^2}{m_D^2} \frac{1}{k^2} \approx \frac{1}{m_D^2} \int_{\gamma - \delta_1} \frac{dk}{\gamma - \delta_1}$$

(107)

the integral has no infrared divergence. In fact it is unlikely that the main contribution to $\Phi$ comes from the momenta $k < \gamma/3$. Indeed, as discussed in the appendix, there is an imaginary part of $\Pi'$ in the range $|k_0| < k$ analogous to (101) i.e. to the Landau-damping term. However it only exists for $k > \gamma/3$.

One may look at the contribution from the momenta $k \sim g^2 T \ln 1/g$ (i.e $I_0$ is as in (106)) to two other quantities

- the damping rate of a hard point-like gluon. Then in $\Phi$, one ($\Pi_R - \Pi_A$) is as in (105) and the other ($\Pi_R - \Pi_A$) as in (101), and $\Phi$ has an infrared log divergence.

- the collision cross section of two point-like hard gluons. Then in $\Phi$ both ($\Pi_R - \Pi_A$) are as in (101), and $\Phi$ has a linear divergence, as stated in the introduction.

The expression obtained for the collision term $\Phi'_(3g)$ suggests that the point-particle aspect looses its relevance when the scale $g^2 T \ln 1/g$ is integrated out.
6 The one-loop self-energy diagram with a 4-gluon effective vertex

In the HTL case, the contribution to $\Pi^{\mu\nu}$ of the diagram drawn on Fig. 3(a) is important. Adding it to the diagram with 3-gluon vertices, the polarization tensor is transverse [1, 4, 18]

$$Q_\mu \Pi^{\mu\nu}_{(3g)} + Q_\mu \Pi^{\mu\nu}_{(4g)} = 0$$

(108)

and the collision operator has a zero mode

$$\int \frac{d\Omega_{v_2}}{4\pi} \left[ \Phi_{(3g)}(v_1, v_2) + \Phi_{(4g)}(v_1, v_2) \right] = 0 = \hat{C}$$

(109)

In the HTL case the one-loop 4-point function basic amplitude can be written as a sum of three trees and there are four equivalent forms. Then, taking into account the color factors, one has to add all the permutations of the external legs. In the self-energy diagram there enters the forward 4-point amplitude with two identical colors.

In order to generalize to the collision-resummed case, we are on a less firm ground because the four possible forms as a sum of three trees are no more equivalent, and the Ward Identities do not take the HTL-like form. Our method will be: we shall consider one tree diagram, the basic one in the HTL case before enforcing symmetry. Its contribution to the polarization tensor $\Pi^{\mu\nu}_{(4g)}$ will be computed in the R/A formalism. It will give the structure in $v$ space arising from the 4-point function.

The basic tree to be considered is diagram (b) on Fig. 3, and its contribution to the self-energy-type soft exchange $K$ is diagram (c) (Arrows on the propagators give the orientation of $v$ and $K$).

In the R/A formalism, the needed 4-point function in diagram (c) has two legs of type $R$ and two legs of type $A$. With Bödeker’s approximation to drop terms whose singularities lie on one side of the real axis of the $k_0$ variable, only one possibility is left for the causality flow, the one drawn on diagram (c) (as a vertex of type $RAA$ should not be a vertex whose one leg is carrying the momentum $(-Q)_A$). With the rules of Sec. 3.2 and with (67,68), leaving out the color factor $\delta_{ab}$

$$\Pi^{\mu\nu}_{(4g)}(Q) = i g^2 N \left( \frac{2}{m_D^2} \right) \int \frac{d^4k}{i(2\pi)^4} D_{\rho\sigma}((-K)_R) \left[ n(k_0) + n(-k_0 - q_0) + 1 \right]$$

