Contracting the Wigner-Kernel of a Spin to the Wigner-Kernel of a Particle

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Abstract

A general relation between the Moyal formalisms for a spin and a particle is established. Once the formalism has been set up for a spin, the phase-space description of a particle is obtained from the ‘contraction’ of the group of rotations to the group of translations. This is shown by explicitly contracting a spin Wigner-kernel to the Wigner kernel of a particle. In fact, only one out of $2^{2s}$ different possible kernels for a spin shows this behaviour.

1 Introduction

To represent quantum mechanics in terms of $c$-number valued functions has various appealing properties. It becomes possible to situate the quantum mechanical description of a system in a familiar frame, namely the phase space of its classical analog. Similarities and differences of the two descriptions can be visualized particularly well in such an approach. Further, from a structural point of view, to calculate expectation values of operators by means of ‘quasi-probabilities’ in phase space, is strongly analogous to the determination of mean values in classical statistical mechanics [1]. The basic ingredient to set up such a symbolic calculus is a one-to-one correspondence between (self-adjoint) operators $\hat{A}$ (acting on a Hilbert space $\mathcal{H}$) on the one hand, and (real) functions $W_A$ defined on the phase-space $\Gamma$ of the classical system on the other.

The quantum mechanics of spin and particle systems can be represented faithfully in terms of functions defined on the surface of a sphere with radius $s$, and on a plane, respectively. Intuitively, one expects these phase space-formulations to approach each other for increasing values of the spin quantum number since the surface of a sphere is
then approximated by a plane with increasing accuracy. Therefore, appropriate Wigner functions of a spin, say, should go over smoothly into particle Wigner-functions in the limit of large $s$. It will be shown how this transition can be transformed in a rigorous and general way. The derivation is based on the group theoretical technique of contraction. The group $SU(2)$ of quantum mechanical rotations is contracted to the Heisenberg-Weyl group $HW_1$ associated with the particle. In this procedure, rotations go over into translations. Subsequently, the operator kernel which defines the spin Wigner-formalism in a condensed manner will be shown to contract to the operator kernel for a particle in the limit of infinite $s$.

2 Wigner-kernel for a particle

Consider a particle on the real line $\mathbb{R}^1$, with position and momentum operators satisfying $[\hat{q}, \hat{p}] = i\hbar$. The Stratonovich-Weyl correspondence, associating operators with functions in phase space, can be characterized elegantly by means of a kernel $[2, 3]$,

$$\hat{\Delta}(\alpha) = 2\tilde{T}(\alpha)\hat{\Pi}\tilde{T}^\dagger(\alpha), \quad \alpha = \frac{1}{\sqrt{2}}(q + ip) \in \Gamma \equiv \mathcal{C},$$

which has an interpretation as a parity operator displaced by $\alpha$. The unitary $[4]$

$$\tilde{T}(\alpha) = \exp[\alpha a^+ - \alpha^* a],$$

effects translations in phase space $\Gamma$,

$$a \rightarrow \tilde{T}(\alpha)a\tilde{T}^\dagger(\alpha) = a - \alpha,$$

where $a^- \equiv a = (\hat{q} - i\hat{p})/\sqrt{2}$ and $a^+ = a^\dagger$ are the standard annihilation and creation operators ($\hbar = 1$). At the origin $\alpha = 0$, the kernel equals (two times) the unitary, involutive parity operator $\hat{\Pi}$,

$$\hat{\Pi}a\hat{\Pi}^\dagger = -a,$$

corresponding to a reflection at the origin of $\Gamma$. Using the number operator $\hat{N} = a^+a$ and its eigenstates,

$$\hat{N}|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots,$$

parity can be given a simple form which will be useful later,

$$\hat{\Pi} = \exp[i\pi\hat{N}] = \sum_{n=0}^{\infty}(-)^n|n\rangle\langle n|.$$

The kernel $\hat{\Delta}(\alpha)$ can be derived from the Stratonovich-Weyl postulates $[5]$ which are natural conditions on a quantum mechanical phase-space representation. The correspondence between a (self-adjoint) operator $\hat{A}$ and a (real) function is defined by

$$W_A(\alpha) = \text{Tr} \left[\hat{\Delta}(\alpha)\hat{A}\right],$$
while its inverse reads

$$\hat{A} = \int d\alpha W_A(\alpha) \hat{\Delta}(\alpha).$$

(8)

If $\hat{A}$ is the density operator of a pure state, $\hat{\rho} = |\psi\rangle \langle \psi|$, the symbol defined in (7) is the \textit{Wigner function} of the state $|\psi\rangle$,

$$W_\psi(p, q) = \frac{2}{\hbar} \int dx \psi^*(q + x)\psi(q - x) \exp[2i px/\hbar].$$

(9)

It is important to note that the kernel $\hat{\Delta}(\alpha)$ is entirely defined in terms of the operators $a^\pm$ and $\hat{N}$, forming a closed algebra under commutation if the identity is included:

$$[a, a^+] = 1, \quad [\hat{N}, a^\pm] = \pm a^\pm.$$  

(10)

This algebra generates the \textit{Heisenberg-Weyl group} $HW_1$, and the kernel $\hat{\Delta}(\alpha)$ is an element of it (apart from the factor of two).

