44

Prépublication de
l' Institut de
Recherche
Mathématique
Avancée

Strasbourg

TILTING MODULES AND THEIR APPLICATIONS

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1998/34

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Mathematics subject classification : 17B10.

Mots clés : tilting modules, modular Verlinde formula.

Tilting Modules and their Applications. *

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0. Introduction: The aim of the paper is the theory of tilting modules for algebraic groups over an algebraically closed field K of characteristic p . In order to be convenient to the reader, the paper is basically self-content, what explains its lenght. Also we try to make it accessible to a "characteristic zero" reader. In the introduction, we will describe the content of the paper.

The first three sections contains the basic material on algebraic groups. We recall the basic theory of restricted Lie algebras (due to Jacobson), the definition of Chevalley groups and the basic theorems of representation theory: classification of simple modules [St], simplicity of the Steinberg's module [St], Kempf's vanishing theorem [Ke]. Because of its technical difficulty, it has not been possible to include a proof of the linkage principle [A4]. Otherwise we included most proofs. A full account of the subject can be found in Jantzen's book [J3].

In the setting of finite dimensional algebras, tilting modules have been introduced by various authors with different meanings. The notion of tilting modules for algebraic groups has been found by Donkin [D2], following Ringel's work [Ri] on quasi-hereditary algebras. It is based on the notion of good filtrations. Therefore we give an elementary treatment of these filtrations in Section 4. Then the reader can consult the appendix for the usual cohomological approach of good filtrations [FP].

A crucial fact about tilting modules is their stability under tensor product: this follows from a similar result for good filtrations. The proof of this result [M2] uses a refinement of the notion of Frobenius splittings, invented by Metha-Ramana-Ramanathan [MR][RR], namely the canonical Frobenius splittings [M2]. Our treatment of Frobenius splittings, based on a formula of [M1], seems more intuitive than the original presentation [MR]. In Section 6, we give a full proof of the stability by tensor product of good filtration, in

* Research supported by UA1 du CNRS at Strasbourg.

a clearer way than the original paper. Section 7 is devoted to the definition of tilting modules, following the work of Ringel [Ri] and Donkin [D2].

Starting with tilting modules, we can form a tensor category which is very similar to a category considered by Moore and Seiberg [MS] in the context of Conformal Field Theory. This similarity, explained in Section 9 following [GM1], has suggested to the authors of [GM1] a formula for certain tensor product multiplicities of tilting modules. For this reason, we call it the *modular Verlinde's formula*. In Section 10, we give a proof of the formula which is simpler than the original proof.

The applications involves the *stable representations theory*, i.e. the representation theory of S_n and $GL(n)$ with n large. This subject does not intersect Lusztig conjecture [Lu1] for $n > p$ or the conjectures of Broué and James for $n \geq p^2$. To apply the modular Verlinde's formula, we use various dual pairs in the sense of Howe [Ho]. The abstract theory is explained in Section 11. The section 12, concerning $GL(V)$ -theory, follows closely [PM]. The section 14, concerning the symmetric group, follows [M3] but it also contains some unpublished results connected with the works of Erdmann [E] and Kleshchev [Kl1].

Acknowledgements: We thank very much T. Kashiwara, T. Kobayashi, T. Matsuki, K. Nishiyama and T. Oshima for the remarkable organization of the Okayama-Kyoto conference on Representation Theory. Since some of these topics are more than 10 years old, I would like to thank many mathematicians for discussions. Among them we should mention G. Georgiev, G. Papadopoulos, J. Jensen, H. Andersen, S. Donkin, R. Dipper, K. Erdmann, J. Jantzen, M. Kaneda, A. Kleshchev, R. Rouquier and W. Soergel.

1. Lie algebras and algebraic groups in finite characteristics. Let K be an algebraically closed field of characteristic p . In this section, we recall a few facts about Lie algebras and algebraic groups over K .

The Jacobson polynomial: A Lie polynomial $H(x, y)$ in the two variables x, y is an element of the free Lie algebra generated by x and y . Ordinary polynomials can be evaluated in any commutative algebra. Similarly the Lie polynomial H can be evaluated in any Lie algebra: if $\mathfrak{g}$ is a Lie algebra and $a, b \in \mathfrak{g}$, $H(a, b)$ is a well-defined element of $\mathfrak{g}$.

Theorem 1.1: (Jacobson's identity) *There is a two variables Lie polynomial J_p such that: $(x + y)^p = x^p + y^p + J_p(x, y)$, for any elements x, y in an associative K -algebra A .*

Proof: Let $\mathcal{A}$ be the free associative $\mathbf{Z}$ -algebra in two variables x, y , and let $\mathfrak{g} \subset \mathcal{A}$ be its Lie subalgebra generated by x and y . The Campbell-Hausdorff formula:

$$\exp x \exp y = \exp \sum_{k \geq 1} \frac{1}{k} H_k(x, y)$$

is an identity which holds in a suitable completion of $\mathbf{Q} \otimes \mathcal{A}$, where each $H_k \in \mathfrak{g}$ is a homogenous Lie polynomial of degree k with integral coefficients. Multiplying by $p!$ its homogenous component of degree p , one gets:

$$x^p + y^p = (x + y)^p + (p - 1)! H_p(x, y) + p\tau,$$

where $\tau \in \mathcal{A}$. Reducing modulo p , one gets Jacobson's identity. Q.E.D.

Remark: For example, we have $J_2(x, y) = [x, y]$, $J_3(x, y) = [x, [x, y]] + [y, [y, x]]$. A convenient reference for the original proof (based on identities of binomial coefficients) is [J]. Indeed we have $J_p = H_p$ modulo p , thanks to Stirling's identity $(p - 1)! = -1$ modulo p .

Restricted Lie algebras. Another special feature of characteristic p is the following: if δ is a derivation of a K -algebra, then δ^p is again a derivation. By definition, a Lie algebra $\mathfrak{g}$

is called *restrictable* if $\text{ad}(x)^p$ is an inner for all $x \in \mathfrak{g}$. If $\mathfrak{g}$ is restrictable, a p -structure is a map, $x \in \mathfrak{g} \mapsto x^{[p]} \in \mathfrak{g}$ such that:

- (i) $\text{ad}(x^{[p]}) = \text{ad}(x)^p$,
- (ii) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$,
- (iii) $x^{[p]} + y^{[p]} = (x + y)^{[p]} + J_p(x, y)$,

for all $x, y \in \mathfrak{g}$, $\lambda \in K$. A restrictable Lie algebra endowed with a p -structure is called *restricted*. Let $\mathfrak{g}$ be a restricted Lie algebra and let $U(\mathfrak{g})$ be its enveloping algebra. For any $x \in \mathfrak{g}$, $x^p - x^{[p]}$ is a central element of $U(\mathfrak{g})$. By definition, the *restricted enveloping algebra* $u(\mathfrak{g})$ is the quotient of $U(\mathfrak{g})$ by the ideal generated by all the elements $x^p - x^{[p]}$. For a vector space V , let $\bar{S}V$ be the quotient of SV by the ideal generated by all x^p , $x \in V$. Then $\bar{S}V$ is the restricted enveloping algebra of the abelian Lie algebra V endowed with a trivial p -structure. If $x_1, \dots, x_n$ is a basis of V , then the elements $x_1^{m_1} \dots x_n^{m_n}$, where $m_k < p$ for any k , form a basis of $\bar{S}V$. In particular $\bar{S}V$ has dimension p^n . Let $\mathfrak{g}$ be a restricted Lie algebra. As for ordinary enveloping algebras, there is a natural filtration:

$$K = u_0(\mathfrak{g}) \subset u_1(\mathfrak{g}) \subset u_2(\mathfrak{g}) \subset \dots$$

of the algebra $u(\mathfrak{g})$, where $u_n(\mathfrak{g})$ the linear subspace generated by products of less than n elements of $\mathfrak{g}$. The restricted version of Poincaré-Birkoff-Witt Theorem is the following:

Theorem 1.2: (Jacobson) *The graded algebra associated with the filtered algebra $u(\mathfrak{g})$ is isomorphic to $\bar{S}\mathfrak{g}$. In particular, its dimension is $p^{\dim \mathfrak{g}}$.*

The proof, which is standard, is omitted. Although there are no Haar measures, a certain bilinear form on $u(\mathfrak{g})$ is quite similar.

Corollary 1.3: *Let $\mathfrak{g}$ be a restricted Lie algebra of dimension n .*

- (i) *There is an isomorphism: $L : u(\mathfrak{g})/u_k(\mathfrak{g}) \simeq (\bigwedge^n \mathfrak{g})^{\otimes p-1}$, where $k = n(p-1) - 1$.*
- (ii) *The bilinear form of $u(\mathfrak{g})$: $x, y \mapsto L(xy)$ is non-degenerated.*

Proof: For a vector space V of dimension n , $\bar{S}^{(p-1)n}V$ has dimension 1 and it is isomorphic with $(\bigwedge^n V)^{\otimes p-1}$ as a $GL(V)$ -module. Moreover $\bar{S}^m V = 0$ for $m > (p-1)n$. Therefore, $u(\mathfrak{g})/u_k(\mathfrak{g})$ is isomorphic to $(\bigwedge^n \mathfrak{g})^{\otimes p-1}$ by Theorem 1.2. Moreover, as the corresponding bilinear form on $\bar{S}V$ is non-degenerated, the bilinear form $x, y \mapsto L(xy)$ is non-degenerated. Q.E.D.

Simple algebraic groups. A detailed account of the theory of algebraic groups can be found in [J3]. By *group scheme*, we mean a finitely generated affine K -scheme G endowed with a group structure. This definition is equivalent to require that $K[G]$ is a commutative Hopf algebra which is finitely generated as a commutative algebra. An *algebraic group* G is a reduced group scheme. Indeed, in characteristic zero any group scheme is reduced. In contrast, in finite characteristics, there are non-reduced schemes with a group structure: the most simple example is $\text{Spec } K[t]/(t^p)$. Therefore the hypothesis that G is reduced is part of the definition of an algebraic group. As it is defined, an algebraic group is automatically smooth.

Let G be a connected algebraic group. Its *hyperalgebra* $\mathcal{H}_G$ is the algebra of left G -invariant differential operators on G (or equivalently the convolution algebra of distributions supported at 1, i.e. linear maps $K[G] \rightarrow K$ whose kernel contains a power of the maximal ideal defining 1). The Lie algebra $\mathfrak{g}$ of G consists of left G -invariant derivations of $K[G]$. It is a restricted Lie algebra, whose p -structure is $x \mapsto x^p$. It follows easily that the

subalgebra of $\mathcal{H}_G$ generated by $\mathfrak{g}$ is isomorphic to $u(\mathfrak{g})$. In contrast with the characteristic zero case, $\mathcal{H}_G$ is not generated by $\mathfrak{g}$, because $\mathcal{H}_G$ is infinite dimensional whenever $\dim G > 0$. Things are even worse: if $\dim G > 0$, the algebra $\mathcal{H}_G$ is not finitely generated.

The most basic example is the additive group $\mathbf{A}_1$. We have $K[\mathbf{A}_1] = K[t]$. For all $n \geq 0$, define the differential operator $e^{(n)} : K[t] \rightarrow K[t]$ by $e^{(n)} t^m = \binom{m}{n} t^{m-n}$. As $e^{(n)}$ is the reduction modulo p of $(1/n!) d^n/dt^n$, the elements $e^{(n)}$ are called the *reduced powers* of $e = d/dt$. The set $\{e^{(n)}\}$ is a basis of the hyperalgebra $\mathcal{H}_{\mathbf{A}_1}$; it follows that $\mathcal{H}_{\mathbf{A}_1}$ is the commutative algebra generated by the infinite set of variables $(x_n)_{n \geq 1}$ and defined by the relations $x_n^p = 0$ for all $n \geq 1$. To find such a presentation of $\mathcal{H}_{\mathbf{A}_1}$, set $x_n = e^{(p^n)}$.

We would like to define the iterated Frobenius kernels, following [J2]. Let G be an algebraic groups. It turns out that the subalgebra $(K[G])^p$ is an Hopf subalgebra of $K[G]$ and therefore it defines an algebraic group G_p . The corresponding map $Fr : G \rightarrow G_p$ is called the *absolute Frobenius map*. The kernel of this morphism is called the *Frobenius kernel*. It is a group scheme whose underlying Hopf algebra is $u(\mathfrak{g})^*$. At the set theoretical level, Fr is bijective and the Frobenius kernel is an infinitesimal group scheme, i.e. its unique point is 1. In general G_p is not isomorphic to G as an algebraic group: this occurs exactly when G is defined over $\mathbf{F}_p$. More generally, one gets the iterated Frobenius kernels by using the subalgebras $(K[G])^q \subset K[G]$, where q is a power of p .

Recall that in characteristic zero, the connected simple algebraic groups are not simple, but they are simple up to a finite center. In characteristic p , the notion of a simple algebraic group is more involved. Any algebraic group contains a lot of invariant group subschemes, like the iterated Frobenius kernels. Thus it is more difficult to give an axiomatic to define the simple algebraic groups: roughly speaking a simple algebraic group is a connected algebraic group whose all smooth invariant subgroups are contained in a finite center. Indeed, Chevalley and Steinberg [St] defined the connected simply connected simple algebraic groups as reductions modulo p of some $\mathbf{Z}$ -forms of the corresponding objects in characteristic zero. Hence they are classified by the indecomposable Dynkin diagrams.

The simplest way to describe a connected simply connected simple group G is to start by defining its Hopf algebra $K[G]$. Let I be an indecomposable Dynkin diagram and let $(a_{i,j})_{i,j \in I}$ be its Cartan matrix. Let $\mathfrak{g}_{\mathbf{Q}}$ be the simple split Lie $\mathbf{Q}$ -algebra generated by the symbols $(e_i, f_i, h_i)_{i \in I}$ and defined by the standard relations:

- (i) $[h_i, h_j] = 0$
- (ii) $[e_i, f_i] = h_i$,
- (iii) $[e_i, f_j] = 0$
- (iv) $[h_i, e_j] = a_{i,j} e_j$,
- (v) $[h_i, f_j] = -a_{i,j} f_j$,
- (vi) $ad(e_i)^{1-a_{i,j}}(e_j) = 0$,
- (vii) $ad(f_i)^{1-a_{i,j}}(f_j) = 0$,

for any $i, j \in I, i \neq j$. Set $\mathcal{H}^{\mathbf{Z}}$ be the $\mathbf{Z}$ -subalgebra of $U(\mathfrak{g})$ generated by all divided powers $e_i^{(n)}$ and $f_i^{(n)}$ (where the symbol $x^{(n)}$ means $(1/n!) x^n$). By definition, $\mathbf{Z}[G_{\mathbf{Z}}]$ is the Hopf $\mathbf{Z}$ -algebra dual of $\mathcal{H}^{\mathbf{Z}}$, i.e. it consists of linear forms $L : \mathcal{H}^{\mathbf{Z}} \rightarrow \mathbf{Z}$ whose kernel contains a right ideal I such that $\mathcal{H}^{\mathbf{Z}}/I$ is a finitely generated $\mathbf{Z}$ -module. Equivalently: there is an integer N such that $L(e_i^{(n)} u) = L(f_i^{(n)} u) = 0$ for any $u \in \mathcal{H}^{\mathbf{Z}}, i \in I$ and $n > N$.

For any field K , set $G_K = \text{Spec } K \otimes \mathbf{Z}[G_{\mathbf{Z}}]$. By definition, $K[G_K]$ is a commutative

Hopf algebra: however it requires some work to prove that $K[G_K]$ is a finitely generated domain and that $\mathcal{H}_{G_K} = K \otimes \mathcal{H}^Z$. By definition, G_K is the connected simply connected simple group over K with Dynkin diagram I .

When there is no ambiguity on the ground field K , we simply set $G = G_K$. As usual, the group G contains a series of remarkable subgroups (the Borel subgroups, their radicals, and so on), but their definition requires some care because of the involved definition of G . By definition the group U (respectively U^-) is the group whose associated hyperalgebra $\mathcal{H}_U$ is the subalgebra generated by all $e_i^{(n)}$ (respectively by all $f_i^{(n)}$). The Borel subgroups B, B^- are the normalizers of U, U^- . As usual U is the radical of B . The Cartan subgroup is $H = B \cap B^-$. Denote by $\mathfrak{g}, \mathfrak{u}, \mathfrak{u}^-$ and $\mathfrak{h}$ be the Lie algebras of the algebraic groups G, U, U^- and H . As usual, there is a Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{u}^-$. Similarly, there is a PBW decomposition $\mathcal{H}_G$:

Lemma 1.4: (PBW Decomposition) *The multiplication induces a linear isomorphism:*

$$\mathcal{H}_G \simeq \mathcal{H}_U \otimes \mathcal{H}_H \otimes \mathcal{H}_{U^-}.$$

The proof of this classical lemma can be found in Bourbaki [Bo]. The definition of the set of roots Δ , the set of positive roots Δ^+ , the Weyl group W , the root lattice Q , the monoid $Q^+ = \mathbb{N}\Delta^+$, the weight lattice P and the set of dominant weights P^+ are easy and all these discrete objects are independent of the ground field K . For example P is the group of characters of H . It should be noted that $P \simeq \mathbb{Z}^l$, where $l = \dim H$, therefore the natural map $P \rightarrow \mathfrak{h}^*$ is not injective. Therefore the roots should be understood as elements of P , not as elements of $\mathfrak{h}^*$. Similarly the coroots are elements of $\text{Hom}(P, \mathbb{Z})$. Simple roots, coroots are denoted by α_i, h_i , for any $i \in I$.

From now on, K will denote an algebraically closed field of characteristic $p > 0$ and G will be a simple algebraic group as defined before.

2. Representations of simple algebraic groups. We will recall some basic facts, due to Steinberg [St], about representations of algebraic groups. Let L be any algebraic group. A finite dimensional L -module M is a finite dimensional vector space M together with an algebraic group morphism $L \rightarrow GL(M)$. Another equivalent definition is given by a structural map $\Delta_M : M \rightarrow M \otimes K[G]$, which satisfies an obvious cocycle condition (i.e. the two induced maps $M \rightarrow M \otimes K[L] \otimes K[L]$ coincide). It turns out that the second definition allows to define infinite dimensional L -modules: a typical example of infinite dimensional L -module is $K[L]$, when $\dim L > 0$. We denote by $\text{Mod}(L)$ the category of L -modules. Unless stated otherwise, we will assume that any L -module is finite dimensional.

Let G be a simple algebraic group. For any $\mu \in P$, we will denote by $K(\mu)$ the corresponding one dimensional H -module. Sometimes $K(\mu)$ will be considered as a B -module or as a B^- -module with a trivial action of U or U^- . For any H -module M , we will denote by M_μ its μ -weight space. A weight λ of M is called an *highest weight* of M if $M_\mu = 0$ for any $\mu > \lambda$ (i.e. for any $\mu \neq \lambda$ with $\mu - \lambda \in Q^+$). For any B -module N , denote by $\mathcal{L}(N)$ the sheaf of sections of the bundle $G \times_B N \rightarrow G/B$ (for simplicity, we will set $\mathcal{L}(\mu) = \mathcal{L}(K(\mu))$). We set $DN = \Gamma(G/B, \mathcal{L}(N))$. Then $N \mapsto DN$ is a left exact functor $D : \text{Mod}(B) \rightarrow \text{Mod}(G)$. The functor D can be described as the induction functor Ind_B^G , i.e. it satisfies the following universal property. First, the evaluation of a global section on

G/B at the point B/B provides a natural transformation $DN \rightarrow N$ for any $N \in \text{Mod}(B)$. Next, for any $M \in \text{Mod}(G)$ and any $N \in \text{Mod}(B)$, any B -morphism $M \rightarrow N$ factors through a unique G -morphism $M \rightarrow DN$.

For $\lambda \in P^+$, we set $\nabla(\lambda) = H^0(G/B, \mathcal{L}(w_0\lambda))$, where w_0 is the highest element of W . Set $\Delta(\lambda) = \nabla(-w_0\lambda)^*$. With these definitions, λ is the highest weight of the modules $\nabla(\lambda)$ and $\Delta(\lambda)$ and their character is given by Weyl character formula, as we will see later: therefore, the G -modules $\Delta(\lambda)$, $\nabla(\lambda)$ are called the *Weyl module*, the *dual of the Weyl module*¹ with highest weight λ .

Lemma 2.1: *Let $\lambda \in P^+$.*

- (i) *As H -modules, $H^0(U, \nabla(\lambda)) \simeq K(\lambda)$.*
- (ii) *The weights of $\nabla(\lambda)$ are in the convex hull of $W\lambda$.*
- (iii) *λ is the unique highest weight of $\nabla(\lambda)$, and $\dim \nabla(\lambda)_\lambda = 1$.*
- (iv) *$\nabla(\lambda)$ contains a unique simple submodule $L(\lambda)$, and $\dim L(\lambda)_\lambda = 1$.*

Proof: By semi-continuity principle, $\nabla(\lambda)$ is "at least as big" as its characteristic zero counterpart. Therefore $H^0(U^-, \nabla(\lambda))$ contains a non-zero vector of weight $w_0\lambda$. As in characteristic zero, Bruhat Decomposition $G = \coprod_{w \in W} BwB$ holds. The U^- -orbit Ω of B/B is a dense open subset of G/B , and therefore $H^0(U^-, \nabla(\lambda))$ is one-dimensional. It follows that all weights of $\nabla(\lambda)$ are in $w_0\lambda + Q^+$. Then the first three assertions follows by W -invariance of $\nabla(\lambda)$. Let $L(\lambda)$ be the G -module generated by a non-zero highest weight vector v_λ of $\nabla(\lambda)$. By Lie's theorem, any non-zero B -submodule of $\nabla(\lambda)$ contains v_λ . Therefore any non-zero G -submodule contains $L(\lambda)$, what amounts to the fact that $L(\lambda)$ is the unique simple G -submodule of $\nabla(\lambda)$. Q.E.D.