(110)

$$(-k_0 + q_0) < v^\mu (-v.Q - i\hat{C})^{-1} v^\rho (-v.(K + Q) - i\hat{C})^{-1} v^\sigma (-v.Q - i\hat{C})^{-1} v^\nu >$$

and

$$n(k_0) + n(-k_0 - q_0) + 1 = n(k_0) - n(k_0 + q_0)$$

(111)

$$\approx \frac{T}{k_0} - \frac{T}{k_0 + q_0} = \frac{Tq_0}{k_0(k_0 + q_0)}$$

(The tree’s numerator corresponds to the choice, the leg $K + Q$ is never cut, i.e. one chooses to write the 4-point function as a sum of three trees, one of which is diagram (b), the two others are tadpole diagrams that do not depend on $Q$)

Adding the case where $K, \rho$ and $-K, \sigma$ are exchanged, completing with irrelevant terms (singularities on same side), one obtains

$$\Pi^{\mu\nu}_{(4g)}(Q) = i g_0 \frac{g^2 N T}{2} \left( \frac{2}{m_D^2} \right) \int \frac{d^4k}{i(2\pi)^4} \frac{1}{k_0} \left[ D_{\rho\sigma}((-K)_R) - D_{\rho\sigma}(K_R) \right]$$

$$< v^\mu (v.Q + i\hat{C})^{-1} [v^\rho (v.(K + Q) + i\hat{C})^{-1} v^\sigma$$

$$+ v^\sigma (v.(Q - K) + i\hat{C})^{-1} v^\rho] (v.Q + i\hat{C})^{-1} v^\nu >$$

(112)
The contribution from the vertex-type diagram (\(- K \) and \(- Q \) exchanged) is subdominant as it exhibits only one factor (\( v.Q + \hat{\mathcal{C}} \))\(^{-1} \).

Neglecting \( Q \) in front of \( K \), the resulting \( \Pi^{\mu\nu}_{(4g)} \) is

\[
\Pi^{\mu\nu}_{(4g)}(Q) = -ig_0 \frac{g^2 NT}{2m_D^2} \int \frac{d_k}{i(2\pi)^4} \frac{1}{k_0} [D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A)] \\
< v^\mu (v.Q + i\hat{\mathcal{C}})^{-1} \{v^\rho v^{\gamma} - v^\rho (v.K - i\hat{\mathcal{C}})^{-1} v^\gamma \} (v.Q + i\hat{\mathcal{C}})^{-1} v^\nu >
\]

A consequence is

\[
Q_v^{\mu\nu}_{(4g)}(Q) = -ig_0 \frac{g^2 NT}{2m_D^2} \int \frac{d_k}{i(2\pi)^4} \frac{1}{k_0} [D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A)] \\
< v^\mu (v.Q + i\hat{\mathcal{C}})^{-1} \{v^\rho (v.K + i\hat{\mathcal{C}})^{-1} v^\gamma - v^\rho (v.K - i\hat{\mathcal{C}})^{-1} v^\gamma \} >
\]

where Eq. (16) has been used. One sees that the factor that depends on \( v \) in (114) is \( V^{\mu\nu\rho}(K, Q) / k_0 \) as in Eq. (83). From (63,66)

\[
D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A) = -i \ D_{\rho\rho'}(K_A) D_{\sigma\sigma'}(K_R) (\Pi^{\rho\sigma'}(K_R) - \Pi^{\rho\sigma'}(K_A))
\]

so that comparing with (86) one obtains

\[
Q_v^{\mu\nu}_{(4g)} = -Q_v^{\mu\nu}_{(3g)}
\]

Moreover, comparing \( \Pi^{\mu\nu}_{(4g)}(Q) \) as in (113) with (59) for the case \( \mu = j \), \( \nu = i \) one extracts a collision term (see (88))

\[
\Phi^{(4g)}(v, v') = -i \int \frac{d_k}{(2\pi)^4} \frac{1}{k_0 m_D^2} [D_{\rho\sigma}(K_R) - D_{\rho\sigma}(K_A)] \\
< v^\rho (v.K + i\hat{\mathcal{C}})^{-1} v^\sigma - v^\rho (v.K - i\hat{\mathcal{C}})^{-1} v^\sigma v^\rho >
\]