3 \hspace{1cm} \textbf{Wigner-kernel for a spin}

For a quantum spin, the symbol associated with an operator is a continuous function defined on the \textit{sphere} $S^2$, being the phase space of the classical spin. When setting up a phase-space formalism, rotations take over the role of translations. The group $SU(2)$ is generated by the components of the spin operator $\hat{S}$. The three operators $\hat{S}^\pm = (\hat{S}^x \pm i\hat{S}^y)$ and $\hat{S}^z$, satisfy the commutation relations

$$[\hat{S}^+, \hat{S}^{-}] = 2\hat{S}^z, \quad [\hat{S}^z, \hat{S}^\pm] = \hat{S}^\pm.$$  

(11)

The standard basis

$$\mathbf{n}_z \cdot \hat{\mathbf{S}}|s, m\rangle = m|s, m\rangle, \quad m = -s, \ldots, s,$$

(12)

is given by the eigenstates of the z component $\hat{S}^z$ of the spin.

For a quantum spin, it is natural to expect that the elements of the Wigner kernel will be labeled by points of the sphere $S^2$, corresponding to unit vectors $\mathbf{n} = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$, parametrized by standard spherical coordinates. Replacing intuitively translations in (1) by rotations leads to the expression

$$\hat{\Delta}(\mathbf{n}) = \hat{U}(\mathbf{n}) \hat{\Pi}_s \hat{U}^+(\mathbf{n}),$$

(13)

where

$$\hat{U}(\mathbf{n}) = \exp[-i\vartheta \mathbf{k} \cdot \hat{\mathbf{S}}]$$

(14)

with a unit vector $\mathbf{k} = (-\sin \varphi, \cos \varphi, 0)$ in the $xy$ plane. Thus, $\hat{U}(\mathbf{n})$ represents a finite rotation which maps the operator $\hat{S}^z = \mathbf{n}_z \cdot \hat{\mathbf{S}}$ into $\mathbf{n} \cdot \hat{\mathbf{S}}$, i.e. $\mathbf{n}_z \rightarrow \mathbf{n}$. What are natural choices for the operator $\hat{\Pi}_s$?
Two possibilities come to one’s mind. First, try to transfer the concept of reflection about some point in phase space. Introduce canonical coordinates \((q, p) = (\varphi, \cos \vartheta)\) on the sphere. Then, ‘parity’ would correspond to the map \((\varphi, \cos \vartheta) \rightarrow (-\varphi, -\cos \vartheta)\), or \((\varphi, \vartheta) \rightarrow (2\pi - \varphi, \pi - \vartheta)\). This is just a rotation by \(\pi\) about the \(x\) axis. Since all points of the sphere are equivalent, one could also chose a rotation by \(\pi\) about the \(z\) axis as candidate for parity. Second, \(\hat{\Pi}_s\) might be considered to generate reflections about the center of the sphere, \(n \rightarrow -n\), that is, \((\varphi, \vartheta) \rightarrow (\varphi + \pi, \pi - \vartheta)\). It can be shown that both possibilities do not give rise to a symbolic calculus on the sphere [6], violating bijectivity between operators and phase-space functions, for example.

Nevertheless, acceptable operator kernels \(\hat{\Delta}_s(n)\) do exist as shown by Stratonovich [5], Várlilly and Gracia-Bondía [7], and by Amiet and Cibils [8]. For example, the condition that the kernel should satisfy appropriate Stratonovich-Weyl postulates implies [7] that

\[
\hat{\Delta}_s(n) = \sum_{m, m' = -s}^{s} Z_{mm'}^s(n) |s, m\rangle \langle s, m'|. \tag{15}
\]

The coefficients,

\[
Z_{mm'}^s(n) = \frac{\sqrt{4\pi}}{2s + 1} \sum_{l=0}^{2s} \varepsilon_l \sqrt{2l + 1} \left\langle \begin{array}{ccc} s & l & s \\ m & m' - m & m' \end{array} \right\rangle Y_{lm'm-m}(n), \tag{16}
\]

where \(\varepsilon_0 = 1\) and \(\varepsilon_l = \pm 1, l = 1, \ldots, 2s\), are linear combinations of Clebsch-Gordan coefficients multiplied by spherical harmonics \(Y_{lm}(n), l = 0, 1, \ldots, 2s, m = -l, \ldots, l\). Note that there is no unique kernel but, due to the factors \(\varepsilon_l\), one can define \(2^{2s}\) different Stratonovich-Weyl correspondence rules.