Theorem 2.2: (Steinberg) *The map $\lambda \in P^+ \mapsto L(\lambda)$ is a bijection from P^+ to the set of simple G -module.*

Proof: Let S be a simple G -module. As the action of U is unipotent, $H_0(U, S)$ is non-zero. As in characteristic zero, $SL(2)$ -theory implies that any weight of $H_0(U, S)$ is antidominant. Therefore there exists a dominant weight λ and a non-zero B -morphism $S \rightarrow K(w_0\lambda)$. By the universal property of the functor D , one gets a non-zero G -morphism $S \rightarrow \nabla(\lambda) = DK_{w_0\lambda}$. By Lemma 2.1, S is isomorphic to $L(\lambda)$. Q.E.D.

Therefore, the classification of simple G -modules is the same as in characteristic zero. As the space $H^0(U, \nabla(\lambda))$ is one-dimensional, the G -module $\nabla(\lambda)$ is indecomposable. However, in general $\nabla(\lambda)$ is reducible. A simple example is provided by the following remark: for any $n \geq 1$, the n -power map $\Sigma_n : \nabla(\lambda) \rightarrow \nabla(n\lambda), f \mapsto f^n$ is a G -equivariant polynomial map and its image is a G -invariant cone. However, when $n = p$ its image is a linear subspace, with the same dimension as $\nabla(\lambda)$. We can deduce that the module $\nabla(p\lambda)$ is reducible for any $\lambda \neq 0$. For a more complicated example, note that the trivial module K is always a quotient of $\nabla(2(p-1)\rho)$. Indeed, with the definitions of Section 5, $\nabla(2(p-1)\rho)$ can be identified with the space of $\mathcal{O}_{G/B}^p$ -linear maps $\sigma : \mathcal{O}_{G/B} \rightarrow \mathcal{O}_{G/B}^p$: therefore the map $\sigma \mapsto \sigma(1)$ is a G -morphism from $\nabla(2(p-1)\rho)$ to K . As G/B is Frobenius split, this map is not zero.

¹ This well-established terminology could be a bit confusing, because $\nabla(\lambda)$ is only the dual of $\Delta(\lambda)$ twisted by the Cartan involution.

Next, Steinberg [St] showed how to reduce the computation all simple modules to a finite problem, namely the determination of those whose highest weight is restricted. A dominant weight λ is called *restricted* if $\lambda(h_i) < p$ for any $i \in I$. The number of restricted weights is p^l . In order to introduce Steinberg's method, we first state a simple lemma.

Lemma 2.3: *Let λ be a restricted weight. Denote indistinctly by v_λ a non-zero highest weight vector of $\Delta(\lambda)$ or of $\nabla(\lambda)$. We have $\Delta(\lambda) = u(\mathbf{u}^-).v_\lambda$, and $H^0(\mathbf{u}, \nabla(\lambda)) = Kv_\lambda$.*

Proof: By Lemma 2.1 (i), one gets $\Delta(\lambda) = \mathcal{H}_{U^-}.v_\lambda$. Moreover, we have: $f_i^{(n)}.u.v_\lambda = \sum_{a+b=n} \text{Ad}(f_i^{(a)})(u) f_i^{(b)}.v_\lambda$, for any $i \in I$, $n \geq 0$ and $u \in u(\mathbf{u}^-)$. As λ is restricted, we have $f_i^{(b)}.v_\lambda = 0$ for any $b \geq p$, therefore $u(\mathbf{u}^-).v_\lambda$ is stable by $\mathcal{H}_{U^-}$. Hence $\Delta(\lambda) = u(\mathbf{u}^-).v_\lambda$. The second point follows by duality. Q.E.D.

It follows from its definition that $\mathbf{F}_p[G_{\mathbf{F}_p}]$ is a $\mathbf{F}_p$ -form of the algebra $K[G]$, see Section 1. The *Frobenius map* $F : K[G] \rightarrow K[G]$ is the K -linear algebra morphism such that $F(f) = f^p$ for any $f \in \mathbf{F}_p[G_{\mathbf{F}_p}]^2$. We will also denote by $F : G \rightarrow G$ the corresponding group morphism. For $G = SL(n)$, the Frobenius morphism F is the map $(a_{i,j}) \mapsto (a_{i,j}^p)$. In general, the F is a group morphism which is pointwise bijective, but as a morphism of algebraic varieties, it is a finite morphism³ of degree $p^{\dim G}$. Its image is the subring $K[G]^p$, which is stable by $\mathcal{H}_G$. Hence F induces an algebra morphism $F : \mathcal{H}_G \rightarrow \mathcal{H}_G$, and we have $F(e_i^{(n)}) = F(f_i^{(n)}) = 0$ if n is not divisible by p , and $F(e_i^{(n)}) = e_i^{(n/p)}$, $F(f_i^{(n)}) = f_i^{(n/p)}$ otherwise. It follows easily that $F(\mathfrak{g}) = 0$. As F induces the multiplication by p on P , we have $F_*L(\lambda) = L(p\lambda)$.

Any non-negative integer n admits a finite p -adic decomposition: $n = n_0 + p.n_1 + p^2.n_2 + \dots$, where $0 \leq n_k < p$ and almost all n_k are zero. Similarly, any dominant weight admits a p -adic decomposition $\lambda = \lambda_0 + p.\lambda_1 + p^2.\lambda_2 + \dots$, where all λ_k are restricted and almost all of them are zero.

Theorem 2.4: (Steinberg's tensor product Theorem) *Let λ as before. We have:*

$$L(\lambda) \simeq L(\lambda_0) \otimes F_*L(\lambda_1) \otimes F_*^2L(\lambda_2) \dots \quad ^4$$

Proof: It should be noted that $\lambda_k = 0$ for $k \gg 0$, therefore $F_*^k L(\lambda_k) = K$ for $k \gg 0$ and the tensor product is indeed finite.

By induction, it is enough to prove $L(\lambda) \simeq L(\lambda_0) \otimes F_*L(\lambda_1)$, for any $\lambda = \lambda_0 + p.\lambda_1$ with λ_0 restricted and $\lambda_1 \in P^+$. Let v_0^+, v_1^+ be non-zero highest weight vectors of $L(\lambda_0)$, $F_*L(\lambda_1)$. By Lemma 2.3, we have $H^0(\mathbf{u}, L(\lambda_0)) = K.v_0^+$. Moreover $\mathbf{u}$ acts trivially on $F_*L(\lambda_1)$. Hence $H^0(\mathbf{u}, L(\lambda_0) \otimes F_*L(\lambda_1)) = K.v_0^+ \otimes F_*L(\lambda_1)$, from which one deduces $H^0(U, L(\lambda_0) \otimes F_*L(\lambda_1)) = K.v_0^+ \otimes v_1^+$. Similarly, one proves $H_0(U^-, L(\lambda_0) \otimes F_*L(\lambda_1)) = K.v_0^+ \otimes v_1^+$. Therefore $v_0^+ \otimes v_1^+$ generates $L(\lambda_0) \otimes F_*L(\lambda_1)$ as a U^- -module and any non-

² The Frobenius morphism is also called the *K-linear Frobenius morphism*. It coincides with the absolute Frobenius morphism Fr on the subring $\mathbf{F}_p[G_{\mathbf{F}_p}]$.

³ It is a purely characteristic p phenomenon that a finite morphism of degree > 1 can be bijective. This arises in the presence of purely inseparable extensions.

⁴ There is a similar statement for finite dimensional continuous simple representations of $G_{\mathbf{C}}$; they are of the form $L \otimes \sigma_*L'$, where L, L' are holomorphic and where σ is the involution of $Gal(\mathbf{R})$

zero U -submodule contains $v_0^+ \otimes v_1^+$. Hence the G -module $L(\lambda_0) \otimes F_* L(\lambda_1)$ is simple. Its highest weight being λ , it is isomorphic to $L(\lambda)$ by Theorem 2.2. Q.E.D.

Use again the notation F for the additive map $\mathbf{Z}[P] \rightarrow \mathbf{Z}[P]$, $e^\lambda \mapsto e^{p\lambda}$. By Theorem 2.4, the character of any $L(\lambda)$ is determined by those of restricted highest weight modules:

Corollary 2.5: (Steinberg) *Let λ as before. We have $\text{ch } L(\lambda) = \prod_{k \geq 0} F^k \text{ch } L(\lambda_k)$.*

(Almost all terms of the previous product equals to 1 and the product is actually finite.) Set $\rho = 1/2 \sum_{\alpha \in \Delta^+} \alpha$. As usual, ρ is a weight because $\rho(h_i) = 1$ for all $i \in I$. The next result of Steinberg is so important that the module $\Delta((p-1)\rho)$ is now called the *Steinberg module* and it deserves the special notation St .

Theorem 2.6: (Steinberg)

- (i) *As a $u(\mathfrak{u}^-)$ -module, St is free of rank one.*
- (ii) *The G -module St is simple, and $St \simeq \nabla((p-1)\rho)$.*
- (iii) $\text{ch } St = e^{(p-1)\rho} \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} \dots + e^{(1-p)\alpha})$.

Proof: We claim:

- (i) $u(\mathfrak{u}^-)$ contains a unique (up to scalar) vector $X \neq 0$ of weight $2(1-p)\rho$, and
- (ii) any non-zero H -invariant left ideal I of $u(\mathfrak{u}^-)$ contains X .

By Jacobson Theorem 1.2, we get:

$$\text{ch } u(\mathfrak{u}^-) = \text{ch } \bar{S}\mathfrak{u}^- = \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} \dots + e^{(1-p)\alpha}),$$

therefore any weight of $u(\mathfrak{u}^-)$ is of the form $\sum_{\alpha \in \Delta^+} m_\alpha \alpha$, where $-p < m_\alpha \leq 0$. Thus the weight $2(1-p)\rho$ occurs with multiplicity 1, what shows the unicity of X . Let I be any non-zero H -invariant left ideal of $u(\mathfrak{u}^-)$. Let $L : u(\mathfrak{u}^-) \rightarrow (\bigwedge^{(p-1)N} \mathfrak{u}^-)^{\otimes p-1} \simeq K(2(1-p)\rho)$ be the canonical map defined in Corollary 1.3. As the associated bilinear map is non-degenerated, we have $L(I) \neq 0$. In particular $2(1-p)\rho$ is a weight of I . Therefore $X \in I$ and the claim is proved.

It follows from Lemma 2.3 that $L((p-1)\rho) = u(\mathfrak{u}^-).v^+$, where v^+ is an highest weight vector of $L((p-1)\rho)$. The lowest weight of $L((p-1)\rho)$ is $w_0(p-1)\rho = (1-p)\rho$. Hence $X.v^+$ is a non-zero lowest weight vector of $L((p-1)\rho)$. Therefore X does not belongs to the annihilator I in $u(\mathfrak{u}^-)$ of v^+ . It follows from the previous claim that $I = 0$. Therefore, the $u(\mathfrak{u}^-)$ -module $L((p-1)\rho)$ is free of rank one and $\Delta((p-1)\rho) = L((p-1)\rho)$. By duality we also have $\nabla((p-1)\rho) = L((p-1)\rho)$, what proves assertions (i) and (ii). The last assertion follows from the fact that

$$\text{ch } L((p-1)\rho) = e^{(p-1)\rho} \text{ch } u(\mathfrak{u}^-). \quad \text{Q.E.D.}$$

3. Kempf's vanishing theorem and Weyl character formula. In this section we will prove Kempf's vanishing Theorem [Ke], following the beautiful proof of Andersen [A3] and Haboush [Ha]. Its main corollary is the Weyl character formula for $\nabla(\lambda)$. We will also state Jantzen linkage Principle [J1][HJ] and Andersen strong linkage Principle [A4]. However, we will only sketch the proof of the linkage principles.

Any sheaf $\mathcal{M}$ of $\mathcal{O}_{G/B}$ -modules can be considered as a sheaf of $\mathcal{O}_{G/B}^p$ -modules, hence we get a natural map $\mathcal{O}_{G/B}^p \otimes_K \Gamma(G/B, \mathcal{M}) \rightarrow \mathcal{M}$. Recall that $St = \Gamma(G/B, \mathcal{L}((1-p)\rho))$.

Lemma 3.1: (Andersen-Haboush formula [A3][Ha])

The natural map $\theta : \mathcal{O}_{G/B}^p \otimes St \rightarrow \mathcal{L}((1-p)\rho)$ is an isomorphism.

Proof: The map $\mathcal{O}_{G/B}^p \rightarrow \mathcal{L}((1-p)\rho)$, $f \mapsto f v^+$ (where v^+ is an highest weight vector of St) is clearly injective. By Lemma 2.3 we have: $H^0(\mathfrak{u}, \mathcal{O}_{G/B}^p \otimes St) = \mathcal{O}_{G/B}^p \otimes k.v^+$. Hence by $\mathfrak{u}$ -equivariance, θ is injective. Moreover, the source and the goal of θ are locally free $\mathcal{O}_{G/B}^p$ -sheaves of same rank, namely $p^{\dim G/B}$. Hence θ is an isomorphism on some dense open subset. By G -equivariance, θ is an isomorphism. Q.E.D.

Theorem 3.2: (Kempf's vanishing theorem [Ke]) *Let $\lambda \in P^+$. We have:*

$$H^k(G/B, \mathcal{L}(-\lambda)) = 0, \text{ for any } k > 0.$$

Proof: For any line bundle $\mathcal{L}$ over G/B , denote by $\mathcal{L}^p$ the image of the p -power map $\Sigma_p : \mathcal{L} \rightarrow \mathcal{L}^{\otimes p}$. Thus $\mathcal{L}^p$ is an invertible $\mathcal{O}_{G/B}^p$ -sheaf and $\mathcal{L}^{\otimes p} = \mathcal{O}_{G/B} \otimes_{\mathcal{O}_{G/B}^p} \mathcal{L}^p$. It follows Andersen-Haboush formula that $\mathcal{L}(-p\lambda + (1-p)\rho) \simeq \mathcal{L}(-\lambda)^p \otimes St$. As sheaves of abelian groups, $\mathcal{L}(-\lambda)$ and $\mathcal{L}(-\lambda)^p$ are isomorphic. Hence we get:

$$H^k(G/B, \mathcal{L}(-\lambda)) \neq 0 \Rightarrow H^k(G/B, \mathcal{L}(-p\lambda + (1-p)\rho)) \neq 0.$$

By induction, we get:

$$H^k(G/B, \mathcal{L}(-\lambda)) \neq 0 \Rightarrow H^k(G/B, \mathcal{L}_1^{\otimes q/p} \otimes \mathcal{L}_2^{\otimes q-q/p-1}) \neq 0,$$

for all power q of p , where $\mathcal{L}_1 = \mathcal{L}(-p\lambda - \rho)$ and $\mathcal{L}_2 = \mathcal{L}(-\rho)$. Therefore, the proof follows from the following three assertions:

- (i) G/B is a projective variety,
- (ii) $\mathcal{L}(-\mu - \rho)$ is ample for any $\mu \in P^+$,
- (iii) For any two ample invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ over a projective variety X , there are only finitely many $m_1, m_2 \geq 0$ such that $H^k(X, \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2}) \neq 0$ for some $k > 0$.

Assertion (iii) is a simple refinement of Serre's vanishing theorem whose proof is standard. Let us prove Assertion (i) and (ii).

First one claims that G/B is projective and $\mathcal{L}((1-p)\rho)$ is very ample. Let v^+ be the highest weight vector of St . First, consider the map $\xi : G/B \rightarrow \mathbf{P}St$, $g \mapsto g.Kv^+$. We observe that the differential at 1 of ξ can be identified with the map $\mathfrak{u}^- \rightarrow St/Kv^+$, $u \mapsto u.v^+$. By Theorem 2.6 (i), this map is injective. By G -invariance, one proves that ξ is an embedding. Moreover, by Lemma 2.1 and Theorem 2.6, Kv^+ is the unique B -invariant line of $\mathbf{P}St$. Thus, by Lie's theorem, any closed B -invariant subset of $\mathbf{P}St$ contains Kv^+ . Therefore the image of ξ is closed and isomorphic to G/B . Thus the claim is proved.

In particular Assertion (i) holds. For any $\mu \in P^+$, $\mathcal{L}(-\mu)$ has non-zero global sections. By G -invariance, it implies that $\mathcal{L}(-\mu)$ is generated by its global sections. Therefore the sheaf $\mathcal{L}(-\mu - \rho)^{\otimes p-1} \simeq \mathcal{L}((1-p)\mu) \otimes \mathcal{L}((1-p)\rho)$ is very ample, what proves Assertion (ii). Q.E.D.

Remarks: As G/B is projective, the functor D sends finite dimensional B -modules to finite dimensional G -modules (this fact can be proved more elementarily). Indeed, a refinement of Assertion (i) is true: $\mathcal{L}(-\mu - \rho)$ is very ample whenever μ is dominant. Moreover, the Kempf's vanishing theorem holds under the weaker hypothesis that $\lambda + \rho$ is dominant. It should be noted that Kempf's theorem is a characteristic free generalization of Bott's vanishing theorem. However the part of Bott's theorem involving non-dominant weights fails in finite characteristics: there are weights λ such that $H^k(G/B, \mathcal{L}(\lambda)) \neq 0$ for at least two values of k . See [A5] for a counterexample involving $G = SL(3)$. The original Kempf's proof is based on a case-by-case analysis.

Corollary 3.3: (Weyl character formula) *Let $\lambda \in P^+$. We have:*

$$\text{ch } \Delta(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e^{w\rho}}.$$

Proof: By semi-continuity theorem, the character of the virtual G -module $\sum_k (-1)^k H^k(G/B, \mathcal{L}(-\lambda))$ is independent of the characteristic of the field K , hence given by Weyl character formula. Hence, by Kempf's vanishing theorem, the character of $\Delta(\lambda)$ is given by the Weyl character formula. Q.E.D.

As noted before, there are indecomposable G -modules which are not simple. Therefore the category $\text{Mod}(G)$ of G -modules is not semi-simple. As for any artinian category, we can decompose $\text{Mod}(G)$ into blocks. We define now the notion of a block. Let Γ be the non-oriented graph whose vertices are the (isomorphism classes of) simple G -modules and whose edges connect two simple G -modules L, L' whenever $\text{Ext}_G^1(L, L') \neq 0$ (or equivalently: $\text{Ext}_G^1(L', L) \neq 0$). A *block* is a connected component of Γ . Any G -module M can be decomposed as $\bigoplus_{b \in \pi_0(\Gamma)} M(b)$, where any simple subquotient of $M(b)$ belongs to the block b .

By definition, the affine Weyl group W_{aff} is the group of affine transformations of P generated by W and the additional affine symmetry $s_0 : \lambda \mapsto \lambda - (\lambda(h_0) - p)\alpha_0$, where α_0 is the highest short root and $h_0 = h_{\alpha_0}$ (thus h_0 is the highest coroot).

Theorem 3.4: (Jantzen-Andersen linkage Principles [J1][A4])

(i) Let $\lambda, \mu \in P^+$. If $L(\lambda)$ and $L(\mu)$ are in the same block, then $\lambda + \rho$ and $\mu + \rho$ are conjugated by W_{aff} .

(ii) Let $\lambda \in P^+$, $\mu \in P$. If $L(-w_0\lambda)$ is a composition factor of $H^*(G/B, \mathcal{L}(-\mu))$, then $\lambda + \rho$ and $\mu + \rho$ are conjugated by W_{aff} .

The linkage Principle (namely 3.4 (i)) has been conjectured by Verma and it has been proved by Jantzen [J1] for $p \geq h$ and by Andersen [A4] in general. The strong Linkage principle (namely 3.4 (ii)) is due to Andersen [A4]. The sketch of proof given below is based on the Andersen's proof.

Corollary 3.5: If $(\lambda + \rho)(h_0) \leq p$, then the module $\nabla(\lambda)$ is simple.

The simplest and standard proof of Corollary 3.5 is based the following fact. For any $\lambda \in P^+$ with $(\lambda + \rho)(h_0) \leq p$, there are no dominant weight $\mu < \lambda$ with $\mu + \rho \in W_{\text{aff}}(\lambda + \rho)$. Therefore $L(\lambda)$ is the unique subquotient of $\nabla(\lambda)$, i.e. $\nabla(\lambda)$ is simple.

However, we will sketch a more sophisticated proof in order to explain the idea of Andersen's proof of the linkage principle. First recall Demazure's trick of its "very simple proof" of Bott's theorem over a field of characteristic zero. Therefore assume for a moment that K is a field of characteristic zero. Demazure's trick is the isomorphism:

$$H^k(G/B, \mathcal{L}(-\mu)) \simeq H^{k+1}(G/B, \mathcal{L}(-s_i(\mu + \rho) + \rho)),$$

for any $k \geq 0$, $i \in I$ and $\mu \in P$ with $\mu(h_i) \geq 0$. Starting with a weight $\lambda \in P^+$, one gets by iterating an isomorphism:

$$H^k(G/B, \mathcal{L}(-\lambda)) \simeq H^{k+l(w)}(G/B, \mathcal{L}(-w(\lambda + \rho) + \rho)),$$

for any $w \in W$. However, by Serre's duality, $H^{l(w_0)}(G/B, \mathcal{L}(-w_0(\lambda + \rho) + \rho)) \simeq \Delta(\lambda)$, because $w_0\rho = -\rho$ and $\mathcal{L}(2\rho)$ is the sheaf of top forms of G/B . Thus for $k = 0$ and $w = w_0$, one gets $\nabla(\lambda) \simeq \Delta(\lambda)$ what amounts to the fact that the G -module $\nabla(\lambda)$ is simple. Of course, there simpler ways to prove this fact⁵.