\( \Phi^{(4g)} \) is an operator. When applied on a function \( \hat{\mathcal{O}} \) that does not depend on \( v' \)

\[
\hat{\Phi}^{(4g)} \hat{\mathcal{O}} = \int \frac{d \Omega_{v_1}}{4\pi} \Phi^{(4g)}(v_1, v'_1) = - \int \frac{d \Omega_{v_2}}{4\pi} \Phi^{(3g)}(v_1, v_2)
\]

from (95) and (115), i.e. the collision operator has a zero mode

\[
\hat{\mathcal{C}}' \hat{\mathcal{O}} = + \hat{\Phi}^{(4g)} \hat{\mathcal{O}} = 0
\]

If one restrict oneself to transverse gluon exchange, (117) may be written

\[
\Phi^{(4g)}(v, v') = -i \int \frac{d_k}{(2\pi)^4} \frac{1}{k_0 m_D^2} [\Delta^t(K_R) - \Delta^t(K_A)] \\
v^\rho \ (v.K + i\hat{\mathcal{C}})^{-1} (v.K - i\hat{\mathcal{C}})^{-1} v^\rho
\]

To go back to the expression that corresponds to momentum exchange \( K \sim gT \), one replaces \( \hat{\mathcal{C}} \) by \( \epsilon \) in (117) and one gets back the local term in \( \hat{\mathcal{C}} \) (see (4)). Indeed, the identity operator in \( v \) space is \( I = \delta S_2(v - v') \), and one recognizes in (115,117) the hard gluon damping rate \( \gamma \) if \( \hat{\mathcal{C}} \) is replaced by \( \epsilon \). \( \Phi^{(4g)} \) may be associated with the damping of the collective excitations, its scale is \( \Phi \) discussed in Sec. 5.3.
The color conductivity

The resulting collision term is

$$\dot{C}' = C'(v, v') = -m_D^2 g^2 N T \frac{\Phi'(3g)(v, v') + \Phi'(4g)(v, v')}{2}$$ (121)

with $\Phi'(3g)(v, v')$ as in (89), or (94) for transverse gluon exchange, $\Phi'(4g)(v, v')$ as in (117), or (120). Both are functions of $v, v'$ as there is no other vector available, they are invariant under rotations in $v$ space so that the eigenvalues $c'_l$ of the operator $\dot{C}'$ only depend on the angular momentum $l$. The eigenvalue is zero for $l = 0$.

We now examine how those results are related to the pioneer’s work of Arnold and Yaffe [11]. They extract the conductivity from the static longitudinal part of the polarization tensor $\Pi_{00}(q_0 = 0, q)$ expanded to order $q^2$. $\Pi_{00}$ is obtained from an effective theory dealing mostly with static quantities. The resulting $\Pi_{00}$, the 3-gluon and 4-gluon vertices have structures similar to ours. However the detailed comparison requires further work. Their approach is static, ours is basically causal i.e. time ordered. They encounter a linear infrared divergence which disappears upon dimensional regularization, we don’t have any infrared divergence.

In the collision approach, the conductivity tensor is [10]

$$\sigma_l(q_0, q) = \frac{i}{q_0} \Pi_l(q_0, q), \quad \sigma_l(q_0, q) = -\frac{i q_0}{q^2} \Pi_l(q_0, q)$$ (122)

At the scale $Q \sim g^2 T$, from (55) and (51)

$$\Pi_l = q_0 m_D^2 < v^i (vQ + i\dot{C} + i\dot{C}')^{-1} v^j > \frac{1}{2} p_{ij}$$ (123)

and a similar one for $\Pi_l$ (see the appendix). $\Pi_l$ is, up to a factor, the $l = 1 m = 1$ eigenvalue of the operator $(vQ + i\dot{C} + i\dot{C}')^{-1}$. In the explicit form for $\Pi_l$ in the appendix, one just has to make the substitution $c_l \rightarrow c_l + c'_l$. The scale of $c_l$ and $c'_l$ has been discussed in Sec. 5.3. It remains to be seen how the presence of $c'_l$ affects the pattern of singularities of $\Pi_l$. In the complex $q_0$ plane and whether at sufficiently small $q_0, q$ there is no Landau damping for the mean $W$ field, so that the naive limit $Q << ||\dot{C}|| + ||\dot{C}'||$ is valid, i.e.