Unfortunately, the expression (15) does not admit a simple interpretation of the operator in analogy to (1). It follows from an independent derivation [9] of \(\hat{\Delta}(n)\) that (15) can be written in the form (13) where

\[
\hat{\Pi}_s = \hat{\Delta}_s(n_z) = \sum_{m = -s}^{s} \Delta_{s}(m)|s, m\rangle \langle s, m|, \tag{17}
\]

with coefficients

\[
\Delta_{s}(m) = \sum_{l=0}^{2s} \varepsilon_l \frac{2l + 1}{2s + 1} \left\langle \begin{array}{ccc} s & l & s \\ m & 0 & m \end{array} \right\rangle. \tag{18}
\]

Still, the operator \(\hat{\Pi}_s\) does not have an obvious interpretation but a new strategy to justify its form emerges. Consider a plane tangent to the sphere at its north pole. For increasing radius, the sphere is approximated locally better and better by the plane. Therefore, one might expect that for \(s \rightarrow \infty\) objects defined on the sphere turn into objects defined on the plane. It has been conjectured in [9] that in this limit the Wigner kernel of a spin goes over into the kernel for a particle. It is the purpose of this paper to show that

\[
\lim_{s \rightarrow \infty} \hat{U}(n) \hat{\Delta}(n_z) \hat{U}^\dagger(n) = \hat{\Delta}(\alpha), \tag{19}
\]
is indeed true for the kernel $\tilde{\Delta}_z(n_z)$ with parameters $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{2s} = 1$, denoted by $\tilde{\Delta}(n_z)$ for short. Thus, while the rotations $\tilde{U}(n)$ should go over into translations, the operator $\tilde{\Delta}(n_z)$ corresponds, in one way or another, to parity for a spin. A convenient framework to prove (19) is the *contraction* of groups [10] as shown in the next section.

## 4 Contracting $SU(2)$ to $HW_1$

Introduce three operators $\hat{A}^\pm$ and $\hat{A}^z$ defined as linear combinations of the generators of the algebra $su(2)$ in polar form,

$$
\hat{A}^\pm = c S^\pm, \quad \hat{A}^z = -S^z + \frac{1}{2c^2}, \quad (20)
$$

plus the identity $1_s$. This transformation is invertible for each value of the parameter $c > 0$. The non-zero commutators of the new generators are given by

$$
[\hat{A}^-, \hat{A}^+] = 1_s - 2c^2 \hat{A}^z, \quad [\hat{A}^\pm, \hat{A}^\mp] = \hat{A}^\pm. \quad (21)
$$

These relations have a well defined limit if $c \to 0$, notwithstanding that the transformation (20) is not invertible for $c = 0$. In fact, they reproduce the commutation relations (10) of the Heisenberg-Weyl algebra after identifying

$$
\lim_{c \to 0} \hat{A}^\pm = a^\pm, \quad \lim_{c \to 0} \hat{A}^z = N, \quad \lim_{c \to 0} 1_s = 1. \quad (22)
$$

How do rotations behave in this limit? Any finite rotation $\tilde{U}(n) \in SU(2)$ in (14) can be written in the form

$$
\tilde{U}(n) = \exp \left[ \xi_- \tilde{S}^- - \xi_+ \tilde{S}^+ \right], \quad \xi_- = \frac{\vartheta}{2} e^{i\varphi}, \quad \xi_+ = \xi_-^*, \quad (23)
$$

or, expressed in terms of the operators (20),

$$
\tilde{U}(n) = \exp \left[ c(\xi_- \hat{A}^+ - \xi_+ \hat{A}^-) \right]. \quad (24)
$$