⁵ In characteristic zero, the interest of Demazure's trick lies in the case $k > 0$: one

Demazure's trick uses that the $SL(2)$ -modules $H^0(\mathbf{P}^1, \mathcal{L}(-n\rho))$ and $H^0(\mathbf{P}^1, \mathcal{L}((n+2)\rho))$ are isomorphic for all $n \geq 0$. In characteristic p , it is true for all $n < p$ and some other special values of n . Under the condition $(\lambda + \rho)(h_0) \leq p$, each step of Demazure's trick uses the previous isomorphism for some value of $n < p$. Therefore Demazure's proof applies to such a weight, and one gets a proof of Corollary 3.5.

Andersen's proof of the strong linkage principle is based on this idea. He is using a map $H^k(G/B, \mathcal{L}(-\lambda)) \rightarrow H^{k+l(w)}(G/B, \mathcal{L}(-w(\lambda + \rho) + \rho))$, and he determines by induction what are the possible composition factors of its kernel and cokernel.

4. Good filtrations. In this section, we will define the good filtrations. To my best knowledge, this notion has been invented by Jantzen and Humphreys (see [Hu]), and it has been used for the first time in Upadhyaya's work [U]. The G -modules have a canonical filtration: originally this idea comes from [Fr] but we will follow the approach of [M2], which is simpler and is better adapted. A simple proof of the characteristic p version of Peter-Weyl Theorem will be provided: this result is often attributed to Donkin (unpublished) and Koppinen [Ko]. At the end, we will describe two different criteria [CPSV][M2] for the existence of a good filtration.

Choose a one-to-one linear map $E : P \rightarrow \mathbf{R}$ such that $E(\alpha_i) > 0$ for any $i \in I$; the injectivity condition simply means that the real numbers $E(\alpha_i)$ are $\mathbf{Q}$ -linearly independent. By definition, we have $E(\lambda) < E(\mu)$ whenever $\lambda < \mu$. Hence E induces a total ordering of P which extends the usual partial ordering $<$. For any non-zero dominant weight $\lambda = \sum_{i \in I} x_i \alpha_i$, it is well known that all coefficients x_i are positive. Hence the set $E(P^+)$ is discrete, and we can uniquely write $P^+ = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$, where $E(\lambda_0) < E(\lambda_1) < E(\lambda_2) \dots$.

For any B -module M , denote by $M(k)$ its biggest submodule such that any weight μ of $M(k)$ satisfies $E(\mu) \leq E(\lambda_k)$. By this way, we get a filtration $M(0) \subset M(1) \subset \dots$ of M . The filtration $(M(k))_{k \geq 0}$ of M is called the *canonical filtration* of M . However, one should be aware that the filtration depends effectively on E . As a matter of notation, we set $h^0(M, \lambda) = \dim H^0(U, M)_\lambda$.

Lemma 4.1: [M2] *Assume that M is a G -module.*

- (i) *Each $M(k)$ is indeed a G -submodule,*
- (ii) *There is a canonical inclusion $M(k)/M(k-1) \subset \nabla(\lambda_k)^{h^0(M, \lambda_k)}$.*

Proof: Proof of Point (i): Use the PBW Decomposition 1.4: $U_G = U_{U^-} \otimes U_H \otimes U_U$. It follows that the G -submodule generated by $M(k)$ is $U_{U^-} \cdot M(k)$. However, any weight of U_{U^-} is ≤ 0 . Thus $U_{U^-} \cdot M(k) \subset M(k)$, hence $M(k)$ is a G -module.

Proof of Point (ii): Clearly, λ_k is an highest weight of $M(k)$, with multiplicity $h^0(M, \lambda_k)$. Therefore the weight $w_0 \lambda_k$ is a lowest weight of $M(k)$ and the H -equivariant projection $\pi : M(k) \rightarrow M(k)_{w_0 \lambda_k}$ is B -equivariant. By the universal property of the functor D , π induces a morphism $D\pi : M(k) \rightarrow D M(k)_{w_0 \lambda_k} \simeq \nabla(\lambda_k)^{h^0(M, \lambda_k)}$, whose kernel is obviously $M(k-1)$. Q.E.D.

gets Bott's vanishing theorem: $H^k(G/B, \mathcal{L}(-\lambda)) \simeq H^{k+l(w_0)}(G/B, \mathcal{L}(-w(\lambda + \rho) + \rho)) = 0$, because any cohomology in degree $> \dim G/B$ vanishes.

A filtration $\mathcal{F}_0 M \subset \mathcal{F}_1 M \subset \mathcal{F}_2 M \dots$ of a G -module M is called *good*⁶ if for all $k \geq 0$, $\mathcal{F}_k M / \mathcal{F}_{k-1} M \simeq \nabla(\lambda_k)^{m_k}$, for some $\lambda_k \in P^+$ and some $m_k \geq 0$.

Lemma 4.2: *For any G -module M , we have: $\text{ch } M \leq \sum_{\lambda \in P^+} h^0(M, \lambda) \text{ch } \nabla(\lambda)$. Moreover, the following assertions are equivalent:*

- (i) $\dim M = \sum_{\lambda \in P^+} h^0(M, \lambda) \dim \nabla(\lambda)$,
- (ii) $\text{ch } M = \sum_{\lambda \in P^+} h^0(M, \lambda) \text{ch } \nabla(\lambda)$,
- (iii) *the canonical filtration is good.*

Proof: From the previous lemma, we have:

$$\text{ch } M = \sum_{k \geq 0} \text{ch } M(k) / M(k-1) \leq \sum_{k \geq 0} h^0(M, \lambda_k) \text{ch } \nabla(\lambda_k).$$

Moreover, there is equality if and only if the canonical filtration is good. Q.E.D.

Theorem 4.3: (Donkin-Koppinen [Ko]) *There exist*

- (i) *a $G \times G$ -equivariant filtration $\mathcal{F}_0 K[G] \subset \mathcal{F}_1 K[G] \dots$ of $K[G]$*
- (ii) *a bijection $\mathbb{Z}_{\geq 0} \rightarrow P^+, k \mapsto \mu_k$,*

such that $\mathcal{F}_k K[G] / \mathcal{F}_{k-1} K[G]$ is $G \times G$ -isomorphic to $\nabla(\mu_k) \otimes \nabla(-w_0 \mu_k)$.

Proof: First we claim that $H^0(U \times U^-, K[G]) \simeq \oplus_{\lambda \in P^+} K(\lambda) \otimes K(-\lambda)$ as an $H \times H$ -module. Indeed $\Omega = U.H.U^-$ is a dense open set in G , hence $H^0(U \times U^-, K[G]) \subset H^0(U \times U^-, \Omega) \simeq K[H] \simeq \oplus_{\lambda \in P} K(\lambda) \otimes K(-\lambda)$ as an $H \times H$ -module. However the weight of any U -invariant vector in a G -module is dominant, hence we have $H^0(U \times U^-, K[G]) \subset \oplus_{\lambda \in P^+} K(\lambda) \otimes K(-\lambda)$. Conversely for $\lambda \in P^+$, set $\phi_\lambda(g) = \langle \xi | g^{-1}.v^+ \rangle$, where v^+ is an highest weight vector of $L(\lambda)$ and ξ is a lower weight vector of $L(\lambda)^*$. Then ϕ_λ is a $U \times U^-$ -invariant element of $K[G]$ of weight $(\lambda, -\lambda)$. Therefore the claim is proved.

Now consider $K[G]$ as $G \times G$ -module, and let $F : P \oplus P \rightarrow \mathbf{R}$ be an injective linear form as before. Denote by $g_0 < g_1 < g_2 \dots$ the gaps of the canonical filtration of the $G \times G$ -module $K[G]$, i.e. the integers g with $K[G](g) \neq K[G](g-1)$. Set $\mathcal{F}_k K[G] = K[G](g_k)$ for all $k \geq 0$; up to a change in the indices, the filtration $\mathcal{F}_k K[G]$ is identical to the canonical filtration. It follows from the claim that any highest weight in $H^0(U \times U, \mathcal{F}_k K[G] / \mathcal{F}_{k-1} K[G])$ is of the form $(\mu_k, -w_0 \mu_k)$ for some $\mu_k \in P^+$ and has multiplicity one. Hence from Lemma 4.2, we get:

$$\dim \mathcal{F}_k K[G] \leq \sum_{j \leq k} (\dim \nabla(\mu_j))^2.$$

Over $\mathbf{C}$, we have $\mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}] = \oplus_{j \leq k} L_{\mathbf{C}}(\mu_j) \otimes L_{\mathbf{C}}(-w_0 \mu_j)$ by Peter-Weyl Theorem. By semi-continuity theorem⁷, we have $\dim \mathcal{F}_k K[G] \geq \dim_{\mathbf{C}} \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$. Thus, we get:

$$\dim \mathcal{F}_k K[G] \geq \sum_{j \leq k} (\dim L_{\mathbf{C}}(\mu_j))^2,$$

where $L_{\mathbf{C}}(\mu_j)$ is the simple $G_{\mathbf{C}}$ -module with highest weight μ_j . By Weyl character formula, the previous inequalities are equalities, and the theorem follows from Lemma 4.2. Q.E.D.

Let M, N be two G -modules. Recall that $\text{Ext}_G^1(M, N)$ is the classifying group of all extensions of G -modules: $0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0$.

⁶ The usual definition of good filtration requires that each subquotient is a dual of a Weyl module (i.e. $m_k = 1$). However, our choice does not modify the notion of *module having a good filtration*.

⁷ The filtration $\mathcal{F}$ is defined over $\mathbf{Z}[G_{\mathbf{Z}}]$ as well. We have $\mathbf{C} \otimes \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}] = \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$, hence $\mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}]$ is torsion free of rank $\dim \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$. Moreover $\mathbf{Z}[G_{\mathbf{Z}}] / \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}]$ is torsion free, hence the map $K \otimes \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}] \rightarrow \mathcal{F}_k K[G]$ is one-to-one.

Thus we get $\dim_K \mathcal{F}_k K[G] \geq \text{rk}_{\mathbf{Z}} \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}] = \dim_{\mathbf{C}} \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$.

Lemma 4.4: *Let M be a G -module and let $\lambda \in P^+$.*

(i) if $\text{Ext}_G^1(\Delta(\lambda), M) \neq 0$, there is a weight ν of M with $\nu > \lambda$.

(ii) if $\text{Ext}_G^1(M, \nabla(\lambda)) \neq 0$, there is a weight ν of M with $\nu > \lambda$.

Proof: Given one of its highest weight vector v_λ , the Weyl module $\Delta(\lambda)$ satisfies the following universal property: for any module M and $v \in H^0(U, M)_\lambda$, there is a G -morphism $\phi : \Delta(\lambda) \rightarrow M$ with $\phi(v_\lambda) = v$. Assume $\text{Ext}_G^1(\Delta(\lambda), M) \neq 0$ and consider a non trivial extension of G -modules: $0 \rightarrow M \rightarrow X \rightarrow \Delta(\lambda) \rightarrow 0$. Let $v \in X_\lambda$ be a lifting of v_λ . As the extension does not split, v is not U -invariant. Therefore the U -module generated by v contains a weight $\nu > \lambda$, what proves Assertion (i). The proof of Assertion (ii) is similar. Q.E.D.

The following Lemma is a very special case the Cline-Parshall-Scott-van der Kallen vanishing theorem (see the Appendix).

Lemma 4.5: [CPSV] *For any $\lambda, \mu \in P^+$, we have $\text{Ext}_G^1(\Delta(\lambda), \nabla(\mu)) = 0$.*

Proof: Both inequalities $\mu > \lambda$ and $\lambda > \mu$ cannot hold simultaneously. In any case, the claim follows from Assertion (i) or (ii) of the previous lemma. Q.E.D.

For any G -module M and any $\lambda \in P^+$, set $h^1(M, \lambda) = \dim \text{Ext}_G^1(\Delta(\lambda), M)$.

Proposition 4.6: *Let M be a G -module. We have*

$$\text{ch } M \leq \sum_{\lambda \in P^+} h^0(M, \lambda) \text{ch } \nabla(\lambda),$$

$$\text{ch } M \geq \sum_{\lambda \in P^+} (h^0(M, \lambda) - h^1(M, \lambda)) \text{ch } \nabla(\lambda).$$

Proof: The first inequality follows from Lemma 4.2. Prove the second inequality. The structural morphism $M \rightarrow M \otimes K[G]$ is indeed a G -embedding $M \subset K[G]^m$, where $m = \dim M$. Using Theorem 4.3, there is a module with a good filtration X such that $M \subset X$: indeed a convenient choice is $X = \mathcal{F}_k K[G]^m$ for k big enough. Then we get a short exact sequence: $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$. By Lemma 4.5, we have $\text{Ext}_G^1(\Delta(\lambda), X) = 0$, for any $\lambda \in P^+$. Therefore we get an exact sequence:

$$0 \rightarrow \text{Hom}_G(\Delta(\lambda), M) \rightarrow \text{Hom}_G(\Delta(\lambda), X) \rightarrow \text{Hom}_G(\Delta(\lambda), N) \rightarrow \text{Ext}_G^1(\Delta(\lambda), M) \rightarrow 0$$

for any $\lambda \in P^+$. So we get:

$$(4.6.1) \quad h^0(X, \lambda) - h^0(N, \lambda) = h^0(M, \lambda) - h^1(M, \lambda).$$

Moreover by Lemma 4.2, we get:

$$(4.6.2) \quad \text{ch } X = \sum_{\lambda \in P^+} h^0(X, \lambda) \text{ch } \nabla(\lambda),$$

$$(4.6.3) \quad \text{ch } N \leq \sum_{\lambda \in P^+} h^0(N, \lambda) \text{ch } \nabla(\lambda).$$

As $\text{ch } M = \text{ch } X - \text{ch } N$, the relations (4.6.1 – 3) imply the required inequality. Q.E.D.

In the following statement, the equivalence of (ii) and (iii) is often called the Cline-Parshall-Scott-van der Kallen criterion for good filtrations [CPSV].

Corollary 4.7: *For any G -module M , the following three properties are equivalent:*

(i) the canonical filtration of M is a good filtration,

(ii) M has a good filtration,

(iii) $\text{Ext}_G^1(\Delta(\lambda), M) = 0$ for all $\lambda \in P^+$.

Moreover, if M has a good filtration, any direct summand has a good filtration.

Proof: (i) $\Rightarrow$ (ii) is obvious, (ii) $\Rightarrow$ (iii) follows from Lemma 4.5, and (iii) $\Rightarrow$ (i) follows from Lemmas 4.2 and 4.5. Moreover, if M has a good filtration then $\text{Ext}_G^1(\Delta(\lambda), N) = 0$ for any direct summand N and therefore N has a good filtration. Q.E.D.

We will state a criterion of different nature for the existence of a good filtration for a reduced commutative G -algebra. Until the end of the section, we will no more assume that the G -modules are finite dimensional. Let $\lambda \in P^+$ and set $x = E(\lambda)$. A G -module is called x -isotypical if λ is the unique weight of $H^0(U, M)$. For any x -isotypical G -module M , the H -equivariant projection $M \rightarrow M_{w_0\lambda}$ is actually B -equivariant (here $M_{w_0\lambda}$ is viewed as a B -module with a trivial U -action; this should not be confused with the natural embedding $M_{w_0\lambda} \rightarrow M$ which is B^- -equivariant). Thus set $N_x M = D M_{w_0\lambda}$. It is clear that the map $M \rightarrow N_x M$ is injective, $H^0(U, M) = H^0(U, N_x M)$, and $N_x M$ is a direct sum of modules isomorphic to $\nabla(\lambda)$.

By definition, a graded G -algebra $\bar{A} = \bigoplus_{n \geq 0} \bar{A}_n$ is called x -isotypical if $\bar{A}_n$ is $n\lambda$ -isotypical for all $n \geq 0$. Set $\bar{A}^- = \bigoplus_{n \geq 0} (\bar{A}_n)_{nw_0\lambda}$. The natural projection $\bar{A} \rightarrow \bar{A}^-$ is a morphism of B -algebras. As before $\bar{A}^-$ is viewed as a B -module with a trivial U -action and we should emphasize that the natural embedding $\bar{A}^- \rightarrow \bar{A}$ is B^- -equivariant. Set $C_x \bar{A} = \bigoplus_{n \geq 0} N_{nx} \bar{A}_n$. Equivalently $C_x \bar{A} = D \bar{A}^-$. Therefore $C_x \bar{A}$ is a G -algebra which contains $\bar{A}$. It is easy to prove that $C_x \bar{A}$ is commutative (respectively associative, reduced) if and only if $\bar{A}^U$ is commutative (respectively associative, reduced).

Proposition 4.8: [M2] *Let $\lambda \in P^+$ and set $x = E(\lambda)$. Let $\bar{A}$ be a reduced commutative associative x -isotypical algebra. If $\bar{A}_1$ is not a direct sum of $\nabla(\lambda)$, there is an element $f \in C_x \bar{A} \setminus \bar{A}$ with $f^p \in \bar{A}$.*

Proof: By definition, $N_x \bar{A}_1$ is a direct sum of $\Delta(\lambda)$. The hypothesis implies $\bar{A}_1 \neq N_x \bar{A}_1$. Therefore $N_x \bar{A}_1$ contains a submodule M isomorphic to $\nabla(\lambda)$ such that M is not contained in $\bar{A}_1$. Let $y \in M_{w_0\lambda}$ be a non-zero vector. As $\bar{A}^-$ is reduced, its subalgebra generated by y is isomorphic to $K[y]$. Set $\mathcal{A} = D K[y]$. Therefore $\mathcal{A}$ is a graded subalgebra of $C_x \bar{A}$, and $\mathcal{B} = \mathcal{A} \cap \bar{A}$ is a graded subalgebra of $\bar{A}$. Moreover we have $\mathcal{A}_1 = M$ and $\mathcal{A}$ is isomorphic to the algebra $\bigoplus_{n \geq 0} \Gamma(G/B, \mathcal{L}^{\otimes n})$, where $\mathcal{L} = \mathcal{L}(w_0\lambda)$.

First we claim that the algebras $\mathcal{A}$, $\mathcal{B}$ are finitely generated and the morphism $\text{Spec } \mathcal{A} \rightarrow \text{Spec } \mathcal{B}$ is finite and bijective. Let $\mathcal{E} \subset \mathcal{A}$ be the subalgebra generated by $\bigoplus_w M_{w\lambda}$ and let $\mathcal{E}^+$ be its maximal homogenous ideal. As $\mathcal{L}$ is semi-ample, the algebra $\mathcal{A}$ is finitely generated⁸. The spectrum of $\mathcal{A}$ can be identified with the subset $G.v^+ \cup \{0\}$ of $\Delta(-w_0\lambda)$, where v^+ is any highest weight vector of $\Delta(-w_0\lambda)$. By Bruhat decomposition, for any $g \in G$, we can find $w \in W$ such that $\langle wy | g.v^+ \rangle \neq 0$. As $wy \in \mathcal{E}^+$, the radical of the ideal $\mathcal{E}^+ \mathcal{A}$ is the maximal homogenous ideal of $\mathcal{A}$ and therefore $\mathcal{A}$ is a finitely generated $\mathcal{E}$ -module. As $M_{w\lambda} \subset \bar{A}_1$ for any $w \in W$, we have $\mathcal{E} \subset \mathcal{B}$. Therefore $\mathcal{B}$ is finitely generated and the extension $\mathcal{B} \subset \mathcal{A}$ is finite. In particular $\text{Spec } \mathcal{A} \rightarrow \text{Spec } \mathcal{B}$ is onto. Let v_1^+ be a highest weight vector of $(M \cap \bar{A}_1)^*$. The stabilizers of v^+ and v_1^+ are the same parabolic subgroups. Hence the morphism $\text{Spec } \mathcal{A} \rightarrow \text{Spec } \mathcal{B}$ is bijective and the claim is proved.

We will now prove Proposition 4.8 under the additional assumption that $\mathcal{B}$ is not normal. Denote by $\bar{\mathcal{B}}$ be its normalization, which is again a graded algebra. As $\mathcal{A}$ is normal, we have $\mathcal{B} \subset \bar{\mathcal{B}} \subset \mathcal{A}$. These extensions being finite, the map $\text{Spec } \mathcal{A} \rightarrow \text{Spec } \bar{\mathcal{B}}$

⁸ Using a less elementary approach, one can prove that $\mathcal{A}$ is generated by M . However, we will not use this fact.

is onto. Hence $\pi : \text{Spec } \overline{\mathcal{B}} \rightarrow \text{Spec } \mathcal{B}$ is bijective. Moreover π is an isomorphism on some non-empty open subset. By G -invariance, π is an isomorphism outside $\{0\}$. Therefore we have $\mathcal{B}_n = \overline{\mathcal{B}}_n$ for $n \gg 0$. Hence for any homogenous $f \in \mathcal{B} \setminus \overline{\mathcal{B}}$ of maximal degree, f^p belongs to $\mathcal{A}$. Therefore $f \in C_x \overline{\mathcal{A}} \setminus \overline{\mathcal{A}}$ and $f^p \in \overline{\mathcal{A}}$ as it is required by the Proposition 4.8.

To complete the proof of Proposition 4.8, we can now assume that $\mathcal{B}$ is normal. Let $K_{\mathcal{A}}, K_{\mathcal{B}}$ be the fraction fields of $\mathcal{A}$ and $\mathcal{B}$. By hypotheses, $\mathcal{B} \neq \mathcal{A}$ and therefore $K_{\mathcal{B}} \neq K_{\mathcal{A}}$. However the extension $K_{\mathcal{B}} \subset K_{\mathcal{A}}$ is purely inseparable (because its separability degree is the cardinal of a generic fiber of the map $\text{Spec } \mathcal{A} \rightarrow \text{Spec } \mathcal{B}$). Thus there is a power q of p such that $K_{\mathcal{A}}^q \subset K_{\mathcal{B}}$. Hence we can find $f \in \mathcal{A} \setminus K_{\mathcal{B}}$ such that $f^p \in K_{\mathcal{B}}$. As $\mathcal{B}$ is normal, we have $f^p \in \mathcal{B}$. Therefore $f \in C_x \overline{\mathcal{A}} \setminus \overline{\mathcal{A}}$ and $f^p \in \overline{\mathcal{A}}$. Q.E.D.