$$\Pi_l = q_0 m_D^2 \frac{1}{i(c_1 + c'_1)} = \frac{q_0}{i} \sigma_l$$ (124)

with $c_1 = \gamma - \delta_1$ and

$$c'_l = -m_D^2 g^2 N T \frac{\Phi'(3g)(v, v') + \Phi'(4g)(v, v')}{2}$$ (125)

7 Summary and Conclusions

The results of the paper are summarized. The framework is pure glue, thermal equilibrium, $\ln 1/g >> 1$. The approach has been quantum, perturbative. One has computed the contribution from the one-loop soft gluon exchange of momenta $g^2 T \ln 1/g$ to the gluon self-energy. It emerges a candidate for the collision operator $\dot{C}'$ that should allow to integrate out the scale $g^2 T \ln 1/g$ via a Boltzman-type transport equation, just as
the operator \( \hat{\mathcal{C}} \) does for the scale \( gT \). \( \hat{\mathcal{C}}' \) is expressed in terms of \( \hat{\mathcal{C}} \), both operators have similar features. \( \hat{\mathcal{C}}' \) is made of two terms

\[
\mathcal{C}'(\mathbf{v}_1, \mathbf{v}_2) = -m_D^2 \frac{g^2 NT}{2} (\Phi'(3g)(\mathbf{v}_1, \mathbf{v}_2) + \Phi'(4g)(\mathbf{v}_1, \mathbf{v}_2))
\]  

(126)

\[
\int \frac{d\Omega_{\mathbf{v}_2}}{4\pi} \mathcal{C}'(\mathbf{v}_1, \mathbf{v}_2) = 0
\]  

(127)

If one restricts oneself to the magnetic sector, the results are

\[
\Phi'_R(3g)(\mathbf{v}_1, \mathbf{v}_2) = \int \frac{d\eta k}{(2\pi)^3} \left| \Delta^i(K_R)(\mathbf{v}_1, \mathbf{P}_i, \mathbf{v}_2) \right|^2 (-)[W_R^i(K, \mathbf{v}_1) - W_A^i(K, \mathbf{v}_1)] [W_R^i(K, \mathbf{v}_2) - W_A^i(K, \mathbf{v}_2)]
\]  

(128)

The integration over the space momentum is limited to the range \( \mu_2 < k < \mu_3 \) where \( \mu_2 \) separates the scale \( gT \) from the scale \( g^2 T \ln 1/g \), and \( \mu_3 \) the scale \( g^2 T \ln 1/g \) from the scale \( gT \). \( \Delta^i \) is the \( \hat{\mathcal{C}} \)-collision-resummed transverse gluon propagator, \( \mathcal{P}_i \) the projector on the direction transverse to \( \mathbf{k} \). The field \( W(K, \mathbf{v}) \) describe the average distribution of the fast particles, in other words their collective excitations at the scale \( (K)^{-1} \). The \( \mathbf{v} \) dependance is due to collisions at the scale \( (gT)^{-1} \). \( \Phi'_R(3g) \) may be interpreted as the near-forward collision cross-section of two collective excitations of the fast particles. The \( W \) field is characterized by the equation

\[
W^i(K, \mathbf{v}) = v_i^j W^j + \hat{k}^i W^i
\]  

(129)

\[
(v.K + i\hat{\mathcal{C}}) W^i_R = v^i
\]  

(130)

whose solution may be written

\[
W^i_R(K, \mathbf{v}) = (v.K + i\hat{\mathcal{C}})^{-1} v^i
\]  

(131)