Consequently, if the coefficients $\xi_\pm$ shrink with the parameter $c$ according to

$$
\lim_{c \to 0} \frac{\xi_-}{c} = \lim_{c \to 0} \frac{\vartheta e^{i\varphi}}{2c} = \alpha, \quad \lim_{c \to 0} \frac{\xi_+}{c} = \lim_{c \to 0} \frac{\vartheta e^{-i\varphi}}{2c} = \alpha^*, \quad (25)
$$

a rotation $\tilde{U}(n)$ tends to a well-defined element of the Heisenberg-Weyl group, Eq. (2):

$$
\lim_{c \to 0} \tilde{U}(n) = T(\alpha) \cdot (26)
$$

5
For consistency, the limit $c \to 0$ must correctly reproduce the eigenvalues of the operator $\hat{N}$, given by the non-negative integers. Let us look at the fate of the eigenvalue equation (12) for $m = s$, which is expected to give $\hat{N}|n\rangle = 0$. One has

$$\lim_{c \to 0} (\hat{A}^z|s, s\rangle) \lim_{c \to 0} \left[ (-\hat{S}^z + \frac{1}{2c^2})|s, s\rangle \right] = \lim_{c \to 0} \left[ (s + \frac{1}{2c^2}) \lim_{c \to 0} |s, s\rangle \right] = 0$$

(27)

implying $2c^2s = 1$ for $\lim_{c \to 0} |s, s\rangle = |n = 0\rangle$. Consequently, the radius of the sphere, $s$, increases with decreasing values of $c$. The state $|s, s\rangle$ turns indeed into the ground state associated with the operator $\hat{N}$ since one has in general

$$\lim_{c \to 0} |s, m\rangle = \lim_{c \to 0} |s, s - n\rangle = |n\rangle, \quad n = s - m \in \mathbb{N}_0,$$

(28)

as follows from

$$\hat{N}|n\rangle = \lim_{c \to 0} \left[ (-\hat{S}^z + \frac{1}{2c^2})|s, s - n\rangle \right] = \lim_{c \to 0} \left( (s - m) + \left( \frac{1}{2c^2} - s \right) \right)|n\rangle = n|n\rangle.$$

(29)

Now it is obvious why one needs to associate the creation operator $\hat{S}^+$ with the annihilation operator $a$ (cf. (20)): the eigenstates with maximal $s$ are linked to the oscillator ground state with minimal $n = 0$. In [10], a different convention has been used. Nevertheless, it remains true that not only spin eigenstates are mapped into number eigenstates but many other expressions related to the group $U(2)$ turn into an equivalent expression for the group $HW_1$.

This is good news for the present purpose to establish a relation between the Moyal formalism of a particle and a spin. Consider the limit of the kernel (13) under contraction using (26),

$$\lim_{c \to 0} \widehat{\Delta}(n) = \hat{T}(\alpha) \left( \lim_{c \to 0} \widehat{\Pi}_s \right) \hat{T}^\dagger(\alpha).$$

(30)

The middle term can be written as

$$\lim_{c \to 0} \widehat{\Pi}_s = \lim_{c \to 0} \sum_{m = -s}^{s} \Delta_\varepsilon(m)|s, m\rangle \langle s, m| = \sum_{n = 0}^{\infty} \left( \lim_{c \to 0} \Delta_\varepsilon(s - n) \right) |n\rangle\langle n|.$$

(31)

Upon comparison with (6), the Wigner kernel of a spin is seen to turn into the Wigner kernel of the particle if

$$\lim_{s \to \infty} \sum_{l = 0}^{2s} \varepsilon_l \left( \frac{2l + 1}{2s + 1} \right)^{1/2} \left| \begin{array}{cc} s & s \\ s - n & n - s \end{array} \right|_l = 2$$

(32)

holds for all non-negative integers $n$. In the next section, this will be shown to be true for the choice $\varepsilon_l = +1$, $l = 1, \ldots 2s$. 

6
5 Summing the series

Evaluating the sum (32) in the limit $s \to \infty$ proceeds in two steps. First, the asymptotic form of the terms

$$\Delta_{l,n}^s = \left( \frac{2l + 1}{2s + 1} \right)^{1/2} \left\langle \begin{array}{ccc}s & s & l \\ s - n & n - s & 0 \end{array} \right\rangle$$

(33)

to be summed is determined with the help of a recurrence formula for Clebsch-Gordan coefficients. Then, the sums are transformed into integrals which can be evaluated. All approximations drop terms of the order $1/s$ at least, hence the result is exact in the limit of infinite $s$.