5. Canonical Frobenius splittings. The notion of Frobenius splittings is due to Metha, Ramanan and Ramanathan [MR][RR]. Somehow, their beautiful idea originated in Andersen-Haboush formula 3.1. Here we will not follow their original paper, based on Cartier operators [C]. Instead we will follow another approach, based on the characteristic p version of the change of variables formula of [M1]: this approach is very elementary and more intuitive. The notion of canonical Frobenius splittings comes from [M2].

The basic change of variables formula:

$$\int_X \phi dx_1 \wedge \dots \wedge dx_n = \int_X \phi \det\left(\frac{\partial x_i}{\partial y_j}\right) dy_1 \wedge \dots \wedge dy_n$$

is valid in the context of the Differential Geometry. Here X is a n -dimensional oriented manifold, $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are two systems of parameters and ϕ is any test function. As the integral is somehow the inverse of the derivation, we can write this formula as follows:

$$\left(\frac{\partial^n}{\partial x_1 \dots \partial x_n}\right)^{0-1} = \left(\frac{\partial^n}{\partial y_1 \dots \partial y_n}\right)^{0-1} \circ [\det\left(\frac{\partial y_i}{\partial x_j}\right)]^{0-1}.$$

Of course, this is only a formal identity. The same equality holds in characteristic p , where 0 is changed to p : this is why we wrote $0 - 1$ instead of -1 . Before stating the theorem, we need to comment the notations. It is usual to write differential operators under the form $\sum_{\alpha} f_{\alpha} \partial^{\alpha}$, where the f_{α} are functions and ∂^{α} are partial derivatives. However, it will be more convenient here to write functions on the right side of partial derivatives. For example, the differential operator $d/dx \circ f$ should be understood as the operator sending an arbitrary function ϕ to $d/dx(f\phi)$: with the usual notation, this differential operator is denoted by $f' + f d/dx$.

Theorem 5.1: ([M1]) *Let X be a smooth algebraic variety over K of dimension n , and let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be two systems of parameters on some open subset U . Then as differential operators, we have:*

$$\left(\frac{\partial^n}{\partial x_1 \dots \partial x_n}\right)^{p-1} = \left(\frac{\partial^n}{\partial y_1 \dots \partial y_n}\right)^{p-1} \circ [\det\left(\frac{\partial y_i}{\partial x_j}\right)]^{p-1}.$$

See the proof in [M1]. Let X be an affine variety and set $A = K[X]$. Note that A^p is a subring of A . A F -map is a A^p -linear map $\sigma : A \rightarrow A^p$. Such a map is called a *Frobenius splitting* if $\sigma(1) = 1$, i.e. if σ is a splitting of the embedding $A^p \subset A$. The localization relative to a non-zero element $f \in A$ is the same as the localization relative to f^p . Hence any F -map σ commutes with localizations and it defines a morphism of $\mathcal{O}_X^p$ -sheaves $\sigma : \mathcal{O}_X \rightarrow \mathcal{O}_X^p$. Therefore the notion of F -maps and Frobenius splittings can

be defined for any reduced scheme, not only for an affine variety⁹. For any reduced scheme X denote by $\mathcal{FM}_X$ the sheaf of F -maps, i.e. the sheaf of $\mathcal{O}_X^p$ -linear maps from $\mathcal{O}_X$ to $\mathcal{O}_X^p$. Also set $FM(X) = \Gamma(X, \mathcal{FM}_X)$. We will consider $\mathcal{FM}_X$ as a sheaf of $\mathcal{O}_X$ -modules (by multiplication at the source).

Let us compute the sheaf $\mathcal{FM}_X$ for a smooth variety X of dimension n . Let $x_1, \dots, x_n$ be a system of parameters at a point $x \in X$. We have $(\frac{\partial}{\partial x_i})^m x_i^n = n(n-1)\dots(n-m+1)x_i^{n-m}$. Thus we get:

(i) $(\frac{\partial}{\partial x_i})^p = 0$,

(ii) A function $f \in \mathcal{O}_{X,x}$ belongs to $\mathcal{O}_{X,x}^p$ if and only if $\partial/\partial x_i f = 0, \forall i$.

Hence the image of the differential operator $(\frac{\partial^n}{\partial x_1 \dots \partial x_n})^{p-1}$ consist of p -power only. It follows easily that any F -map $\sigma : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}^p$ is a differential operator of the form $(\frac{\partial^n}{\partial x_1 \dots \partial x_n})^{p-1} \circ \phi$, where $\phi \in \mathcal{O}_{X,x}$. Therefore $\mathcal{FM}_X$ is invertible. Moreover the previous theorem gives a formula for the transition functions of the line bundle define by $\mathcal{FM}_X$. Thus we get:

Theorem 5.2: (Metha-Ramanathan formula [MR]) *For a smooth variety X , we have $\mathcal{FM}_X \simeq \mathcal{K}_X^{\otimes(1-p)}$, where $\mathcal{K}_X$ is the sheaf of top differential forms on X .*

Let X be a variety, and let $\mathcal{L}$ be an invertible sheaf. Denote by $\mathcal{L}^p$ the image of p -power map $\Sigma_p : \mathcal{L} \rightarrow \mathcal{L}^{\otimes p}$. As X is reduced, the map $\Sigma_p : \mathcal{L} \rightarrow \mathcal{L}^p$ is a isomorphism of sheaves of abelian groups, and we have $\mathcal{L}^{\otimes p} = \mathcal{O}_X \otimes_{\mathcal{O}_X^p} \mathcal{L}^p$. Hence any F -map σ induces a map $\sigma_{\mathcal{L}} : \mathcal{L}^{\otimes p} \rightarrow \mathcal{L}^p$, namely $\sigma_{\mathcal{L}} = \sigma \otimes 1_{\mathcal{L}^p}$. We will also denote by $\sigma_{\mathcal{L}}^{1/p} : \mathcal{L}^{\otimes p} \rightarrow \mathcal{L}$ the map $\Sigma_p^{-1} \circ \sigma_{\mathcal{L}}$. Therefore $\sigma_{\mathcal{L}}$ is a $\mathcal{O}_X^p$ -morphism and we have $\sigma_{\mathcal{L}}^{1/p}(f^p x) = f \sigma_{\mathcal{L}}^{1/p}(x)$ for all $f \in \mathcal{O}_X, x \in \mathcal{L}^{\otimes p}$.

For a B -variety X , there are usually no non-trivial B -invariant F -maps. For example by Theorem 5.2, $FM(G/B) \simeq \nabla(2(p-1)\rho)$ and any B -invariant F -map on G/B is zero. Therefore we consider a weaker form of B -invariance, namely the notion of canonical F -maps. From this viewpoint the terminology "almost B -invariant F -maps" could have been better. However the terminology "canonical" is explained by Proposition 5.5: e.g. it is easy to show that G/B admits a *unique* canonical Frobenius splitting. A F -map σ is called *canonical* if and only if σ is H -invariant and $Ad(e_i^{(n)})(\sigma) = 0$ for any $i \in I$ and any $n \geq p$. Recall that the adjoint action Ad is defined in the context of Hopf algebras, i.e. $Ad(e_i^{(n)})(\sigma) = \sum_{a+b=n} (-1)^a e_i^{(a)} \circ \sigma \circ e_i^{(b)}$. One should not confuse Ad with the commutator: e.g. there are canonical F -maps σ such that $[\sigma, e_i^{(p+1)}] \neq 0$. However, we have:

Lemma 5.3: (Commutation relations [M2]) *Let X be a B -variety and let $\sigma \in FM(X)$. Then σ is canonical if and only if we have: $e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}$, for any $i \in I$, any $n > 0$ and any B -equivariant invertible sheaf $\mathcal{L}$.*

⁹ In the original framework [MR], the notion of Frobenius splittings is defined for any scheme, including non-reduced: the definition given here is varies from the original one, but it is essentially equivalent whenever the scheme is reduced (moreover a Frobenius split scheme is automatically reduced, see[Ra]). The present approach is intuitive and more adapted to the notion of canonical Frobenius splittings.

Proof: First assume that σ is canonical, and let $\mathcal{L}$ be any B -equivariant invertible sheaf. As any section of $\mathcal{L}^p$ is a p -power, we have $e_i \mathcal{L}^p = 0$. Moreover we have $e_i^{(m)} = e_i^{(pa)} e_i^{(b)}$, if $m = pa + b$ where $0 \leq b < p$. So we have: $e_i^{(m)} \circ \sigma_{\mathcal{L}} = 0$ for any m not divisible by p , and the hypothesis $Ad(e_i^{(pn)})(\sigma) = 0$ can be restated as:

$$\sum_{a+b=n} (-1)^a e_i^{(pa)} \circ \sigma_{\mathcal{L}} \circ e_i^{(pb)} = 0, \forall n > 0.$$

We will prove now the commutation relation $e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}$ by induction over $n > 0$. Consider first the case where n is a power of p . Then the previous relation can be written as:

$$e_i^{(pn)} \circ \sigma_{\mathcal{L}} - \sigma_{\mathcal{L}} \circ e_i^{(pn)} + \sum_{0 < a < n} (-1)^a e_i^{(pa)} \circ \sigma_{\mathcal{L}} \circ e_i^{(p(n-a))} = 0.$$

By the induction hypothesis, we have: $e_i^{(pa)} \circ \sigma_{\mathcal{L}} \circ e_i^{(pb)} = e_i^{(pa)} \circ e_i^{(pb)} \circ \sigma_{\mathcal{L}} = \binom{n}{a} e_i^{(pn)} \circ \sigma_{\mathcal{L}} = 0$, for any $a, b > 0$ with $a + b = n$; indeed as n is a power of p , $\binom{n}{a} = 0$ modulo p for all $a \neq 0, n$. Therefore $e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}$. When n is not a power of p , we can find a , with $0 < a < n$ such that $\binom{n}{a} \neq 0$ modulo p , hence by induction we get:

$$e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \binom{n}{a}^{-1} e_i^{(pa)} \circ e_i^{(p(n-a))} \circ \sigma_{\mathcal{L}} = \binom{n}{a}^{-1} \sigma_{\mathcal{L}} \circ e_i^{(pa)} \circ e_i^{(p(n-a))} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}.$$

Therefore the commutation relations are proved by induction.

Conversely, assume that the commutation relations hold. In particular they hold for the trivial line bundle $\mathcal{O}_X$, for which we have $\sigma_{\mathcal{L}} = \sigma$. By the same proof as before, one proves $Ad(e_i^{(n)})(\sigma) = 0$ whenever $n > 1$ is a power of p . For any $n \geq p$, there are two integers a, b with $a + b = n$, $b > 1$, b is a power of p and $\binom{n}{b} \neq 0$ modulo p . Hence $Ad(e_i^{(n)})(\sigma) = \binom{n}{b}^{-1} Ad(e_i^{(a)}) \circ Ad(e_i^{(b)})(\sigma) = 0$. Q.E.D.

Let us explain why a canonical F -map is "almost B -invariant":

Proposition 5.4: (M2)] *Let X be a B -variety, let $\mathcal{L}$ be a B -equivariant invertible sheaf and let σ be a canonical F -map. Then the induced map $\sigma_{\mathcal{L}}^{1/p} : \Gamma(X, \mathcal{L}^{\otimes p}) \rightarrow \Gamma(X, \mathcal{L})$ sends B -modules to B -modules.*

Proof: By definition, the map $\sigma_{\mathcal{L}}^{1/p} : \Gamma(X, \mathcal{L}^{\otimes p}) \rightarrow \Gamma(X, \mathcal{L})$ is the composition of $\sigma_{\mathcal{L}} : \Gamma(X, \mathcal{L}^{\otimes p}) \rightarrow \Gamma(X, \mathcal{L}^p)$ and of the inverse of $\Sigma_p : \Gamma(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L}^p)$. We have $e_i^{(pn)} \circ \Sigma_p = \Sigma_p \circ e_i^{(n)}$, for any $n \geq 0$ and any $i \in I$. From the commutation relations 5.3, we get $e_i^{(n)} \circ \sigma_{\mathcal{L}}^{1/p} = \sigma_{\mathcal{L}}^{1/p} \circ e_i^{(pn)}$. It follows that $\sigma_{\mathcal{L}}^{1/p}$ sends U -modules to U -modules. Moreover by H -invariance of σ , $\sigma_{\mathcal{L}}^{1/p}$ sends H -modules to H -modules. Hence $\sigma_{\mathcal{L}}$ sends B -modules to B -modules. Q.E.D.

Remark: If X is a G -variety, and if σ is canonical, it follows from $SL(2)$ -theory that we have $Ad(f_i^{(n)})(\sigma) = 0$ for all $i \in I$ and all $n \geq p$. Hence $\sigma_{\mathcal{L}}$ sends also G -modules to G -modules.

For any B -variety X , denote by $CFM(X)$ the set of canonical F -maps of X .

Proposition 5.5: (M2)] *Let X a B -variety. If X has a canonical Frobenius splitting, then $G \times_B X$ also admits a canonical Frobenius splitting. More precisely, there is a natural map $I_X : CFM(X) \rightarrow CFM(G \times_B X)$ which sends canonical Frobenius splittings of X to canonical Frobenius splittings of $G \times_B X$.*

Proof: As a matter of notations, we set $\omega = (p-1)\rho$, $St = \nabla(\omega)$, we will denote by u a non-zero element of $u(\mathfrak{u})$ of weight 2ω , and by v^- a non-zero lowest weight vector of

the Steinberg module St . Set $v^+ = u.v^-$. Then v^+ is a non-zero highest weight vector of St . The elements u and v^- are unique up to a scalar, but one can check that the map I_X defined below is independent of the choices of u and v^- .

There is a natural identification $\tau : CFM(X) \simeq Hom_B(St, FM(X) \otimes K(-\omega))$. Indeed St is the $\mathcal{H}_B$ -module generated by one vector of weight $-\omega$, namely v^- , and defined only by the relations $e_i^{(n)}.v^- = 0$ for any $i \in I$ and any $n \geq p$. Therefore any $\sigma \in CFM(X)$ induces the map $\tau(\sigma) : St \rightarrow FM(X) \otimes K(-\omega)$ which is the unique B -equivariant map sending v^- to $\sigma \otimes 1$.

Set $Y = G \times_B X$, let X_1 be the variety homeomorphic to X whose structural sheaf is $\mathcal{O}_X^p$ and set $Y_1 = G \times_B X_1$. Thus Y_1 is homeomorphic to Y and $\mathcal{O}_{Y_1} = \mathcal{O}_{G/B} \otimes_{\mathcal{O}_{G/B}^p} \mathcal{O}_Y^p$. Denote by $i : X_1 \rightarrow Y_1$ and $\pi : Y_1 \rightarrow G/B$ the natural maps. Set $\mathcal{L} = \pi^* \mathcal{L}(-\omega)$. As $\pi(X_1) = B/B$, the sheaf $i^* \mathcal{L}$ is B -isomorphic to $\mathcal{O}_X^p \otimes K(-\omega)$. More precisely, we identify St with the subspace $\Gamma(G/B, \mathcal{L}(-\omega))$ of $\Gamma(Y_1, \mathcal{L})$ and we normalize this isomorphism in a such way that the global section v^- of $\mathcal{L}$ goes to $1 \otimes 1$. Thus we have $i^* Hom_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L}) = \mathcal{FM}_X \otimes K(-\omega)$, and $\Gamma(Y_1, Hom_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L})) = D(FM(X) \otimes K(-\omega))$. Therefore, the map τ induces a map $D\tau : CFM(X) \rightarrow Hom_G(St, Hom_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L}))$.

For $\sigma \in CFM(X)$, set $\Theta_\sigma = D\tau(v^-)$. By definition, $\Theta_\sigma : \mathcal{O}_Y \rightarrow \mathcal{L}$ is a morphism of $\mathcal{O}_{Y_1}$ -modules and we have $\Theta_\sigma(1) = \sigma(1)v^-$, Θ_σ has weight $-\omega$ and we have $Ad(e_i^{(n)})(\Theta_\sigma) = 0$ for all $i \in I$ and all $n \geq p$. Next we define a $\mathcal{O}_Y^p$ -linear map $\Theta : \mathcal{L} \rightarrow \mathcal{O}_Y^p$. Using Andersen-Haboush formula 3.1, we have $\mathcal{L} = \mathcal{O}_Y^p \otimes St$. Hence $u.\mathcal{L} = \mathcal{O}_Y^p \otimes Kv^+$, and we define Θ by the requirement $u.s = \Theta(s) \otimes v^+$ for any section s of $\mathcal{L}$. As u is an highest weight vector of $u(\mathfrak{u})$, u is $Ad(U)$ -invariant and therefore the map Θ is U -equivariant, of weight ω . Moreover $\Theta(v^-) = 1$.

We define I_X by $I_X(\sigma) = \Theta \circ \Theta_\sigma$. By definition, $I_X(\sigma)$ is a canonical F -map of Y . Moreover $I_X(\sigma)(1) = \sigma(1)$, hence $I_X(\sigma)$ is a Frobenius splitting of $G \times_B X$ whenever σ is a Frobenius splitting of X . Q.E.D.

Let X be a variety, let Z be a subvariety of X , and let $\mathcal{I}_Z$ be the its defining ideal. By assumption, subvarieties are reduced, thus $\mathcal{I}_Z^p = \mathcal{I}_Z \cap \mathcal{O}_X^p$. Following [MR], we say that a F -map σ is *compatible* with Z if $\sigma(\mathcal{I}_Z) \subset \mathcal{I}_Z^p$. Therefore, a compatible F -map σ induces a F -map of Z .

Lemma 5.6: *Let X be a B -variety and let Z be a B -invariant subvariety. Let $\sigma \in CFM(X)$ be compatible with Z . Then $I_X(\sigma)$ is compatible with the subvariety $\overline{BwB} \times_B Z$ for any $w \in W$. Moreover its restriction to X is σ .*

Proof: First prove that $I_X(\sigma)$ is compatible with X . With the notations of the previous proof, this follows easily from the following two observations:

- (i) the map $\Theta_\sigma : \mathcal{O}_{G \times_B X} \rightarrow \mathcal{L}$ is a lifting of the map $\sigma : \mathcal{O}_X \rightarrow \mathcal{O}_X^p \simeq i^* \mathcal{L}$.
- (ii) the ideal $\mathcal{I}_X$ defining X in $G \times_B X$ is U -invariant and therefore it is invariant under the element $u \in u(\mathfrak{u})$.

Next prove the lemma by induction on w . Let $\mathcal{I}_w$ be the ideal defining $\overline{BwB} \times_B Z$. For $w = 1$, the assertion is already proved. For $w \neq 1$, write $w = s_i v$, where s_i is a simple reflexion and $v \leq w$, and let $\mathcal{H}_i$ be the subalgebra of $\mathcal{H}_G$ spanned by all $f_i^{(n)}$. By induction, we can assume that $I_X(\sigma)(\mathcal{I}_v) \subset \mathcal{I}_v^p$. However, it is clear that $\mathcal{I}_w$ is the biggest subsheaf of $\mathcal{I}_v$ stable by $\mathcal{H}_i$. Hence it follows from the commutation relations 5.3 that

$I_X(\sigma)(\mathcal{I}_w) \subset \mathcal{I}_w^p$. Thus $I_X(\sigma)$ is compatible with $\overline{BwB} \times_B Z$. Q.E.D.

Indeed the map I_X defined in Proposition 5.5 establishes a bijection between the canonical Frobenius splittings of X and the canonical Frobenius splittings of $G \times_B X$ which are compatible with X . In particular, we see that for any parabolic group $P \supset B$, σ induces a canonical Frobenius splitting of $P \times_B X$. Therefore, starting with the trivial Frobenius of the point B/B , we get by induction:

Corollary 5.7: *Any Demazure variety admits a unique canonical Frobenius splitting which is compatible to any Demazure subvariety.*

6. Tensor products of good filtrations. In this section, B -modules and G -modules are of arbitrary dimension. Let $E : P \rightarrow \mathbf{R}$ as in Section 4. We will use a different convention for the canonical filtrations defined in Section 4. For any B -module M , we denote by $\mathcal{F}_x M$ the biggest B -submodule whose any weight λ satisfies $E(\lambda) \leq x$ and set $\mathcal{F}_x^- M = \cup_{y < x} \mathcal{F}_y M$. For $\lambda \in P$, denote by $I(\lambda)$ the injective envelope of the B -module $K(\lambda)$: indeed, we have $I(\lambda) \simeq K[B/H] \otimes K(\lambda)$. We have $\mathcal{F}_x I(\lambda) = I(\lambda)$ if $E(\lambda) \leq x$ and $\mathcal{F}_x I(\lambda) = 0$ otherwise. Any injective B -module is a direct sum of $I(\lambda)$'s. Therefore for any injective B -module M , $\mathcal{F}_x M / \mathcal{F}_x^- M$ is again injective.

By definition, a $\mathcal{H}_G - B$ -module is a B -module M endowed with a compatible $\mathcal{H}_G$ -structure: the compatibility means that the $\mathcal{H}_B$ -structure comes from the B -action. Denote by M_{int} the subspace of all vectors $m \in M$ such that $\dim \mathcal{H}_G.m < \infty$. It is easy to prove that M_{int} is the biggest submodule on which the $\mathcal{H}_G$ -action integrates to a G -action.