\( \Phi'_R(4g) \) is

\[
\Phi'_R(4g)(\mathbf{v}_1, \mathbf{v}_2) = \int \frac{d\eta k}{(2\pi)^3} \left| \Delta^i(K_R)(\mathbf{v}_1, \mathbf{P}_i, \mathbf{v}_2) \right|^2 \frac{1}{m_D^2 k_0} \left( \Pi^i_R(K) - \Pi^i_A(K) \right)
\]

\[
v_i^j \left[ (v.K + i\hat{\mathcal{C}})^{-1} - (v.K - i\hat{\mathcal{C}})^{-1} \right] v^i_{12}
\]  

(132)

where

\[
\Pi^i(K) = k_0 m_D^2 \int \frac{d\Omega_{\mathbf{v}_1}}{4\pi} v_i^j W^j(K, \mathbf{v})
\]  

(133)

\( \Phi'_R(4g) \) may be associated with the damping rate of the collective excitations.

Two important features are
- one may go back from \( \hat{\mathcal{C}}' \) to \( \hat{\mathcal{C}} \) by replacing \( \hat{\mathcal{C}} \) by \( \epsilon \) in the above expressions.
- the quantity

\[
\int \frac{d\Omega_{\mathbf{v}_1}}{4\pi} \frac{d\Omega_{\mathbf{v}_2}}{4\pi} \Phi'_R(3g)(\mathbf{v}_1, \mathbf{v}_2)
\]

has no infrared divergence (in contradistinction to the case of \( \hat{\mathcal{C}}' \)). It is expressed in terms of the eigenvalues of \( \hat{\mathcal{C}} \), whose scale is \( \gamma = g^2 NT/2 \ln \mu_1/\mu_2 \) where \( \mu_1 \) separates the scale \( T \) from the scale \( gT \).

Those results exhibits the central role played by the mean field \( W \) at the scale \( g^2 T \ln 1/g \).

If one now turns to prospective, the eigenvalues’ spectrum of the operator \( \mathcal{C}' \) has to be found, in order to know how the operator \( \hat{\mathcal{C}} + \hat{\mathcal{C}}' \) differs from the operator \( \hat{\mathcal{C}}' \).
This perturbative approach has required the 3-gluon and 4-gluon vertices when the scales \( T \) and \( gT \) are integrated out. Those amplitudes are similar to the hard thermal loop ones, as they may be interpreted as tree diagrams in terms of the collision-resummed \( W \) field. The 3-gluon vertex has been constructed by imposing QED-like Ward identities on the three legs. This naive approach calls for a more systematic approach to the construction of the effective \( n \)-gluon amplitude when the momenta \( T \) and \( gT \) are integrated out, to the study of their properties: symmetry, Ward identities, gauge dependence.

The quantum Retarded/Advanced formalism has been forced upon by the \( W \) field solution. It has provided short, elegant answers. It may not come out as a surprise; as one moves into softer physics, classical aspects dominate.

The above results will likely be found by the other approaches that have brought their own interpretation of the physics when the scale \( gT \) is integrated out. [8, 7, 9]

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Appendix : The analytic properties of \( \Pi \) in the complex \( k_0 \) plane

.1 An explicit expression for \( \Pi^r \) and \( \Pi^i \)

Arnold and Yaffe [11] have written down the characteristics of the operator \([k_0 - \mathbf{v} \cdot \mathbf{k} + iC(\mathbf{v}, \mathbf{v}')]^{-1}\) for the case \( k_0 = 0 \). The extension to the case \( k_0 \neq 0 \) is straightforward. \( C(\mathbf{v}, \mathbf{v}') \) commutes with the rotations in \( \mathbf{v} \) space, its eigenvectors are the spherical harmonics \( Y^m_l(\mathbf{v}) = |lm> \)

\[
\hat{C} |lm> = c_l |lm> = (\gamma - \delta_l)|lm>
\]

\[
c_l = <C(\mathbf{v}, \mathbf{v}')P_l(\mathbf{v}, \mathbf{v}')>_{vv'} \quad c_0 = 0, \quad c_l > 0 \quad \text{for} \quad l > 0
\]

\( C(\mathbf{v}, \mathbf{v}') \) is given in (4).