Clebsch-Gordan coefficients satisfy the following recursion relation [11]:

$$[l(l + 1) - 2s(s + 1) + 2m^2] \left\langle \begin{array}{ccc}s & s & l \\ m & -m & 0 \end{array} \right\rangle$$

$$= [s(s + 1) - m(m + 1)] \left\langle \begin{array}{ccc}s & s & l \\ m + 1 & -(m + 1) & 0 \end{array} \right\rangle$$

$$+ [s(s + 1) - m(m - 1)] \left\langle \begin{array}{ccc}s & s & l \\ m - 1 & -(m - 1) & 0 \end{array} \right\rangle,$$

(34)

implying that

$$(n + 1) \left( 1 - \frac{n + 1}{2s + 1} \right) \Delta_{l,n+1}^s + \left( 2n + 1 - \frac{2n^2 + 2n + 1}{2s + 1} \right) \Delta_{l,n}^s +$$

$$+ n \left( 1 - \frac{n}{2s + 1} \right) \Delta_{l,n-1}^s = \frac{l(l + 1)}{2s + 1} \Delta_{l,n}^s.$$

(35)

For any finite $n$ the terms subtracted on the left-hand-side become less and less important if $s \to \infty$. Assume now that one can write the terms with large values of $n$ in the form

$$\Delta_{l,n}^s(x_l) = \Lambda_n(x_l) \Delta_{l,0}^s, \quad \Lambda_0(x_l) = 1, \quad x_l = \frac{l(l + 1)}{2s + 1}.$$  (36)

The polynomial $\Lambda_n(x_l)$ of order $n$ in $x_l$ satisfies a three-term recursion relation,

$$(n + 1)\Lambda_{n+1}(x_l) + (2n + 1)\Lambda_n(x_l) + n\Lambda_{n-1}(x_l) = x_l \Lambda_n(x_l),$$

(37)

where terms of order $1/s$ have been dropped in (37). Its solutions [12] are proportional to the Laguerre polynomials, and the 'normalization' condition $\Lambda_0(x_l) = 1$ implies that

$$\Lambda_n(x_l) = (-)^n L_n(x_l) = (-)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-x_l)^k}{k!}, \quad n = 0, 1, 2, \ldots$$

(38)
The term $\Delta_{l,0}^s$ in (36) can be determined in the following way. If $s$ is large, one writes for each finite $k$

$$
\left(1 - \frac{k}{2s + 1}\right)^{2s+1} \sim \exp[-k],
$$

(39)

which leads to the approximation

$$
\Delta_{l,0}^s = \left(\frac{2l+1}{2s+1}\right)^{1/2} \left(\begin{array}{cc}
s & s \\
s & -s \\
0 & 0
\end{array}\right) = \frac{2l+1}{2s+1} \left(\begin{array}{c}
(2s)!\\
(2s-l)!(2s+l+1)!
\end{array}\right)^{1/2}
$$

$$
= \frac{2l+1}{2s+1} \left(\prod_{k=0}^{l} (1 - k/(2s+1))\right)^{1/2} \sim \frac{2l+1}{2s+1} \exp\left[-\frac{1}{2} \frac{l(l+1)}{2s+1}\right].
$$

(40)

Collecting the results, one has

$$
\lim_{s \to \infty} \sum_{l=0}^{2s} \Delta_{l,n}^s \sim (-)^n \lim_{s \to \infty} \sum_{l=0}^{2s} \Delta x_{l} L_n(x_{l}) e^{-x_{l}/2},
$$

(41)

where $\Delta x_{l} = (x_{l+1} - x_{l}) = (2l+1)/(2s+1) + O(1/s)$. Transforming now the Riemann sum into an integral, one obtains the final result

$$
\lim_{s \to \infty} \sum_{l=0}^{2s} \Delta_{l,n}^s = (-)^n \int_0^\infty dx L_n(x) e^{-x/2} = 2,
$$

(42)

using the formula

$$
\int_0^\infty dx L_n(x) e^{-x/t} = t(1-t)^n,
$$

(43)

for $t = 2$. This identity is proven easily by means of the expansion in (38).

6 Discussion

Starting from a new form of the kernel defining the familiar Wigner formalism for a spin, its limit for infinite values of $s$ has been shown to be the Wigner kernel of a particle. As the kernel defines entirely a phase-space representation, this result guarantees that the Moyal formalism for a particle is reproduced automatically and in toto, if the limit $s \to \infty$ of the spin Moyal formalism is taken.

In fact, slightly more has been shown. The result removes an ambiguity of the Moyal formalism for a spin: the Stratonovich-Weyl postulates are compatible with a discrete family of $2^{2s}$ distinct kernels $\Delta_{l}(n)$. However, only one of these kernels turns into the particle kernel. This kernel had been singled out before for other reasons [8]. In summary, the group theoretical contraction shows that the phase-space representations à la Wigner for spin and particle systems are structurally equivalent.
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References