Let $\lambda \in P$ and set $x = E(\lambda)$. Generalizing the definition of Section 4, we say that a B -module M is x -isotypical if λ is the unique weight of $H^0(U, M)$. If we identify B/H with the open B -orbit of G/B , we have $I(\lambda) \simeq H^0(B/H, \mathcal{L}(w_0\lambda))$, hence $I(\lambda)$ is a $\mathcal{H}_G - B$ -module. Indeed its structure of $\mathcal{H}_G - B$ -module is unique, as it is proved now:

Lemma 6.1: [M2]

(i) *Let M, N be two x -isotypical $\mathcal{H}_G - B$ -modules. Any B -morphism $L : M \rightarrow N$ is $\mathcal{H}_G$ -equivariant.*

(ii) *Any x -isotypical B -module admits at most one structure of $\mathcal{H}_G - B$ -module.*

(iii) *Let M be a $\mathcal{H}_G - B$ -module. Then $\mathcal{F}_x M$ and $\mathcal{F}_x^- M$ are $\mathcal{H}_G - B$ -submodules.*

Proof: Prove Assertion (i). Assume that L is not $\mathcal{H}_G$ -invariant. There is some $i \in I$ and $n > 0$ such that $Ad(f_i^{(n)})(L) \neq 0$. Moreover we can assume $Ad(f_i^{(m)})(L) = 0$ for any positive integer $m < n$. Set $L' = Ad(f_i^{(n)})(L)$ and let $k > 0$. The fact that $Ad(e_j^{(k)})(L') = 0$ is obvious for any $j \neq i$ and follows from $SL(2)$ -computation for $j = i$. Hence L' is U -invariant of weight $-n\alpha_i$, and $L'(M)$ is a B -submodule whose weights ν satisfy $E(\nu) \leq x - nE(\alpha_i) < x$. Therefore $L' = 0$ what means that L is $\mathcal{H}_G$ -invariant. Assertion (i) is proved. Assertion (ii) follows. The proof of Assertion (iii) is the same as the proof of Lemma 4.1. Q.E.D.

Theorem 6.2: (M2)] *Let X be a G -variety which admits a canonical Frobenius splitting. For any G -equivariant line bundle $\mathcal{L}$, the G -module $\Gamma(X, \mathcal{L})$ has a good filtration.*

Proof: First, we claim that we can make the following additional hypothesis: X contains a B -invariant dense open subset Ω such that $\Omega \simeq B \times_H Y$ for some H -variety Y . Set $X' = G/B \times X$ and $\mathcal{L}' = \mathcal{O}_{G/B} \otimes \mathcal{L}$. We have $X' \simeq G \times_B X$, and $\Omega = Bw_0B \times_B X$ is a

dense open subset of X' which is isomorphic to $B \times_H w_0 X$. Moreover, by Proposition 5.5, X' admits a canonical Frobenius splitting. Also, we have $\Gamma(X', \mathcal{L}') = \Gamma(X, \mathcal{L})$. Therefore using X' and $\mathcal{L}'$ instead of X and $\mathcal{L}$, we can assume that the additional hypothesis holds.

For $n \geq 0$, set $\overline{\mathcal{A}}_n = \Gamma(X, \mathcal{L}^{\otimes n})$, $\mathcal{A}_n = \Gamma(\Omega, \mathcal{L}^{\otimes n})$, $\overline{\mathcal{A}} = \bigoplus_{n \geq 0} \overline{\mathcal{A}}_n$ and $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$. Let $\lambda \in P^+$ and set $x = E(\lambda)$. It is clear that $\overline{\mathcal{A}}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx} \overline{\mathcal{A}}_n$ (respectively $\mathcal{A}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx} \mathcal{A}_n$) is a subalgebra of $\overline{\mathcal{A}}$ (respectively: a subalgebra of $\mathcal{A}$). Moreover $\overline{\mathcal{I}}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx}^- \overline{\mathcal{A}}_n$ (respectively $\mathcal{I}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx}^- \mathcal{A}_n$) is an ideal of $\overline{\mathcal{A}}_x$ (respectively: an ideal of $\mathcal{A}_x$). Thus set $\overline{A} = \overline{\mathcal{A}}_x / \overline{\mathcal{I}}_x$ and $A = \mathcal{A}_x / \mathcal{I}_x$. By Lemma 4.1, $\overline{\mathcal{I}}_x$ and $\overline{\mathcal{A}}_x$ are G -submodules of $\overline{\mathcal{A}}$. By Lemma 6.1, $\mathcal{I}_x$ and $\mathcal{A}_x$ are $\mathcal{H}_G - B$ -submodules of $\mathcal{A}$. Therefore $\overline{A}$ is an x -isotypical G -algebra and A is a $\mathcal{H}_G - B$ -subalgebra of the $\mathcal{H}_G - B$ -algebra $\mathcal{A}$.

Let σ be a canonical Frobenius splitting of X . Define a map $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ as follows: for $f \in \mathcal{A}_n$, set $\Theta(f) = \sigma_{\mathcal{L}^{\otimes n/p}}^{1/p}(f)$ if n is divisible by p and $\Theta(f) = 0$ otherwise. By Proposition 5.4, Θ sends B -modules to B -modules. As Θ is H -invariant, we have $\Theta(\mathcal{A}_x) \subset \mathcal{A}_x$ and $\Theta(\mathcal{I}_x) \subset \mathcal{I}_x$. Moreover by construction we have $\Theta(\overline{\mathcal{A}}) \subset \overline{\mathcal{A}}$. Hence Θ induces a map $\theta : A \rightarrow A$ such that:

- (i) $\theta(f^p) = f$, for all $f \in A$,
- (ii) $\theta(\overline{A}) \subset \overline{A}$.

We claim that $\overline{A}_1$ is a direct sum of $\nabla(\lambda)$. To prove it, we will apply Proposition 4.8 to the algebra $\overline{A}$. First note that $H^0(U, A)$ can be realized as a subalgebra of $H^0(U, \mathcal{A})$. Hence $H^0(U, A)$ is reduced and therefore $\overline{A}$ is reduced. Moreover A_n is a nx -isotypical injective B -module for any $n \geq 0$. By Lemma 6.1, the $\mathcal{H}_G - B$ -module A_n is a direct sum of $I(n\lambda)$. It is clear that $I(n\lambda)_{int} = \nabla(n\lambda)$. Therefore, the algebra $C_x \overline{A}$ (defined in Section 4) can be identified with the subalgebra A_{int} of A . Let $f \in C_x \overline{A} \subset A$ with $f^p \in \overline{A}$. Then $f = \theta(f^p)$ belongs to $\overline{A}$. Hence the claim follows Proposition 4.8.

As $x \in E(P^+)$ is any, $\overline{A}_1 = \mathcal{F}_x \Gamma(X, \mathcal{L}) / \mathcal{F}_x^- \Gamma(X, \mathcal{L})$ is one arbitrary subquotient of the canonical filtration of $\Gamma(X, \mathcal{L})$. Therefore $\Gamma(X, \mathcal{L})$ has a good filtration. Q.E.D.

The following statement has been proved in [Wa] for $G = SL(n)$, by a very long case-by-case analysis in [D1] for all groups, except E_7 and E_8 and characteristic 2. The general proof based on Frobenius splitting comes from [M2]. A nice account of it can be found in [Kn][vdK].

Corollary 6.3: *For any $\lambda, \mu \in P^+$, $\nabla(\lambda) \otimes \nabla(\mu)$ has a good filtration.*

Proof We have $\nabla(\lambda) \otimes \nabla(\mu) = \Gamma(G/B \times G/B, \mathcal{L})$, where $\mathcal{L} = \mathcal{L}(w_0 \lambda) \otimes \mathcal{L}(w_0 \mu)$. Moreover $G/B \times G/B \simeq G \times_B (G \times_B B/B)$, hence by Proposition 5.5, $G/B \times G/B$ admits a canonical splitting. Thus the proof follows from the previous theorem. Q.E.D.

7. Tilting modules. The notion of tilting objects has been invented by Ringel [Ri], in the context of quasi-hereditary algebras. However its first appearance for algebraic groups is due to Donkin [D2]. Let M be a G -module. A *Weyl filtration* of M is the dual of a good filtration. The G -module M is called *tilting* if M has both a good filtration and a Weyl filtration. Usually, these two filtrations are distinct.

Lemma 7.1: *Let M be a G -module.*

(i) *There exists an exact sequence $0 \rightarrow M \rightarrow G(M) \rightarrow W(M) \rightarrow 0$ such that $G(M)$ has a good filtration and $W(M)$ has a filtration by Weyl modules.*

(ii) Moreover, we can assume that the highest weights of M and $G(M)$ are the same and $M_\lambda = G(M)_\lambda$ for any highest weight λ of M .

Proof: Choose a linear form $E : P \rightarrow \mathbf{R}$ as in Section 4 and write accordingly $P^+ = \{\lambda_k | k \geq 0\}$, where $E(\lambda_0) < E(\lambda_1) < E(\lambda_2) \dots$. There is an integer $n > 0$ such that $E(\lambda) \leq E(\lambda_n)$ for any dominant weight μ of M . We define a sequence of modules $M = G_n(M) \subset G_{n-1}(M) \subset \dots \subset G_0(M)$ by decreasing induction as follows. Assume that $G_r(M)$ is already defined. Then set $m_{r-1} = \dim \text{Ext}_G^1(\Delta(\lambda_{r-1}), G_r(M))$, and define $G_{r-1}(M)$ by choosing any exact sequence:

$$0 \rightarrow G_r(M) \rightarrow G_{r-1}(M) \rightarrow \Delta(\lambda_{r-1})^{m_{r-1}} \rightarrow 0,$$

such that, the connecting homomorphism:

$$\delta : \text{Hom}_G(\Delta(\lambda_{r-1}), \Delta(\lambda_{r-1})^{m_{r-1}}) \rightarrow \text{Ext}_G^1(\Delta(\lambda_{r-1}), G_r(M))$$

is an isomorphism.

We claim $\text{Ext}_G^1(\Delta(\lambda_s), G_r(M)) = 0$, for all $s \geq r$. First consider the case $r = n$: by Lemma 4.4, $\text{Ext}_G^1(\Delta(\lambda_s), G_n(M)) = 0$ for any $s \geq n$. Then the proof runs by decreasing induction over r : this follows from $\text{Ext}_G^1(\Delta(\lambda_s), \Delta(\lambda_r)) = 0$ for any $s \geq r$ (by Lemma 4.4) and the fact that the connecting homomorphism δ is bijective.

Set $G(M) = G_0(M)$. By the previous claim $\text{Ext}_G^1(\Delta(\lambda), G(M)) = 0$ for any $\lambda \in P^+$. Therefore by Corollary 4.7, the G -module $G(M) = G_0(M)$ has a good filtration. By its definition, the G -module $W(M) = M/G(M)$ has a Weyl filtration, hence Assertion (i) is proved. Moreover, it follows by induction from Lemma 4.4 that for any weight ν of $W(M)$, we have $\nu < \mu$ for some weight μ of M . Hence Assertion (ii) hold. Q.E.D.

Theorem 7.2: (Donkin [D2])

(i) For any $\lambda \in P^+$, there is a unique indecomposable tilting module $T(\lambda)$ such that $\dim T(\lambda)_\lambda = 1$ and any weight of $T(\lambda)$ is in the convex hull of $W\lambda$.

(ii) $T(\lambda) \simeq T(\mu)$ if and only if $\lambda = \mu$,

(iii) Any indecomposable tilting module is isomorphic to some $T(\lambda)$.

For any G -module M , denote by $T(M)$ the image of the composite map: $H^0(U, M) \rightarrow M \rightarrow H_0(U^-, M)$. As $T(M)$ is an H -module, we will denote by $T_\lambda(M)$ its weight spaces. Set $t(M, \lambda) = \dim T_\lambda(M)$ for any $\lambda \in P^+$.

Lemma 7.3: ([MP1]) Let M be a tilting module. Then $M \simeq \bigoplus_{\lambda \in P^+} T(\lambda)^{t(M, \lambda)}$.

Proofs: It will be more convenient to prove Theorem 7.2 and Lemma 7.3 together. By Lemma 7.1, there is an exact sequence:

$$0 \rightarrow \Delta(\lambda) \rightarrow G(\Delta(\lambda)) \rightarrow W(\Delta(\lambda)) \rightarrow 0,$$

such that $G(\Delta(\lambda))$ has a good filtration, $W(\Delta(\lambda))$ has a Weyl filtration and any weight of $W(\Delta(\lambda))$ is $< \lambda$. Thus $G(\Delta(\lambda))$ contains a unique indecomposable direct summand $T(\lambda)$ with $\dim T(\lambda)_\lambda = 1$. Note that $G(\Delta(\lambda))$ has also a Weyl filtration. Hence $G(\Delta(\lambda))$ is tilting and by Corollary 4.7 its direct summand $T(\lambda)$ is tilting. Thus the existence of $T(\lambda)$ is proved. Note that Assertion (ii) is obvious.

Make further remarks about the module $T(\lambda)$. The last quotient of its the canonical filtration is $\nabla(\lambda)$, hence there is an exact sequence:

$$0 \rightarrow X \rightarrow T(\lambda) \rightarrow \nabla(\lambda) \rightarrow 0,$$

where X has a good filtration. Similarly, we can find an exact sequence:

$$0 \rightarrow \Delta(\lambda) \rightarrow T(\lambda) \rightarrow Y \rightarrow 0,$$

where Y has a Weyl filtration.

We will now prove together the remaining statements. Let M be an indecomposable tilting module. For any highest weight μ of M , we have $M_\mu \simeq T_\mu(M)$. Therefore $T(M) \neq 0$. Let λ be any weight of $T(M)$, and choose $v \in H^0(U, M)_\lambda$ whose image in $T(M)$ is not zero. There is a B^- -equivariant morphism $L : M \rightarrow K(\lambda)$ with $L(v) = 1$. Denote by the same symbol v_λ some highest weight vector of $\nabla(\lambda)$, $\Delta(\lambda)$ and $T(\lambda)$. By the universal property of Weyl modules, there is a map $\psi_1 : \Delta(\lambda) \rightarrow M$ sending v_λ to v . Similarly, there is a map $\psi_2 : M \rightarrow \nabla(\lambda)$ such that $\psi_2(m) = L(m)v_\lambda$ for any $m \in M_\lambda$. Since M has a good filtration, we have $\text{Ext}_G^1(T(\lambda)/\Delta(\lambda), M) = 0$ (by Corollary 4.7). Thus, the map ψ_1 can be extended to a map $\phi_1 : T(\lambda) \rightarrow M$. In the same way, the map ψ_2 can be lifted to a map $\phi_2 : M \rightarrow T(\lambda)$. So we get the following commutative diagram:

$$\begin{array}{ccccc} & & T(\lambda) & & \\ & & \downarrow \nearrow \phi_2 & & \\ \Delta(\lambda) & \longrightarrow & M & \longrightarrow & \nabla(\lambda) \\ & \downarrow \nearrow \phi_1 & & & \\ & T(\lambda) & & & \end{array}$$

By definition, we have $\phi_2 \circ \phi_1(v_\lambda) = v_\lambda$. Therefore, $\phi_2 \circ \phi_1$ is a non nilpotent endomorphism of the indecomposable module $T(\lambda)$. By Fitting's Lemma, $\phi_2 \circ \phi_1$ is invertible. Thus, $T(\lambda)$ is a direct factor of M and so we have $M \simeq T(\lambda)$. Therefore Assertion (iii) of the theorem is proved.

If μ is another weight of $T(M)$, we get $T(\mu) \simeq M \simeq T(\lambda)$, and therefore $\lambda = \mu$. Thus λ is the unique weight of $T(T(\lambda))$ and its multiplicity is 1. Therefore Lemma 7.3 is proved when M is indecomposable, and the general case follows. Q.E.D.

8. The functor T^Γ . In view of the next section, we recall some simple facts about the modular representation theory of an abstract group Γ . Let $\text{Mod}(\Gamma)$ be the category of finite dimensional $K\Gamma$ -modules. For $M \in \text{Mod}(\Gamma)$, there are natural maps $H^0(\Gamma, M) \rightarrow M$ and $M \rightarrow H_0(\Gamma, M)$. We denote by $T^\Gamma(M)$ the image of the composite map $H^0(\Gamma, M) \rightarrow H_0(\Gamma, M)$. (When $\Gamma = G$ and M is a rational representation of G , we have $T^G(M) = T_0(M)$.) Similarly for $M, N \in \text{Mod}(\Gamma)$, set $T^\Gamma(M, N) = T^\Gamma(\text{Hom}(M, N))$. For $M, N \in \text{Mod}(\Gamma)$, the natural map $H^0(\Gamma, M) \otimes H^0(\Gamma, N) \rightarrow H^0(\Gamma, M \otimes N)$ induces a natural map $T^\Gamma(M) \otimes T^\Gamma(N) \rightarrow T^\Gamma(M \otimes N)$. This elementary property implies that the functor $T^\Gamma(-, -)$ is composable: given $M, N, L \in \text{Mod}(\Gamma)$, there is a natural map $T^\Gamma(M, N) \times T^\Gamma(N, L) \rightarrow T^\Gamma(M, L)$. Therefore $T^\Gamma(M)$ is an algebra.

Lemma 8.1: [Be] *Let M, N be two indecomposable Γ -modules. If M and N are isomorphic and their dimensions is not divisible by p , we have $T^\Gamma(M, N) = K$. Otherwise we have $T^\Gamma(M, N) = 0$.*

Proof: The dual of the Γ -module $\text{Hom}(M, N)$ is $\text{Hom}(N, M)$, and the duality pairing is given by $x, y \in \text{Hom}(M, N) \times \text{Hom}(N, M) \mapsto \text{Tr } yx$, where Tr denotes the trace. Therefore the image of any $x \in \text{Hom}_\Gamma(M, N)$ in $T^\Gamma(M, N)$ is zero if and only if $\text{Tr } yx = 0$ for any

$y \in \text{Hom}_\Gamma(N, M)$. As M is indecomposable, any Γ -endomorphism z of M has a unique eigenvalue, say $\lambda(z)$. Thus we get $\text{Tr } yx = \lambda(yx) \dim M$. Therefore we have $T^\Gamma(M, N) = 0$ if M and N are not isomorphic or if $\dim M$ is divisible by p .

Furthermore the nilpotent radical of the K -algebra $\text{End}_\Gamma(M)$ is the kernel of λ , and we have $\lambda(xy) = \lambda(x)\lambda(y)$, for any $x, y \in \text{End}_\Gamma(M)$. Thus $T^\Gamma(M, M) = \text{End}_K(M)/\text{Ker}(\lambda) \simeq K$ if the dimension of M is not divisible by p . Q.E.D.

For any positive integer n , denote by $M_n(K)$ the K -algebra of n by n matrices.

Corollary 8.2: *Let $M \in \text{Mod}(\Gamma)$ and let $M = \bigoplus P^{m_P}$ be its decomposition into indecomposable modules. Then $T^\Gamma(M, M)$ is a matrix algebra.*

More precisely, we have $T^\Gamma(M, M) = \bigoplus_P M_{m_P}(K)$, where P runs over all indecomposable Γ -modules with $\dim P \neq 0 \pmod{p}$.

Let $\mathcal{C}$ be a full subcategory of $\text{Mod}(\Gamma)$. Define a new category $\mathcal{C}_T$ by the following requirements.

(i) The objects of $\mathcal{C}_T$ are the objects of $\mathcal{C}$.

(ii) The space of morphisms between two objects M and N is $T^\Gamma(M, N)$.

By definition, $\mathcal{C}$ is called a *Karoubi category* if it is additive and subtractive, i.e. for any Γ -modules L, M, N with $L \simeq M \oplus N$, we have:

$$(L \in \mathcal{C}) \text{ if and only if } (M, N \in \mathcal{C}).$$

Lemma 8.3: *Let $\mathcal{C}$ be a full Karoubi subcategory of $\text{Mod}(\Gamma)$. Then $\mathcal{C}_T$ is abelian and semi-simple. Moreover its simple objects are the indecomposable modules of $\mathcal{C}$ whose dimension is not divisible by p .*

The lemma follows from the previous corollary. Assume now that $\mathcal{C}$ is stable by tensor product. Then the category $\mathcal{C}_T$ is endowed with a tensor product and $K_0(\mathcal{C}_T)$ is a ring.

Lemma 8.4: [GM2] *Let $\mathcal{C}$ be a full Karoubi subcategory of $\text{Mod}(\Gamma)$, stable by tensor product. Then the ring $K_0(\mathcal{C}_T)$ is reduced.*

Proof: In order to prove the lemma, we can assume that $\mathcal{C} = \text{Mod}(\Gamma)$. Therefore we can assume that $\mathcal{C}$ is stable by duality. By Lemma 8.3, any $x \in K_0(\mathcal{C}_T)$ can be uniquely written as $\sum_{S \in \mathcal{S}} m_S [S]$, where $\mathcal{S}$ is the set of isomorphism classes of indecomposable $K\Gamma$ -modules of dimension not divisible by p . Let $I(x) = m_K$ be the multiplicity of the trivial representation in x . When y represents an effective $K\Gamma$ -module M , we have $I(y) = \dim T^\Gamma(M)$. Hence by Lemma 8.1, we have $I(x.x^*) = \sum_{S \in \mathcal{S}} m_S^2$. Thus we have $I(x.x^*) > 0$ for any $x \neq 0$.

Let $x \in K_0(\mathcal{C}_T)$ be non-zero and set $z = x.x^*$. As $I(z) > 0$, we get $z \neq 0$, hence $I(z.z^*) > 0$, hence $z.z^* \neq 0$. Therefore $x^2.(x^*)^2 \neq 0$ and $x^2 \neq 0$. Thus the ring $K_0(\mathcal{C}_T)$ is reduced. Q.E.D.

Less formally, the compatibility of the functors T^Γ and $\otimes$ can be stated as follows:

Lemma 8.5: [Be] *Let $M, N \in \text{Mod}(\Gamma)$. Assume that M is indecomposable and its dimension is divisible by p . Then the dimension of any direct summand of $M \otimes N$ is divisible by p .*

Proof: By lemma 8.1, the ring $T^\Gamma(M, M)$ is zero, i.e. $1_M = 0_M$ in $T^\Gamma(M, M)$. Hence $1_{M \otimes N} = 1_M \otimes 1_N = 0$, i.e. $T^\Gamma(M \otimes N, M \otimes N) = 0$. Hence by Corollary 8.2, the dimension of any direct summand of $M \otimes N$ is divisible by p .