If one chooses \( \mathbf{k} \) as the \( z \) axis

\[
\mathbf{v} \cdot \mathbf{k} |lm> = l \quad (b^{(m)}_l |l + 1m> + b^{(m)}_{l-1} |l - 1m>)
\]

\[
b^{(m)}_l = \sqrt{(l + 1)^2 - m^2} \quad \frac{4(l + 1)^2 - 1}{4(l + 1)^2 - 1}
\]

For fixed \( m \), the matrix \([k_0 - \mathbf{v} \cdot \mathbf{k} + iC(\mathbf{v}, \mathbf{v}')]\) is a tri-diagonal matrix whose inverse is known. In particular, the element

\[
\Sigma_m = <l = m \quad m|(k_0 - \mathbf{v} \cdot \mathbf{k} + iC(\mathbf{v}, \mathbf{v}'))^{-1}|l = m \quad m>
\]

is the continued fraction

\[
\Sigma_m(k_0, k) = \frac{1}{k_0 + ic_m} \frac{1}{1 - b^{(m)^2}_m} \frac{1}{\rho^2_m} \frac{1}{1 - b^{(m)^2}_{m-1}} \frac{\rho^2_{m+1}}{\rho^2_{m+2}} \frac{1}{1 - b^{(m)^2}_{m+2}} \ldots
\]

22
where
\[ \rho_m^2 = \frac{k^2}{(k_0 + i\epsilon_m)(k_0 + i\epsilon_{m+1})} \] (139)

the transverse part of the self-energy is
\[ \Pi^t_R(k_0, k) = \frac{m_D^2}{3} k_0 \Sigma_1(k_0, k) \] (140)

since from (25)
\[ \Pi^t_R(k_0, k) = m_D^2 k_0 < v^i (v.K + i\tilde{C})^{-1} v^{ij} >_{\nu\nu'} (\delta_{ij} - \tilde{k}_i\tilde{k}_j) \frac{1}{2} \] (141)

and
\[ \frac{1}{4\pi} v_t v'_t = \frac{1}{3} [Y_1^1(v)Y_1^{1*}(v') + Y_1^{-1}(v)Y_1^{-1*}(v')] \] (142)

Similarly, from (25)
\[ \Pi^l_R = -\Pi^l_{00} = -m_D^2 [k_0 \Sigma_0(k_0, k) - 1] \] (143)

If one restricts oneself to transverse gluon exchange, the eigenvalues of the collision operator, defined in (134), have the properties [11]
\[ \delta_{2l+1} = 0 \quad , \quad 0 < \delta_{2l} \leq \delta_2 = \frac{5}{8} \gamma \quad \delta_{2l} \to \frac{2\gamma}{l} \quad \text{as} \quad l \to \infty \] (144)

so that the upper bound \( \bar{\delta} \) for \( \delta_l \) is \( \delta_2 \). In the following, the discussion is restricted to \( \Pi^t \). The extension to any \( \Sigma_m \) is immediate.

.2 The singularity-free region in the complex \( k_0 \) plane

i) The Legendre Functions
If one sets \( \gamma \neq 0 \) and \( \delta_l = 0 \), the continued fraction (138) is a function of \( \rho^2 = (k/(k_0 + i\gamma))^2 \), its value is well known from the case \( \gamma = 0 \)
\[ \Pi^t_R = \frac{m_D^2}{3} k_0 [Q_0(k_0 + i\gamma) - Q_2(k_0 + i\gamma)] \] (145)

The analytic properties of \( Q_l(z) \) in the complex \( z \) plane are seen from
\[ Q_l(z) = \int_{-1}^{1} dx \frac{P_l(x)}{z - x} \]

\( Q_l(z) \) has a logarithmic singularity at \( z = 1 \) and \( z = -1 \), and for \( z \gg 1 \), \( Q_l(z) \) has a series expansion
\[ Q_l(z) = \sum_{n \geq l} a_{ln} z^{-n} \quad \text{with} \quad a_{ln} \geq 0 \]