Remark: The functor T can be defined for any Hopf algebra $\mathcal{H}$ over an arbitrary field K , and it is interesting when $\mathcal{H}$ admits non-zero representations of zero 'dimension', where the 'dimension' is defined in the setting of Hopf algebras. In characteristic zero, this occurs only for non-commutative Hopf algebras. Here is an example which has not been investigated. Let $\mathfrak{g}$ be a Lie super-algebra over $\mathbf{C}$. The *superdimension* of $\mathfrak{g}$ -module M is $\dim M_0 - \dim M_1$. The analog of Lemma 8.5 can be stated as follows: for any indecomposable $\mathfrak{g}$ -module M of superdimension 0, any summand of $M \otimes N$ has superdimension 0, for any other $\mathfrak{g}$ -module N . It turns out that for a classical Lie superalgebra $\mathfrak{g}$, the simple representations L of superdimension 0 are the most atypical (so a priori the most complicated). Therefore, it seems interesting to investigate similar questions for Lie superalgebras in order to understand these very atypical representations. For other characteristic zero examples see e.g. [A6][GM1][AP].

9. The Verlinde's formula and the modular Verlinde's formula. Most of the analogies of this section are borrowed from [GM1]. Let $G_{\mathbf{C}}$ be a connected simply connected algebraic group over $\mathbf{C}$, let $\mathfrak{g}_{\mathbf{C}}$ be its Lie algebra and let Δ be its root system. Normalize the invariant bilinear form κ of $\mathfrak{g}_{\mathbf{C}}$ by the following requirement: $\kappa(h_{\alpha}, h_{\alpha}) = 2$ for any short root α (i.e. for any root if $\mathfrak{g}_{\mathbf{C}}$ belongs to the *ADE* series). Set $\mathcal{O} = \mathbf{C}((t))$, let $\mathbf{K} = \mathbf{C}((t))$ be its quotient field and let $m = t \cdot \mathbf{C}[[t]]$ be its maximal ideal. The loop algebra $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ is the central extension of $\mathfrak{g}_{\mathbf{C}} \otimes \mathbf{K}$ defined by the 2-cocycle $(x_1 \otimes f_1, x_2 \otimes f_2) \mapsto \kappa(x_1, x_2) \text{Res}_0 f_1 df_2$, for any $x_i \in \mathfrak{g}_{\mathbf{C}}$, $f_i \in \mathbf{K}$ (where 0 denotes the closed point of $\text{Spec } \mathcal{O}$). The central element of $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ is denoted by c and set $\mathcal{L}^+ = \mathfrak{g}_{\mathbf{C}} \otimes m$. A $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -module M is called:

*smooth*¹⁰ if $\mathcal{L}^+$ acts locally nilpotently,

integrable if $x \otimes f$ acts locally nilpotently, for any ad-nilpotent $x \in \mathfrak{g}_{\mathbf{C}}$ and any $f \in \mathbf{K}$,

of level l if c acts as the scalar l .

For a non-negative integer l , let $\mathcal{O}_l^{\text{int}}$ be the category of smooth integrable level l $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -modules of finite length. Denote by h'_0 the highest short coroot, i.e. $h'_0 = h_{\beta}$ where β is the highest root. For $\mathfrak{g}_{\mathbf{C}}$ of type *ADE*, $h'_0 = h_0$ is the highest coroot. However for $\mathfrak{g}_{\mathbf{C}}$ of type *BCFG*, $h'_0 \neq h_0$: for example $h_0 = h_1 + 2h_2$ and $h'_0 = h_1 + h_2$ if $\mathfrak{g}_{\mathbf{C}}$ is of type B_2 . Set $P_l^+ = \{\lambda \in P_l^+ | \lambda(h'_0) \leq l\}$. Note that $h'_0 = \sum_{1 \leq i \leq l} m_i h_i$, where all m_i are positive integers, hence P_l^+ is finite.

Theorem 9.1: (Kac) [Ka]

(i) Let $M \in \mathcal{O}_l^{\text{int}}$ be a simple $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -module. Then $H^0(\mathcal{L}^+, M)$ is a simple finite dimensional $\mathfrak{g}_{\mathbf{C}}$ -module, and its highest weight $\lambda = \lambda(M)$ belongs to P_l^+ .

(ii) The map $M \mapsto \lambda(M)$ is a bijection from simple modules of $\mathcal{O}_l^{\text{int}}$ to P_l^+ .

For $\lambda \in P^+$, denote by $L_{\mathbf{C}}(\lambda)$ the simple $\mathfrak{g}_{\mathbf{C}}$ -module whose highest weight is λ . For $\lambda \in P_l^+$, denote by $L_l(\lambda)$ the simple $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -module such that $H^0(\mathcal{L}^+, L_l(\lambda)) \simeq L_{\mathbf{C}}(\lambda)$.

Theorem 9.2: (Deodhar, Gabber, Kac [DGK]) The category $\mathcal{O}_l^{\text{int}}$ is semi-simple.

¹⁰ the terminology "smooth" is used in [KL] by analogy with representation theory of p -adic groups. However, the terminology "level" has no connection.

For any $\lambda, \mu, \nu \in P^+$, denote by $K_{\lambda\mu}^\nu$ the classical tensor product multiplicities which are defined by: $L_C(\lambda) \otimes L_C(\mu) \simeq \sum_{\nu \in P^+} L_C(\nu)^{K_{\lambda\mu}^\nu}$. They are explicitly determined by Kostant's formula.

Following [MS], we would like to define some modified tensor product multiplicities $V_{\lambda\mu}^{\nu'}$, which are defined for all $\lambda, \mu, \nu \in P_l^+$. However, for any $M, N \in \mathcal{O}_l^{int}$, their tensor product $M \otimes N$ has level $2l$ and the multiplicities of its simple components are usually infinite. Therefore the ordinary tensor product cannot be used to define $V_{\lambda\mu}^{\nu'}$. Indeed, Moore and Seiberg [MS] underlined the possibility to define a new tensor product $\otimes$ in a such way that

(i) $M \otimes N \in \mathcal{O}_l^{int}$,

(ii) the tensor product multiplicities $V_{\lambda\mu}^{\nu'}$ are given by Verlinde's formula [V].

More precisely, in the axiomatic of Moore and Seiberg, we have

$$L_l(\lambda) \otimes L_l(\mu) = \sum_{\nu \in P_l^+} L_l(\nu)^{V_{\lambda\mu}^{\nu'}},$$

where the constant $V_{\lambda\mu}^{\nu'}$ are given by the formula

$$V_{\lambda\mu}^{\nu'} = \sum_{w \in W'_{aff}} \epsilon(w) K_{\lambda\mu}^{w(\nu+\rho)-\rho}.$$

The meaning of symbols used in the previous formula is now explained. The integer h' is the dual Coxeter number $\rho(h'_0) + 1$. The group W'_{aff} is the dual affine Weyl group, generated by the linear reflexion $s_i \in W$ and by the additional affine reflexion s'_0 relative to the hyperplane $\lambda(h'_0) = l + h'$. The multiplicities $K_{\lambda\mu}^\nu$ have been previously defined for $\nu \in P^+$ and we set $K_{\lambda\mu}^\nu = 0$ if $\nu \notin P^+$.

By definition, the *fusion ring* is $K_0(\mathcal{O}_l^{int})$, where the product is induced by the modified tensor product $\otimes$. It follows from Kac's theorem 9.1 that the fusion ring is the free $\mathbf{Z}$ -module with a basis $[L_l(\lambda)]$ indexed by $\lambda \in P_l^+$ and whose structure constants are $V_{\lambda\mu}^{\nu'}$. In addition, Seiberg and Moore described a series of axioms satisfied by the tensor category $(\mathcal{O}_l^{int}, \otimes)$. We will now explain their combinatorial consequences for the structure constants $V_{\lambda\mu}^{\nu'}$. First, the existence of $\otimes$ implies that $[L_l(\lambda)][L_l(\lambda)]$ is an effective representation, hence we have:

$$(AX1) \quad V_{\lambda\mu}^{\nu'} \geq 0.$$

The next axiom is the fact that $\otimes$ is somehow commutative and associative. The associativity axiom implies that the fusion ring is associative, or equivalently the combinatorial identity holds:

$$(AX2) \quad \sum_{\nu \in P_l^+} V_{\lambda\mu}^{\nu'} V_{\nu\pi}^{\sigma'} = \sum_{\nu \in P_l^+} V_{\lambda\nu}^{\sigma'} V_{\mu\pi}^{\nu'}, \text{ for any } \lambda, \mu, \pi, \sigma \in P_l^+.$$

Another axiom is the existence of duals in $\mathcal{O}_l^{int}$, and the dual of $L_l(\lambda)$ is $L_l(-w_0(\lambda))$. Hence:

$$(AX3) \quad \text{the map } \lambda, \mu, \nu \mapsto V_{\lambda\mu}^{\nu'-w_0(\nu)} \text{ is a symmetric function into the three arguments.}$$

$$(AX4) \quad \text{the fusion ring is reduced.}$$

Following [MS], the space $Hom_{\mathcal{O}_l^{int}}(L_l(\nu), L_l(\lambda) \otimes L_l(\mu))$ should be described as a space of fields, which are uniquely determined by their residue valued in $Hom_{GC}(L(\nu), L(\lambda) \otimes L(\mu))$. The injectivity of the residue map implies the inequality:

$$(AX5) \quad V_{\mu,\lambda}^{\nu'} \leq K_{\lambda\mu}^\nu.$$

In what follows, we will explain how to define a category $\mathcal{P}$ in the setting of modular representations of the Chevalley group G , following [GM1][GM2]. This category satisfies

the previous axioms (AX1 – 5) of $\mathcal{O}_{p-h}^{int}$, the main differences are as follows:

- (i) the category $\mathcal{P}$ is K -linear, where K is a field of characteristic p (instead of being $\mathbb{C}$ -linear),
- (ii) Morphisms are modified (instead of being the usual one),
- (iii) The tensor product is the ordinary tensor product (instead of being modified),
- (iv) The structure constants of the tensor product involves the affine Weyl group (instead of the dual affine Weyl group).

Assume that $p \leq h$, where $h = \rho(h_0) + 1$. Define the category $\mathcal{P}$ by $\mathcal{P} = \mathcal{T}_T$ (see Section 8), where $\mathcal{T}$ is the category of tilting modules. By Corollary 4.7, the category $\mathcal{T}$ is a Karoubi category, hence by Lemma 8.3, $\mathcal{P}$ is a semi-simple abelian category. In order to explain why the category $\mathcal{P}$ satisfy the previous axioms, we first state a lemma, whose proof will be postponed until the next section. Set $C^0 = \{\lambda \in P^+ | (\lambda + \rho)(h_0) < p\}$.

Lemma 9.3: ([GM1][GM2]) *Assume $p \geq h$. Let $\lambda \in P^+$.*

- (i) *If $\lambda \in C^0$, then $T(\lambda) = \nabla(\lambda)$ and its dimension is not divisible by p .*
- (ii) *Otherwise, the dimension of $T(\lambda)$ is divisible by p .*

It follows from lemmas 8.3 and 9.3 that $K_0(\mathcal{P})$ has basis $([\nabla(\lambda)])_{\lambda \in C^0}$. For a group G of type ADE , we have $C^0 = P_{p-h}^+$, and the index set of the bases of $K_0(\mathcal{P})$ and of the fusion ring $K_0(\mathcal{O}_i^{int})$ are the same. It follows from Corollary 6.3 that $\mathcal{P}$ is a tensor category. For $\lambda, \mu, \nu \in C^0$, the tensor product multiplicity $V_{\lambda\mu}^\nu$ are the structure constants of the ring $K_0(\mathcal{P})$.

Now we will check that the multiplicities $V_{\lambda\mu}^\nu$ satisfy the previous axioms. By definition we have $\nabla(\lambda) \otimes \nabla(\mu) = \oplus_{\nu \in C^0} V_{\lambda\mu}^\nu \nabla(\nu) \oplus T$, where T is a sum of indecomposable modules of dimension divisible by p . Hence by definition, we get $V_{\lambda\mu}^\nu \geq 0$ (AX1). Similarly, the associativity of the constant structure $V_{\lambda\mu}^\nu$ comes from the fact that the tensor product of $\mathcal{P}$ is truly associative (see (AX2)). We have $V_{\lambda\mu}^\nu = \dim T^G(\nabla(\nu), \nabla(\lambda) \otimes \nabla(\mu))$. However, we have $T(\nu)^* = T(-w_0(\nu))$, hence we get:

$V_{\lambda\mu}^{-w_0(\nu)} = \dim T^G(\nabla(\nu) \otimes \nabla(\lambda) \otimes \nabla(\mu))$, and $V_{\lambda\mu}^{-w_0(\nu)}$ is symmetric into the three arguments (AX3). The fact that the ring $K_0(\mathcal{P})$ is reduced comes from Lemma 8.4, therefore $K_0(\mathcal{P})$ satisfies (AX4). To check the last axiom (AX5), we prove:

Lemma 9.4: *For any $\lambda, \mu, \nu \in C^0$, we have $V_{\lambda\mu}^\nu \leq K_{\lambda\mu}^\nu$.*

Proof: Set $M = \nabla(\lambda) \otimes \nabla(\mu)$. It follows from the definition of T^G that $V_{\lambda\mu}^\nu \leq \dim \text{Hom}_G(\Delta(\nu), M)$, thus we get $V_{\lambda\mu}^\nu \leq h^0(M, \nu)$. By Corollary 6.3, M has a good filtration. By Lemma 4.2, one gets $\text{ch } M = \sum_{\pi \in P^+} h^0(M, \pi) \text{ch } \nabla(\pi)$. In particular, the numbers $h^0(M, \pi)$ are independent of the characteristic. Thus $h^0(M, \nu) = K_{\lambda\mu}^\nu$, and the lemma is proved. Q.E.D.

We still have to compute the number $V_{\lambda\mu}^\nu$ to show the complete similarity between the category $\mathcal{P}$ and the fusion category $\mathcal{O}_{p-h}^{int}$. Indeed we have:

Theorem 9.5: (modular Verlinde's formula [GM1][GM2]): *Assume $p \geq h$. For any $\lambda, \mu, \nu \in C^0$, we have: $V_{\lambda\mu}^\nu = \sum_{w \in W_{aff}} \epsilon(w) K_{\lambda\mu}^{w \cdot (\nu + \rho) - \rho}$.*

The following theorem was proved in [GM1][GM2]. However, we will give a simpler proof in the next section. Indeed the hypothesis $p \geq h$ is useless: however if $p < h$, the set C^0 is empty.

Also, it is easy to realize the ring $K_0(\mathcal{P})$ as a quotient of $K_0(G)$. For any additive category $\mathcal{A}$, denote by $K'_0(\mathcal{A})$ the group generated by the symbols $([M])_{M \in \mathcal{A}}$ submitted to the relations $[M] + [N] - [L]$ whenever $L \simeq M \oplus N$. It follows from Donkin's theorem 4.2 that $K'_0(\mathcal{T}) = \mathbb{Z}[P]^W = K_0(G_{\mathbb{C}})$, where $G_{\mathbb{C}}$ is the simply connected complex group of same type as G . Using the natural morphism $K'_0(\mathcal{T}) \rightarrow K_0(\mathcal{P})$, we get the epimorphism $K_0(G_{\mathbb{C}}) \rightarrow K_0(\mathcal{P})$. As $K_0(\mathcal{P})$ is reduced, it is enough to determine the spectrum of $\mathbb{C} \otimes K_0(\mathcal{P})$, as a subset of $\text{Spec}(\mathbb{C} \otimes K_0(G_{\mathbb{C}}))$. Recall that $\text{Spec}(\mathbb{C} \otimes K_0(G_{\mathbb{C}}))$ is exactly the set of conjugacy classes of semi-simple elements of $G_{\mathbb{C}}$. Denote by $Z^{1/p}$ the set of all regular semi-simple conjugacy classes $[g]$ in $G_{\mathbb{C}}$ such that g^p is central.

Theorem 9.6: ([GM1][GM2]) *Assume $p \geq h$. Then $\text{Spec}(\mathbb{C} \otimes K_0(\mathcal{P})) = Z^{1/p}$.*

Remark: The remarkable paper of Moore and Sciberg [MS] describes the axioms of the tensor product category $\mathcal{O}_l^{int}$. However, their papers did not provide a rigorous mathematical proof of the existence of a tensor product, and at that time there were no rigorous mathematical approach of it. Kazhdan and Lusztig [KL] introduced a tensor product on a category of representation of affine Lie algebras. It was very surprising that these authors use a category of negative level representation instead of $\mathcal{O}_l^{int}$ (see also [HL] for a vertex algebra approach). Using the approach of Kazhdan and Lusztig, Finkelberg [F] provided a tensor product on $\mathcal{O}_l^{int}$, and he gave another approach of the Verlinde's formula for quantum groups. See [Sor] for a recent account of Verlinde's formula in the context of algebraic geometry.

10. Proof of the modular Verlinde's formula. The proof given here is a bit simpler than the original proof (and a bit more precise). Indeed in [GM1][GM2], we were not aware of Jantzen's lemma 10.2, which allows some simplifications.

A weight λ is called *p-singular* if $(\lambda + \rho)(h_\alpha) = 0$ modulo p for some root α . Otherwise, λ is called *p-regular*. A G -module M is called *regular* (respectively *singular*) if the highest weight of any simple subquotient of M is *p-regular* (respectively *p-singular*). By the linkage principle 3.4, a block of $\text{Mod}(G)$ contains only regular modules or only singular modules. Thus any G -module M can be decomposed as $M = M_{reg} \oplus M_{sing}$, where M_{reg} is regular and M_{sing} is singular. Accordingly, we can split the category $\text{Mod}(G)$ as $\text{Mod}_{reg}(G) \oplus \text{Mod}_{sing}(G)$.

Lemma 10.1: *Assume $p \geq h$. For any singular module M , $\dim M = 0$ modulo p .*

Proof: Let $\lambda \in P^+$ be a *p-singular* weight. By Weyl dimension formula, we have $\dim \nabla(\lambda) = \prod_{\alpha \in \Delta^+} (\lambda + \rho)(h_\alpha) / \prod_{\alpha \in \Delta^+} \rho(h_\alpha)$. We have $\rho(h_\alpha) \leq h - 1$ for any root $\alpha \in \Delta^+$, hence the denominator is not divisible by p , but the numerator is divisible by p , thus $\dim \nabla(\lambda)$ is divisible by p . By Linkage Principle 3.4, the character of M is a linear combination of $\text{ch } \nabla(\lambda)$ where λ is *p-singular*. Therefore its dimension is divisible by p . Q.E.D.

Set $P_{\mathbf{R}} = \mathbf{R} \otimes P$, set $P_{\mathbf{R}}^{reg} = \{\lambda \in P_{\mathbf{R}} | (\lambda + \rho)(h_\alpha) \notin p\mathbb{Z}, \forall \alpha \in \Delta^+\}$ and let $P_{\mathbf{R}}^{sub}$ be the set of all $\lambda \in P_{\mathbf{R}}$ such that $(\lambda + \rho)(h_\alpha) \in p\mathbb{Z}$ for exactly one $\alpha \in \Delta^+$. The connected components of $P_{\mathbf{R}}^{reg}$ are called the *alcoves*, those of $P_{\mathbf{R}}^{sub}$ are called the *walls*. The *fundamental alcove* is the alcove $C_{fund} = \{\lambda \in P_{\mathbf{R}} | (\lambda + \rho)(h_0) < p \text{ and } (\lambda + \rho)(h_i) > 0, \forall i \in I\}$. Therefore $C^0 = C_{fund} \cap P$. The *fundamental walls* are the walls contained in

$\overline{C_{fund}}$. Thus there are $l + 1$ fundamental walls, namely $(F_i)_{i \in I \cup \{0\}}$, which are defined as follows:

$$F_0 = \{\lambda \in P_{\mathbf{R}} | (\lambda + \rho)(h_0) = p \text{ and } (\lambda + \rho)(h_i) > 0, \forall i \in I\}.$$

$$F_i = \{\lambda | (\lambda + \rho)(h_i) = 0 \text{ and } (\lambda + \rho)(h_0) < p \text{ and } (\lambda + \rho)(h_j) > 0, \forall j \in I, j \neq i\},$$

for any $i \in I$. Any alcove is conjugated under W_{aff} to C_{fund} . Any wall F is conjugated to exactly one F_i : in such case, we say that i is the *type* of F .

Let $\lambda \in P_{\mathbf{R}}^{reg}$, let $i \in I \cup \{0\}$, and let C be the unique alcove containing λ . There is a unique wall F of type i with $F \subset \overline{C}$. We set $s_i * \lambda = s_F \lambda$, where s_F is the affine reflexion relative to the affine hyperplane containing F . There is another definition of $s_i * \lambda$: write $\lambda = w(\mu + \rho) - \rho$, where $w \in W_{aff}$ and $\mu \in C_{fund}$. Then $s_i * \lambda = w s_i(\mu + \rho) - \rho$. Hence the group generated by the operators s_i is isomorphic to W_{aff} . This action of W_{aff} , noted $*$, is often called the right action of W_{aff} on regular weights.

Lemma 10.2: (Jantzen [J3]) *Assume $p \geq h$. Then any alcove and any wall contains an integral weight.*

Proof: For an alcove, the proof is obvious. For a wall, the proof is given in [J3], claim 6.3 (1).

Set $\overline{C}^0 = \overline{C_{fund}} \cap P$. For $\lambda, \mu \in \overline{C}^0$, let $T_\lambda^\mu : Mod(G) \rightarrow Mod(G)$ be the translation functor (a convenient reference for its definition is [J3], ch. 7). Let ν be the unique dominant weight in the W -orbit of $\mu - \lambda$. It follows from [J3] (remark 1 of section 7.6) that $T_\lambda^\mu M$ is a direct summand of $T(\nu) \otimes M$, for any G -module M .

For any simple reflexion s of W_{aff} , denote by $T_s : Mod_{reg}(G) \rightarrow Mod_{reg}(G)$ the *reflexion functor* invented by Jantzen (sometimes these functors are called crossing wall functors). Roughly speaking, the functor T_s is defined as follows. First assume $p \geq h$ (otherwise $Mod_{reg}(G) = 0$). By Jantzen's lemma 10.2, there is an integral weight μ in the unique fundamental wall fixed by s . Then

$$T_s = \oplus_{\lambda \in C^0} T_\mu^\lambda \circ T_\lambda^\mu.$$

For any $M \in Mod_{reg}(G)$, $T_s M$ is a direct summand of a tensor product of M by tilting modules. Therefore, by Corollary 6.3, $T_s M$ has a good filtration (respectively has a Weyl filtration, is tilting) if M has a good filtration (respectively has a Weyl filtration, is tilting). Moreover, $T_s M$ is a direct summand in a direct sum of type $X \otimes Y$, where $X \in Mod_{sing}(G)$. Therefore it follows from Lemma 8.5 and Lemma 10.1 that the dimension of any direct summand of $T_s M$ is divisible by p .

Corollary 10.3: [J3] *Assume $p \geq h$. Let $\lambda \in P^+$ be p -regular and let s be a simple reflexion of W_{aff} . Then $T_s \Delta(\lambda)$ is indecomposable. Moreover:*

- (i) *If $s * \lambda \notin P^+$, we have $T_s \Delta(\lambda) = 0$.*
- (ii) *If $s * \lambda \in P^+$ and $s * \lambda > \lambda$, there is a non-split extension $0 \rightarrow \Delta(s * \lambda) \rightarrow T_s \Delta(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0$.*
- (iii) *If $s * \lambda \in P^+$ and $s * \lambda < \lambda$, there is a non-split extension $0 \rightarrow \Delta(\lambda) \rightarrow T_s \Delta(\lambda) \rightarrow \Delta(s * \lambda) \rightarrow 0$.*

Proof: Assertion (i) is clear, therefore we can assume that $s * \lambda \in P^+$. The existence of the extension follows from the fact that $T_s \Delta(\lambda)$ has a good filtration and $\text{ch } T_s \Delta(\lambda) = \text{ch } \Delta(\lambda) + \text{ch } \Delta(s * \lambda)$. Moreover, $T_s \Delta(\lambda)$ is indecomposable because any direct summand

has a Weyl filtration and its dimension is divisible by p . Therefore the extensions are non-split. Q.E.D.

For $\lambda, \mu \in P^+$, denote by $[T(\lambda) : \nabla(\mu)]$ the multiplicity of $\nabla(\mu)$ in a good filtration of $T(\lambda)$. By Lemma 4.2, we have $[T(\lambda) : \nabla(\mu)] = h^0(T(\lambda), \mu)$. We extend this notation by requiring $[T(\lambda) : \nabla(\mu)] = 0$ if $\mu \notin P^+$.

Lemma 10.4: ([GM1][GM2]) *Assume $p \geq h$. Let $\lambda \in P^+$.*

(i) *If λ belongs to C^0 , then $T(\lambda) = \nabla(\lambda)$ and its dimension is not divisible by p .*

(ii) *Otherwise, the dimension of $T(\lambda)$ is divisible by p , and $\sum_{w \in W_{aff}} \epsilon(w)[T(\lambda) : \nabla(w(\lambda + \rho) - \rho)] = 0$.*

Proof: *Proof of (i):* If $\lambda \in C^0$, then $\nabla(\lambda)$ is simple and isomorphic to $\Delta(\lambda)$ by Corollary 3.5. Thus $T(\lambda) = \nabla(\lambda)$ and its dimension is $\prod_{\alpha \in \Delta^+} (\lambda + \rho)(h_\alpha) / \prod_{\alpha \in \Delta^+} \rho(h_\alpha)$, which is not divisible by p .

Proof of (ii): If λ is p -singular, $\dim T(\lambda)$ is divisible by p (by Lemma 10.1) and there is a reflexion $s \in W_{aff}$ such that $s(\lambda + \rho) - \rho = \lambda$. Hence, we have:

$$\epsilon(w)[T(\lambda) : \nabla(w(\lambda + \rho) - \rho)] + \epsilon(ws)[T(\lambda) : \nabla(ws(\lambda + \rho) - \rho)] = 0, \text{ for any } w \in W_{aff}.$$

Therefore we get $\sum_{w \in W_{aff}} \epsilon(w)[T(\lambda) : \nabla(w(\lambda + \rho) - \rho)] = 0$.

Henceforth we can assume that λ is regular. Write $\lambda = w(\mu + \rho) - \rho$, where $\mu \in C^0$. By induction, we can assume that Assertion (ii) holds for the weight $x(\lambda + \rho) - \rho$, for any $x \neq 1$ with $l(x) < l(w)$. Write $w = vs$, where s is a simple reflexion, $l(v) = l(w) - 1$ and $v(\lambda + \rho) - \rho \in P^+$ and set $\lambda' = v(\lambda + \rho) - \rho$. Note that λ is an highest weight of the tilting module $T_s T(\lambda')$ (e.g. this follows from Corollary 10.3). Therefore $T(\lambda)$ is a direct summand of $T_s T(\lambda')$. Hence its dimension is divisible by p . Moreover, the other indecomposable summands are some tilting modules $T(x(\lambda + \rho) - \rho)$, with $l(x) < l(w)$. Moreover $T(\mu)$ is not a direct summand because its dimension is not divisible by p . Hence Assertion (ii) follows by induction. Q.E.D.

Remark: In their paper [AP], Andersen and Paradowski found a refinement of the previous Lemma. Indeed they prove that if ν is p -singular and λ is p -regular, then $T_\nu^\lambda T(\nu)$ is an indecomposable tilting module. The previous proof (or the one of [GM2]) only implies that $T(\mu)$ never occurs as a direct summand in $T_\nu^\mu T(\nu)$, whenever $\mu \in C^0$.

The following statement is an immediate consequence of Lemma 10.4:

Corollary 10.5: *Let M be a tilting module and let $\lambda \in C^0$. Then*

$$[M : T(\lambda)] = \sum_{w \in W_{aff}} \epsilon(w)[M : \Delta((w(\lambda + \rho) - \rho))].$$

In what follows, we prove the statements of Section 9, whose proofs were postponed.

Proof of Lemma 9.3: See Lemma 10.4 (i).

Proof of Theorems 9.5: For $\lambda, \mu, \nu \in C^0$, we have:

$$\begin{aligned} V_{\lambda\mu}^\nu &= [T(\lambda) \otimes T(\mu) : T(\nu)] \\ &= \sum_{w \in W_{aff}} \epsilon(w)[T(\lambda) \otimes T(\mu) : \Delta((w(\nu + \rho) - \rho))] \\ &= \sum_{w \in W_{aff}} \epsilon(w) K_{\lambda\mu}^{w(\nu + \rho) - \rho} \end{aligned} \quad \text{Q.E.D.}$$

Proof of Theorems 9.6: Let I_1 be the kernel of the morphism $K_0(G_{\mathbb{C}}) \rightarrow K_0(\mathcal{P})$. Let I_2 be the subgroup of $K_0(G)$ generated by all $\text{ch } \Delta(\lambda)$, where λ is p -singular and all

$\text{ch } \Delta(\lambda) + \text{ch } \Delta(s(\lambda + \rho) - \rho)$, where λ is p -regular and s is a reflexion with $s(\lambda + \rho) - \rho \in P^+$. Let I_3 be the ideal of all $f \in K_0(G)$ such that $f(g) = 0$ for any $g \in Z^{1/p}$. By Lemma 8.3, I_1 is spanned by all $\text{ch } T(\lambda)$, where $\dim T(\lambda) = 0$ modulo p . Therefore, it follows from Lemma 10.4 (ii) that $I_1 \subset I_2$.

We claim that $I_2 \subset I_3$. Let $\lambda \in P^+$, let $s \in W_{aff}$ be an affine reflexion such that $s(\lambda + \rho) - \rho \in P^+$ and let $g \in Z^{1/p}$. For $\mu \in P$, set $\delta_\mu(g) = \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho)}(g)$. By Weyl's denominator formula, we have $\delta_0(g) = e^{-\rho}(g) \prod_{\alpha \in \Delta^+} (1 - e^\alpha(g))$. By regularity of g , we have $\delta_0(g) \neq 0$. Let $\bar{s}$ be the linear reflexion associated with s . For any weight μ , $\bar{s}\mu - s\mu \in pQ$. As g^p is central, we have $e^{\bar{s}\mu}(g) = e^{s\mu}(g)$. Therefore $\delta_{s\lambda}(g) = \delta_{\bar{s}\lambda}(g)$. However $\bar{s}$ belongs to W and $\epsilon(\bar{s}) = -1$. Hence $\delta_{\bar{s}\lambda}(g) = -\delta_\lambda(g)$ and $\delta_{s\lambda}(g) + \delta_\lambda(g) = 0$. Therefore the evaluation of $\text{ch } \Delta(\lambda) + \text{ch } \Delta(s(\lambda + \rho) - \rho)$ at g is $\delta_0(g)^{-1}(\delta_{s\lambda}(g) + \delta_\lambda(g)) = 0$, and $I_2 \subset I_3$.

Moreover $K_0(G)/I_1$ is a free abelian group of rank $\text{Card}(C^0)$ and $K_0(G)/I_1$ has rank $\text{Card } Z^{1/p}$. By a case by case analysis, one checks that X and C^0 have the same cardinality. Therefore $I_1 = I_2 = I_3$, what proves Theorem 9.6. Q.E.D.

11. Tilting modules and commutant algebras. Let M be a tilting G -module and let C be its commutant¹¹. For $\lambda \in P^+$, the C -module $H^0(U, M)_\lambda$ will be denoted $Sp(\lambda)$ and it will be called the *Specht module*. This terminology is not standard: usually a Specht module is only a representation of the symmetric group S_n . We will see in Section 14 why this notion generalizes the classical notion of a Specht module. Also set $S(\lambda) = T_\lambda(M)$. The natural map $H^0(U, M)_\lambda \rightarrow T_\lambda(M)$ is a morphism of C -modules $Sp(\lambda) \rightarrow S(\lambda)$. Let $\mathcal{K}(\lambda)$ be its kernel and let $\mathcal{R}$ be the radical of C . Denote by $[M : T(\lambda)]$ the multiplicity of $T(\lambda)$ as a direct summand in M .

Lemma 11.1: *Let $\lambda \in P^+$ such that $S(\lambda) \neq 0$.*

- (i) *The C -module $S(\lambda)$ is simple and its dimension is $[M : T(\lambda)]$.*
- (ii) *We have $\mathcal{K}(\lambda) = \mathcal{R}.Sp(\lambda)$.*
- (iii) *In particular, any $v \in Sp(\lambda) \setminus \mathcal{K}(\lambda)$ generates the C -module $Sp(\lambda)$ and $S(\lambda)$ is the unique simple quotient of $Sp(\lambda)$.*

Proof: It follows from Lemma 7.3 that the C -module $S(\lambda)$ is simple whenever it is not zero. Thus Assertion (i) is obvious. Set $m = \dim S(\lambda)$. By Lemma 7.3, we have $M = T(\lambda)^m \oplus M'$, where M' has no summand isomorphic to $T(\lambda)$. Therefore $\dim S(\lambda) = [M : T(\lambda)]$. Let $x \in \mathcal{K}(\lambda)$. By assumption, its image in $T_\lambda(M)$ is zero, thus x belongs to M' . There is a unique G -equivariant map $\pi : \Delta(\lambda) \rightarrow M'$ such that $\pi(v_\lambda) = x$, where v_λ is an highest weight vector of $\Delta(\lambda)$. As $T(\lambda)/\Delta(\lambda)$ has a Weyl filtration and M' has a good filtration, we have $\text{Ext}_G^1(T(\lambda)/\Delta(\lambda), M') = 0$ (this follows from Corollary 4.7). Therefore we can extend π to a map $\pi' : T(\lambda) \rightarrow M'$. As $m > 0$, we can write $M = T(\lambda) \oplus M''$, where $M'' = T(\lambda)^{m-1} \oplus M'$. Thus we can define a G -equivariant map $\pi'' : M \rightarrow M$ by requiring that its restriction to $T(\lambda)$ is π' and its restriction to M'' is zero. It follows

¹¹ In principle, one should consider M as a right C -module. However, we have $M^* \simeq M^\omega$, where $\omega : G \rightarrow G$ is the Chevalley involution. Hence the algebras C and C^{opp} are isomorphic. Thus the distinction between right C -modules and left C -modules is not important, and we will simply speak of C -modules.

from its definition that π'' is in the radical of C . Moreover, x is in the image of π'' . Thus $K(\lambda) \subset \mathcal{R}.Sp(\lambda)$, what proves Assertion (ii). Then Assertion (iii) follows. Q.E.D.

Proposition 11.2: *Let $(M_j)_{1 \leq j \leq k}$ be k tilting modules whose commutant algebras are denoted $(C_j)_{1 \leq j \leq k}$. Let $\lambda \in P^+$ such that $\dim T(\lambda) \neq 0$ modulo p . Then $T_\lambda(M_1 \otimes M_2 \otimes \dots \otimes M_k)$ is semi-simple as a $C_1 \otimes C_2 \otimes \dots \otimes C_k$ -module.*

Proof: Set $\overline{C}_j = T^G(M_j, M_j)$. By Lemma 8.3, the algebra $\overline{C}_j$ is a semi-simple quotient of C_j . It follows from Lemmas 7.3 and 8.3 that the functor T_λ factors through the category $\mathcal{P}$. Hence the action of C_j over $T_\lambda(M_1 \otimes M_2 \otimes \dots \otimes M_n)$ factors through $\overline{C}_j$, and the proposition follows. Q.E.D.

Let M be a tilting module whose commutant is C . For the next proposition, it will be convenient to extend the definition of Specht modules $Sp(\mu)$ to all weights μ by setting $Sp(\mu) = 0$ if $\mu \notin P^+$. For $\lambda, \mu \in P^+$, we denote by $[T(\lambda) : \nabla(\mu)]$ the multiplicity of $\nabla(\mu)$ in a good filtration of $T(\lambda)$. Similarly, we will extend the definition of the symbol $[T(\lambda) : \nabla(\mu)]$ by requiring that $[T(\lambda) : \nabla(\mu)] = 0$ if $\lambda \notin P^+$ or if $\mu \notin P^+$. The image of a C -module X in $K_0(C)$ will be denoted by $[X]$.

Proposition 11.3: *In $K_0(C)$ the following equalities hold:*

- (i) *For $\lambda \in P^+$, we have $[Sp(\lambda)] = \sum_{w \in W} \epsilon(w)[M_{\lambda+\rho-w\rho}]$.*
- (ii) *For $\lambda \in C^0$, we have $[S(\lambda)] = \sum_{w \in W_{aff}} \epsilon(w)[Sp(w(\lambda + \rho) - \rho)]$.*
- (iii) *For $\lambda \in C^0$, we have $[S(\lambda)] = \sum_{w \in W_{aff}} \epsilon(w)[M_{w(\lambda+\rho)-\rho}]$.*

Proof: There is a semi-simple associative algebra S such that $C = S \times \mathcal{R}$. Moreover, we have $K_0(C) = K_0(S)$. Therefore, it is enough to establish these equalities in $K_0(S)$. We have $M = \bigoplus_{\mu \in P^+} T(\mu) \otimes S(\mu)$ as a $G \times S$ -modules. By Lemma 7.3, the S -module $Sp(\lambda)$ is isomorphic to $\bigoplus_{\mu \in P^+} S(\mu)^{h^0(T(\mu), \lambda)}$ for any $\lambda \in P^+$. By Lemma 4.2, we have $h^0(T(\mu), \lambda) = [T(\mu) : \nabla(\lambda)]$, hence we have:

$$(*) \quad [Sp(\lambda)] = \sum_{\mu \in P^+} [T(\mu) : \nabla(\lambda)] [S(\mu)].$$

By Weyl character formula we have:

$$[T(\mu) : \nabla(\lambda)] = \sum_{w \in W} \epsilon(w) \dim T(\mu)_{\lambda+\rho-w\rho}.$$

Hence $Sp(\lambda) = \sum_{\mu \in P^+} \sum_{w \in W} \epsilon(w) \dim T(\mu)_{\lambda+\rho-w\rho} [S(\mu)] = \sum_{w \in W} \epsilon(w) [M_{\lambda+\rho-w\rho}]$, what proves Assertion (i).

From now on, assume that $\lambda \in C^0$. It follows from the linkage principle (Theorem 3.4) that $[T(w(\lambda + \rho) - \rho) : \nabla(\mu)] = 0$ unless $\mu + \rho$ and $\lambda + \rho$ are W_{aff} -conjugated. Thus by Lemma 10.4, $\sum_{w \in W} \epsilon(w) [T(\mu) : \nabla(w(\lambda + \rho) - \rho)] = \delta_{\lambda\mu}$. Using (*), one gets:

$$\begin{aligned} \sum_{w \in W_{aff}} \epsilon(w) [Sp(w(\lambda + \rho) - \rho)] &= \sum_{w \in W_{aff}} \sum_{\mu \in P^+} \epsilon(w) [T(\mu) : \nabla(w(\lambda + \rho) - \rho)] [S(\mu)] \\ &= \sum_{\mu \in P^+} \delta_{\lambda\mu} [S(\mu)] \\ &= [S(\mu)]. \end{aligned}$$

Therefore Assertion (ii) follows. From the first two assertions one gets:

$$[S(\lambda)] = \sum_{x \in W_{aff}^+} \sum_{y \in W} \epsilon(x) \epsilon(y) [M_{x(\lambda+\rho)-y\rho}],$$

where W_{aff}^+ is the set of all $x \in W_{aff}$ such that $x(\lambda + \rho) - \rho \in P^+$. By W -invariance, the C -module $M_{x(\lambda+\rho)-y\rho}$ is isomorphic to $M_{y^{-1}x(\lambda+\rho)-\rho}$. Any element $w \in W_{aff}$ can be uniquely written as $w = y^{-1}x$, where $x \in W_{aff}^+$ and $y \in W$. Hence Assertion (iii) follows. Q.E.D.

Let M, N be two tilting modules and set $C = \text{End}_G(M)$. Note that C acts over $M \otimes N^{\otimes n}$ (any $u \in C$ acts as $u \otimes 1$). For any $\lambda \in P^+$, $T_\lambda(M \otimes N^{\otimes n})$ is a C -module and we define a $K_0(C)$ -valued formal series by: $\chi^\lambda(z) = \sum_{n \geq 0} [T_\lambda(M \otimes N^{\otimes n})] z^n$. Recall that $Z^{1/p}$ is the set of regular conjugacy classes of G_C such that g^p is central.

Theorem 11.4: *Let $\lambda \in C^0$. Then the generating series $\chi^\lambda(z)$ is a rational function, and its poles are simple and finite. Moreover, for any pole ζ of $\chi^\lambda(z)$, we have $\zeta^{-1} = \text{ch } N(g)^{-1}$, for some $g \in Z^{1/p}$ with $\text{ch } N(g) \neq 0$. Moreover if $\text{ch } N(g) \neq 0$ for any $g \in Z^{1/p}$, then $\chi^\lambda(\infty) = 0$.*

Proof: Using the semi-simplicity of the category $\mathcal{P}$, we get

$$T_\lambda(M \otimes N^{\otimes n}) \simeq \bigoplus_{\mu, \nu \in C^0} T_\mu(M) \otimes T_\nu(N^{\otimes n}) V_{\mu\nu}^\lambda.$$

For any $\nu \in C^0$, set $\theta^\nu(z) = \sum_{n \geq 0} \dim T_\nu(N^{\otimes n}) z^n$. Thus, we have:

$$\chi^\lambda(z) = \sum_{\mu, \nu \in C^0} V_{\mu, n\lambda}^\lambda \theta^\nu(z) [T_\mu(M)].$$

Therefore, it is enough to show a similar assertion for the scalar-valued formal series $\theta^\nu(z)$. Denote by $A : K_0(\mathcal{P}) \rightarrow K_0(\mathcal{P})$ the multiplication by $[N]$. It follows from Theorem 9.6 that the eigenvalues of A are the scalars $(\text{ch } N)(g)$, where $g \in Z^{1/p}$. By Lemma 8.4, the fusion ring $K_0(\mathcal{P})$ is reduced, thus A is diagonalizable. As its coefficients are matrix coefficients of A^n , the formal series $\theta^\nu(z)$ is a matrix coefficient of $1/(1 - zA)$. Therefore $\theta^\nu(z)$ is a rational series, any pole of $\theta^\nu(z)$ is finite and simple and is the inverse of some non-zero eigenvalue of A . Moreover if $\text{ch } N(g) \neq 0$ for any $g \in Z^{1/p}$, the matrix A is invertible and the limit at ∞ of $1/(1 - zA)$ is zero. Q.E.D.

12. Application of tilting modules to representation theory of $GL(V)$. For simplicity, the results of the previous sections have been stated for simple algebraic groups only. However, it will be more convenient to work with $GL(V)$ instead of $SL(V)$, and all statements of the previous sections can be easily adapted to reductive groups. We will start with a few definition involvig Young tableaux and polynomial weights.