For \( z \gg 1 \) this series is absolutely convergent i.e. \( \sum_n a_{ln} \parallel z \parallel^{-n} \) converges.

ii) The case \( \delta_l \neq 0 \)

The continued fraction (138) can be expanded in powers of \( k^2 \) and the domain of convergence of this expansion delimited. In this expansion there appear products of
\[ \frac{k^2}{(k_0 + i(\gamma - \delta_m))(k_0 + i(\gamma - \delta_{m+1}))} \]

23
Since all numerical coefficients of the expansion in \( k^2 \) are positive, the series has as an upperbound the series of the modulus of its terms. Then one just needs a lower bound for
\[
\| k_0 + i(\gamma - \delta_m) \|^2 = (\text{Re}k_0)^2 + (\text{Im}k_0 + \gamma - \delta_m)^2
\]
If \( \tilde{\delta} \) is the upper bound of the \( \delta_m \) (\( \tilde{\delta} < 2\gamma/3 \)), the lower bounds are
\[
\begin{align*}
\text{Im}k_0 + \gamma - \tilde{\delta} > 0 & \implies \| k_0 + i(\gamma - \delta_m) \| > \| k_0 + i(\gamma - \tilde{\delta}) \| \\
\text{Im}k_0 + \gamma < 0 & \implies \| k_0 + i(\gamma - \delta_m) \| > \| k_0 + i\gamma \| \\
-\gamma < \text{Im}k_0 < -(\gamma - \tilde{\delta}) & \implies \| k_0 + i(\gamma - \delta_m) \| > |\text{Re}k_0|
\end{align*}
\]
For the region \( \text{Im}k_0 + \gamma - \tilde{\delta} > 0 \), the expansion in \( k^2 \) of the continued fraction is certainly convergent in the domain \( \| k/(k_0 + i(\gamma - \tilde{\delta})) \| < 1 \) as it is the domain of absolute convergence of the expansion in \( k^2 \) of \( Q_2((k_0 + i(\gamma - \tilde{\delta}))/k) \), i.e., in the complex \( k_0 \) plane, the expansion is convergent out of a half disk of radius \( k \) centered at \( k_0 = -i(\gamma - \tilde{\delta}) \). Similarly, for the region \( \text{Im}k_0 < -\gamma \), it is certainly convergent out of a half disk centered at \( k_0 = -i\gamma \). And for \( -\gamma < \text{Im}k_0 < -(\gamma - \tilde{\delta}) \), it is convergent for \( |\text{Re}k_0| > k \).

To summarize, the expansion of \( \Pi_R^t \) in powers of \( k^2 \) is certainly convergent in the complex \( k_0 \) plane out of a domain made of two half disks and a rectangle (see Fig. 4). In particular, for \( \text{Im}k_0 \geq 0 \) the expansion is certainly convergent for
\[
(\text{Re}k_0)^2 + (\text{Im}k_0 + \gamma - \tilde{\delta})^2 > k^2 \quad \text{with} \quad \tilde{\delta} = \sup_m \delta_m < 2\gamma/3
\] (146)

3. The Imaginary parts of \( \Pi_R^t \)

\( \Pi_R^t \) has two types of imaginary parts along the real \( k_0 \) axis.

i) all along the axis, \( \Pi_R^t \) gets an imaginary part from the eigenvalues \( c_m \) of the collision operator, i.e. from the damping of the color excitations arising from collisions at the scale \( (gT)^{-1} \).

ii) out of the domain of convergence of the expansion in \( k^2 \), \( \Pi_R^t \) gets another imaginary part similar to the \( i\pi \) term of \( Q_2(z) \). For example, for \( Q_0(z) \) in the complex \( z \) plane,
\[
\begin{align*}
\| z \| > 1 \quad Q_0(z) &= \frac{1}{2} \ln\left(\frac{z + 1}{z - 1}\right) \\
\| z \| < 1 \quad Q_0(z) &= -i\pi + \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right) \quad \text{above the cut} \quad z = -1 \text{ to } z = 1
\end{align*}
\]

From the shape of the domain of convergence of the expansion in \( k^2 \) (see Fig. 4), this imaginary part may only appear in the region \( |\text{Re}k_0| < k \). It may be interpreted as a Landau-type effect for the \( W \) field, i.e. the propagating fluctuating \( W \) field absorbs (emits) a soft gluon \( k \sim g^2T \ln 1/g \) from the plasma.

From the inequality (146) one sees that for \( k < (\gamma - \tilde{\delta}) \), the expansion in powers of \( k^2 \) is convergent all along the \( k_0 \) axis, i.e. the half disk on Fig. 4 dont intersect the real axis. As a consequence, \( \Pi^t \) has no Landau-type imaginary part for \( k < (\gamma - \tilde{\delta}) \sim \gamma/3 \)

This appendix discusses the properties of the retarded amplitude \( \Pi_R^t(k_0, k) \). A general property of a retarded propagator is that it is analytic in the upper \( k_0 \) plane. So it is for \( \Pi_R^t \) (see Fig. 4). The advanced amplitude \( \Pi_A^t(k_0, k) \) is just the mirror picture, it is analytic in the lower \( k_0 \) plane, its singularities are in the upper plane
\[
\Pi_A^t(k_0, k) = [\Pi_R^t(k_0, k)]^*
\]

Note that \( \Pi_R^t(k_0, k) \) and \( \Pi_A^t(k_0, k) \) have no common boundary in the complex \( k_0 \) plane. When one considers along the real axis \( \Pi_R^t(k_0, k) - \Pi_A^t(k_0, k) \) one is subtracting two different functions.


.4 Locations of the singularities

We study the tail of the continued fraction $\Sigma_m$ (see (138)).

As $l \to \infty$, $b_l^{(m)} \to 1/4$

i) case $\gamma \neq 0$, $\delta_l = 0$

The tail of the continued fraction $\Sigma_m$ obeys

$$X = 1 - \frac{\rho^2}{4X} \quad \text{i.e.} \quad X = \frac{1 \pm \sqrt{1 - \rho^2}}{2}$$

Hence one recovers the fact that the continued fraction has a singularity at $\rho^2 = 1 = (k/(k_0 + i\gamma))^2$

ii) case $\gamma \neq 0$, $\delta_l \neq 0$

The same argument gives

$$\rho_l^2 = 1 = \frac{k^2}{[k_0 + i(\gamma - \delta_l)][k_0 + i(\gamma - \delta_{l+1})]} \quad \text{(147)}$$

i.e.

$$k_0 = -i(\gamma - \frac{\delta_l + \delta_{l+1}}{2}) + \sqrt{k^2 - \left(\frac{\delta_l - \delta_{l+1}}{2}\right)^2} \quad \text{(148)}$$

Since $(\delta_l + \delta_{l+1})/2 < \delta$ and $(\delta_l - \delta_{l+1})^2 < \delta^2$ ($\delta$ is the upper bound of $\delta_l$), for $k > \delta/2 \sim \gamma/3$ all the singularities are inside the two regions in the complex $k_0$ plane drawn on Fig. 4

$$\sqrt{k^2 - \delta^2}/4 < |\text{Re}k_0| \leq k, \quad \delta > \text{Im}k_0 + \gamma > 0$$

As $l \to \infty$ the singularities tend towards $k_0 = \pm k - i\gamma$ since $\delta_{2l} \to \gamma/(2l)$ for $l \to \infty$

References


Figure 1: The trees of the 3-gluon one-loop vertex. The arrow on a propagator indicates the $v$ vector’s orientation, the double arrow indicates the causality flow.

Figure 2: The three possible ways of joining two vertices in the R/A formalism.

Figure 3: Contribution to the self-energy of a tree of the 4-gluon vertex. Arrows are as in Fig. 1.
Figure 4: Analytic properties of $\Pi_{k}^{l}(k_{0}, k)$. The singularities are inside the rectangles; the expansion in $k^2$ is convergent out of the domain limited by the dashed lines.