Let V be a vector space of dimension n with basis $x_1, \dots, x_n$. The torus of $GL(V)$ is the subgroup of diagonal matrices. Denote by ϵ_i the weight of x_i . Therefore we have $\epsilon_i(h) = z_i$ for a diagonal matrix $h \in H$ with diagonal entries $z_1, z_2, \dots$. Any weight μ of H can be additively written $\mu = \sum_{i \geq 1} m_i \epsilon_i$, where all m_i are integers. Moreover a weight μ is dominant if and only if $m_1 \geq m_2 \geq m_3 \dots$. A weight μ is called *polynomial* if and only if $m_i \geq 0$ for any $i \geq 1$. Its *degree* is $\sum_{i \geq 0} m_i$. A rational representation ϕ of $GL(V)$ is called *polynomial* if and only if any of its weights is polynomial: this is equivalent to Green's definition [G], namely that the matrix coefficients of $\phi(g)$ are polynomial functions into the entries $g_{i,j}$ of the matrix. Note that for a general rational representation ϕ , its matrix coefficients are element of $K[g_{i,j}, \det g^{-1}]$. Hence any representation of $GL(V)$ can be written as $M \otimes L$, where M is polynomial and L is one dimensional. Therefore, any statement about polynomial representations of $GL(V)$ can be easily extended to all rational representations and we will restrict ourself to polynomial representations.

A Young diagram Y is a finite sequence of integers $(m_1, m_2, \dots)$ with $m_1 \geq m_2 \geq \dots \geq 0$. A Young diagram is often represented by a set of boxes in the plane, with m_1 boxes on the first line, m_2 boxes on the second line and so on. For example, the graphic representation of the Young diagram $Y = (5, 3, 2)$ is:

Let Y be a Young diagram. Its *degree* $\deg Y$ is the total number of boxes. Its *height* $\text{ht } Y$ is the number of boxes on the first column. We denote by $Y^\perp$ its mirror image through the main diagonal. In the previous example, $\deg Y = 10$, $\text{ht } Y = 3$ and $Y^\perp$ is the Young diagram:

A *tableau* T of shape Y is a labeling of the boxes of Y by the integers $1, 2, \dots, n$. Its *weight* $w(T)$ is $\sum_{i \geq 1} n_i \epsilon_i$, where n_i is the number of occurrence of the index i in T . As usual, a tableau is called *semi-standard* if the filling is non decreasing from left to right and increasing from top to bottom. For example the tableau:

is semi-standard of shape Y and weight $\epsilon_1 + \epsilon_2 + 4\epsilon_3 + 3\epsilon_4 + \epsilon_5$.

There is a bijective correspondence $\lambda \mapsto Y(\lambda)$ between dominant polynomial weights of $GL(V)$ and Young tableaux of height $\leq n$. Indeed to each dominant polynomial weight $\lambda = \sum_{1 \leq i \leq n} m_i \epsilon_i$ one associates the Young diagram $Y(\lambda) = (m_1, m_2, \dots)$. Let $Y \mapsto \lambda(Y)$ be its inverse. In what follows, we will often use Y to denote the dominant weight $\lambda(Y)$. We will use various groups $GL(V)$. Therefore, we will denote by $L_V(Y)$ the simple $GL(V)$ -module with highest weight $\lambda(Y)$. It will be convenient to set $L_V(Y) = 0$ if $\text{ht } Y > n$. We will use a similar notation and convention for the Weyl module $\Delta_V(Y)$, the dual of the Weyl module $\nabla_V(Y)$ and the tilting module $T_V(Y)$ with highest weight $\lambda(Y)$. It is easy to prove that $V \mapsto L_V(Y)$ can be realized by a polynomial functor of degree $\deg Y$.

Let M be an auxiliary vector space of dimension m . Set $G = GL(M)$. The functor T_λ defined on $\text{Mod}(G)$ (see Section 7) will be denoted by T_Y^M . There could be no confusion with the functor T^G defined in Section 8: indeed $T^G = T_\emptyset^V$. Set $\mathbf{M} = \bigwedge(V \otimes M)$ and denote by ϕ_V the action of $GL(V)$ on $\mathbf{M}$.

Lemma 12.1: *The G -module $\mathbf{M}$ is tilting.*

Proof: For any $k \geq 0$, the G -module $\bigwedge^k M$ is tilting: indeed it is irreducible and satisfies Weyl character formula. Therefore $\bigwedge^k M$ is both a Weyl module and a dual of a Weyl module. The G -module $\mathbf{M}$ is isomorphic to $(\bigwedge M)^{\otimes n}$. Thus by Corollary 6.3, $\mathbf{M}$ is tilting. Q.E.D.

Theorem 12.2: (Donkin [D2]) *The algebra $\text{End}_G(\mathbf{M})$ is generated by $\phi_V(GL(V))$.*

The reference for this theorem is [D2], Proposition 3.11. Indeed, we obtain a dual statement by exchanging V and M . However, it should be noted that usually $\mathbf{M}$ is not tilting as a $GL(V) \times G$ -module. In Howe's terminology, $(GL(V), G)$ is a dual pair in $GL(\mathbf{M})$. Indeed, for fields of characteristic zero, this duality is due to Howe [Ho]. In this setting, Howe showed that the $GL(V) \times G$ -module $\mathbf{M}$ is isomorphic to $\bigoplus_Y L_V(Y) \otimes L_M(Y^\perp)$,

where Y runs over all Young diagrams contained in the $n \times m$ rectangle (i.e. $\text{ht } Y \leq n$ and $\text{ht } Y^\perp \leq m$). In what follows, we will always consider $\mathbf{M}$ as a G -module. Therefore the associated Specht modules $Sp(Y^\perp)$ are $GL(V)$ -modules. We will see that in our context Specht modules are simply Weyl modules, as it is proved by the next lemma (the second assertion comes from [MP1]):

Lemma 12.3: *Let Y be a Young diagram contained in the $n \times m$ rectangle. As $GL(V)$ -modules, we have $Sp(Y^\perp) = \Delta_V(Y)$ and $T_{Y^\perp}^M = L_V(Y)$.*

Proof: Denote by $(y_i)_{1 \leq i \leq m}$ a basis of M , and set $z_{i,j} = x_i \otimes y_j$. Also let $b_{i,j}$ be the box of the $n \times m$ rectangle located at the intersection of the i^{th} row and the j^{th} column. Denote by U_V (respectively U_V^-) the subgroup of upper (respectively lower) unipotent triangular matrices of $GL(V)$ and denote by U_M , U_M^- the corresponding subgroups of G . We can assume $Y \neq \emptyset$ and we place the Young diagram Y inside the rectangle in a such way that it contains the upper left box $b_{1,1}$. For example:

Let Z (respectively Z^-) be the subspace of $V \otimes M$ generated by all $z_{i,j}$ with $b_{i,j} \in Y$ (respectively with $b_{i,j} \notin Y$). Note that Z is a d -dimensional $U_V \times U_M$ -submodule of $V \otimes M$ and Z^- is a $(nm-d)$ -dimensional $U_V^- \times U_M^-$ -submodule of $V \otimes M$, where $d = \deg Y$. Choose non-zero vectors $v \in \bigwedge^d Z$ and $v^- \in \bigwedge^{nm-d} Z^-$.

We claim that $T_{Y^\perp}^M(\mathbf{M})$ contains a non-zero U_V -invariant vector of weight Y . The vector v is $U_V \times U_M$ -invariant of weight $(Y, Y^\perp)$. Hence it defines a U_V -invariant element $\bar{v} \in T_{Y^\perp}^M(\mathbf{M})$. Since v^- is U_M^- -invariant and $v \wedge v^-$ is a non-zero vector of the trivial U_M^- -module $\bigwedge^{nm}(V \otimes M)$, we have $\bar{v} \neq 0$. Hence $T_{Y^\perp}^M(\mathbf{M})$ contains a non-zero U_V -invariant vector of weight Y , namely $\bar{v}$.

By Lemma 11.1 (i), the $GL(V)$ -module $T_{Y^\perp}^M$ is simple. Moreover it contains a U_V -invariant vector of weight Y . By the classification of simple $GL(V)$ -modules (Theorem 2.2), we get $T_{Y^\perp}^M(\mathbf{M}) = L_V(Y)$. As the G -module $\mathbf{M}$ has a good filtration, it follows from Lemma 4.2 (ii) that the character of the $GL(V)$ -module $Sp(Y^\perp)$ is independent of the characteristic. By the previous result of Howe, we get $\text{ch } Sp(Y^\perp) = \text{ch } \Delta_V(Y)$. Moreover by Lemma 11.1 (iii), the $GL(V)$ -module $Sp(Y^\perp)$ is generated by its highest weight vector v . Therefore $Sp(Y^\perp) = \Delta_V(Y)$. Q.E.D.

Corollary 12.3: ([MP1]) *Let Y be a Young diagram contained in the $n \times m$ rectangle, and let $\mu = \sum_{1 \leq k \leq n} k_i \epsilon_i$ be a polynomial weight. We have:*

$$\dim L_V(Y)_\mu = [\bigwedge^{k_1} M \otimes \bigwedge^{k_2} M \otimes \dots : T_M(Y^\perp)].$$

Proof: The μ -weight space of $\mathbf{M}$ is the subspace $\bigwedge^{k_1} V' \otimes \bigwedge^{k_2} V' \otimes \dots$. Therefore the corollary follows from Lemma 12.2 and Lemma 7.3.

We will now determine when $Y^\perp$ is in the fundamental alcove of G . Note that this last condition depends on m . To explain this condition, we need to introduce the definition of m -special Young diagrams. For a Young diagram Y , denote by $c_i(Y)$ the number of boxes on the i^{th} -column. For any integer $m < p$, say that the Young diagram Y is m special if

the number of columns is $\leq m$ and if $c_1(Y) - c_m(Y) \leq p - m$. The following easy lemma is proved in [MP1].

Lemma 12.4: ([MP1]) *Assume $m < p$. The map $Y \mapsto Y^\perp$ is a bijection from the set of m -special Young diagram to the fundamental alcove C^0 of G .*

Therefore, we can use Corollary 12.3 and the modular Verlinde's formula 9.5 to compute the character of $L_V(Y)$ whenever $Y^\perp$ is in the fundamental alcove of G . Indeed the character formula can be computed in a combinatorial way. To explain the underlying combinatorics, we need to define the notion of a m -semi-standard tableau. For a tableau T , denote by $T[i]$ the subset of boxes with labels $\leq i$. Therefore, when T is semi-standard, $T[i]$ is again a Young diagram. By definition, a semi-standard tableau T is m -semi-standard if all Young diagrams $T[i]$ are m -special. Set $N(Y, \mu)$ the number of m -semi-standard tableaux of shape Y and weight μ . The details of the proof of the following theorem are given in [MP1], Theorem 4.3:

Theorem 12.4: ([MP1]) *Let Y be a m -special Young diagram for some $m < p$. Any weight of $L_V(Y)$ is polynomial and for any polynomial weight μ we have:*

$$\dim L_V(Y)_\mu = N(Y, \mu).$$

As an example of character formula, note that for any fundamental weight ω_i , $Y(m\omega_i)$ is m -special. Therefore, by using Steinberg tensor product Theorem 2.4, one can deduces a character formula for the $GL(V)$ -module $L_V(N\omega_i)$ for any n, N, i and any characteristic p . See [MP1], Theorem 5.3.

There is a general conjecture, due to Lusztig [Lu1], [Lu2] about the character of a simple rational $GL(V)$ -module. The experts believe that this conjecture holds for $p \geq n$ (see e.g. the introduction of [So]) and it has been proved for $p \gg n$ by Andersen, Jantzen and Soergel [AJS]. In contrast, the previous character formula applies only to some peculiar highest weights, but they hold for V of any dimension n and are therefore outside the validity domain of Lusztig's Conjecture. This conjecture does not seem adapted to the investigation of simple polynomial functors. Using Weyl's polarizations, the simple polynomial functor $V \mapsto L_V(Y)$ is entirely determined by the $GL(n)$ -module $L_{K^n}(Y)$, where $n = \deg Y$. Therefore, Lusztig's Conjecture only applies to polynomial functors of degree $\leq p$. In contrast, there are m -special Young diagrams of arbitrary degree.

Let $V = \bigoplus_{1 \leq j \leq k} V_j$ be any decomposition of V . Therefore $\prod_{1 \leq j \leq k} GL(V_j)$ is a subgroup of $GL(V)$. The following theorem has been proved independently (by very different methods) in [BK] and in [MP1]. The proof given below is simpler.

Theorem 12.6: ([BK] [MP1]) *Let Y be a m -special Young diagram for some $m < p$. As a $\prod_{1 \leq j \leq k} GL(V_j)$ -module, $L_V(Y)$ is semi-simple.*

Proof: Set $\mathbf{M}_j = \bigwedge(V_j \otimes M)$. By Theorem 12.1, $GL(V_j)$ generates the commutant algebra $C_j = \text{End}_G(\mathbf{M}_j)$. By Lemmas 10.4 and 12.4, the dimension of the G -module $T_M(Y^\perp)$ is not divisible by p . Therefore by Proposition 11.2, the $\prod_{1 \leq j \leq k} GL(V_j)$ -module $T_{Y^\perp}^M(\bigotimes_{1 \leq j \leq k} \mathbf{M}_j)$ is semi-simple. Therefore the theorem follows from the fact that $\mathbf{M} \simeq \bigotimes_{1 \leq j \leq k} \mathbf{M}_j$ and from the isomorphism $T_{Y^\perp}^M(\mathbf{M}) \simeq L_V(Y)$ proved in Lemma 12.3. Q.E.D.

13. An easy example: the fusion ring of $SL(p-1)$. As before, V is a vector space of arbitrary dimension. In this section, the auxiliary vector space M is K^{p-1} and we

will investigate the fusion ring for $G = SL(p-1)$, which is very simple. Then we recover some well-known multiplicity one results. We denote by ω_i^M the fundamental weights of G , and set $\omega_0^M = 0$. For an integer n , it will be convenient to set $\omega_n^M = \omega_{\bar{n}}^M$ where $\bar{n}$ is its residue modulo $p-1$.

Lemma 13.1: *For $G = SL(p-1)$, the only weights in C^0 are $\omega_0^M, \omega_1^M, \dots, \omega_{p-2}^M$. In the category $\mathcal{P}$, we have: $T(\omega_i^M) \otimes T(\omega_j^M) = T(\omega_{i+j}^M)$.*

This Lemma follows from easy computation. As the tensor product multiplicities are 1, it follows that for any $(p-1)$ -special Young diagram Y the $GL(V)$ -module $L_V(Y)$ is multiplicity free. By this way, we recover a result of Doty. We denote by ω_i the fundamental weights of $GL(V)$.

Theorem 13.2: (Doty [Do]) *Let $\lambda \in P^+$ be of the form $\lambda = a\omega_i + b\omega_{i+1}$ with $a+b = p-1$. Then any weight of $GL(V)$ -module $L_V(\lambda)$ has multiplicity one.*

Proof: The Young diagram $Y(\lambda)$ is $(p-1)$ -special. Therefore by Corollary 12.3 and Lemma 13.1, any any weight of the $GL(V)$ -module $L_V(\lambda)$ has multiplicity one. Q.E.D.

Doty's proof is based on the fact that the module $L_V(\lambda)$ is a quotient of $S^N V$, where $N = ai + b(i+1)$. Indeed with the notations of Section 1, we have $L_V(\lambda) \simeq \bar{S}^N V$. We can also recover the following characteristic zero result:

Theorem 13.3: (Benckart-Britten-Lemire [BBL]) *Let $\lambda \in \mathbf{C} \otimes P$ be a weight of the form $\lambda = a\omega_i + b\omega_{i+1}$ where $a, b \in \mathbf{C}$ and $a+b = 1$. Let $L_{\mathbf{C}}(\lambda)$ be the $\mathfrak{gl}(n, \mathbf{C})$ -module with highest weight λ . Then any weight of $L_{\mathbf{C}}(\lambda)$ has multiplicity one.*

Proof: Let Λ be the set of all weights of the form $a\omega_i + b\omega_{i+1}$ with $a+b = -1$, and $\Lambda_{\mathbf{Z}}$ those which are integral, i.e. such that a, b are integers. As $\Lambda_{\mathbf{Z}}$ is Zariski dense in Λ , it is enough to prove the assertion for the weights in $\Lambda_{\mathbf{Z}}$. Then the result follows easily from the previous theorem and by semi-continuity principle. Q.E.D.

The last proof is based in the idea of [MP2]. See also [MP2] for more complicated examples of combinatorial weight multiplicity formulas in the category $\mathcal{O}$ of $\mathfrak{gl}(n, \mathbf{C})$, which can be deduced from Theorem 12.4.

14. Application of tilting modules to the symmetric group S_n . In this section, we will investigate the representation of the symmetric group S_n . Let M be an auxiliary vector space of dimension m and set $G = GL(M)$. Denote by ϕ the natural action of S_n on $M^{\otimes n}$. By Lemma 12.1, the G -module $M^{\otimes n}$ is tilting. As in Section 12, we will use that $\phi(KS_n)$ is the commutant of the G -module $M^{\otimes n}$. This fact has been proved by Weyl in characteristic zero and it has been extended to finite characteristics by de Concini and Procesi. See [dP] for the proof of the next result:

Theorem 14.1: (de Concini and Procesi [dP]) *We have $End_G(M^{\otimes n}) = \phi(KS_n)$.*

We can also describe a Young diagram Y by a finite sequence $(m_1^{a_1}, m_2^{a_2}, \dots)$ by the following rule: $m_1, m_2, \dots$ are the various lengths of the non-empty lines of Y and a_k is the number of lines of Y of length m_k . Therefore $m_1, m_2, \dots$ are distinct positive integers. We do not require that the sequence $(m_1, m_2, \dots)$ is ordered, therefore $(m_1^{a_1}, m_2^{a_2}, \dots)$ is defined up to permutation. For example, the Young diagram defined by the sequence $(3^2, 1^1)$ is:

A Young diagram is called *p-regular* if $a_k < p$ for any k . It is clear that Y is *p-regular* if and only if the weight $\lambda(Y^\perp)$ is restricted. This usual terminology conflicts with the notion of *p-regular* weights. An element $g \in S_n$ is *p-regular* if its order is not divisible by p .

Lemma 14.2: *There is a natural bijection between the p -regular conjugacy classes of S_n and the p -regular Young diagrams of degree n .*

Proof: To any conjugacy class $[g]$ of S_n , one associates a finite sequence $(m_1^{a_1}, m_2^{a_2}, \dots)$ by the following rule: $m_1, m_2, \dots$ are the various lengths of the cycles of g and a_k is the number of cycles of $[g]$ of length m_k . Therefore, there is a bijection between:

(i) all the p -regular conjugacy classes of S_n , and

(ii) all the sequences $(m_1^{a_1}, m_2^{a_2}, \dots)$ of degree n , with no parts m_k divisible by p .

Any integer $a \geq 1$ admits a p -adic expansion $a = \sum_{r \geq 0} b_r p^r$, where $0 \leq b_r < p$ for any $r \geq 0$. For any one term sequence m^a , set $\psi(m^a) = (m^{b_0}, (pm)^{b_1}, (p^2m)^{b_2}, \dots)$. More precisely, we remove in the sequence the trivial parts $(p^k m)^{b_k}$ whenever $b_k = 0$. For an arbitrary sequence $(m_1^{a_1}, m_2^{a_2}, \dots)$, set $\psi((m_1^{a_1}, m_2^{a_2}, \dots)) = (\psi(m_1^{a_1}), \psi(m_2^{a_2}), \dots)$. By unicity of the p -adic expansions, ψ establishes a bijections between (ii) and

(iii) all the p -regular sequences of degree n .

Therefore the lemma is proved. Q.E.D.

As in section 12, we will identify any Young diagram Y of height $\leq m$ with a dominant polynomial weight of G . Define the S_n -modules: $Sp(Y) = H^0(U, M^{\otimes n})$ and $S(Y) = T_Y(M^{\otimes n})$. Indeed $Sp(Y)$ is the usual Specht module. The following statement is a tilting module version of the classical Schur correspondence, formulated by Green in [G].

Proposition 14.3:

(i) *The S_n -module $Sp(Y)$ is independent of the dimension m of M (provided that $m \geq \text{ht } Y$).*

(ii) *If Y is a p -regular, $S(Y)$ is a non-zero simple S_n module; otherwise $S(Y) = 0$.*

(iii) *If $\dim M \geq n$, then $Y \mapsto S(Y)$ is a bijection from the p -regular Young diagrams Y of degree n to the simple KS_n -modules.*

Proof: Under the provision $m \geq \text{ht } Y$, the weight space $(M^{\otimes n})_Y$ is independent of m , and therefore $Sp(Y)$ is also independent of the dimension of M , what proves the first assertion. In order to prove the last two assertions, we can assume that $m \geq n$. By Lemma 11.1(i) and Theorem 14.1, the S_n -module $S(Y)$ is simple whenever it is not zero.

We claim that $S(Y) = 0$ whenever Y is not p -regular. Set $V = K^n$, $\mathbf{M} = \bigwedge(V \otimes M)$ and $\epsilon = \sum_{1 \leq k \leq n} \epsilon_i$. We can identify $M^{\otimes n}$ with the ϵ -weight space of the $GL(V)$ -module $\mathbf{M}$. By Lemma 12.3, $S(Y)$ is the ϵ -weight space of the simple $GL(V)$ -module $L_V(Y^\perp)$. The weight $Y^\perp$ is not restricted, therefore by Steinberg tensor product Theorem 2.4, ϵ is not a weight of $L_V(Y^\perp)$. Thus $S(Y) = 0$ and the claim is proved.

Thanks to the additional assumption $m \geq n$, $M^{\otimes n}$ contains the regular representation of S_n . Thus any simple S_n -module occurs as a subquotient of $M^{\otimes n}$. Therefore any S_n -module is isomorphic to $S(Y)$, for some p -regular Young diagram Y of degree n . By Brauer's theory the number of simple S_n -module equals the number of p -regular conjugacy

classes in S_n . By Lemma 14.2, this number equals the number of p -regular Young diagrams Y of degree n . Therefore $Y \mapsto S(Y)$ is a bijection from the p -regular Young diagrams Y of degree n to the simple KS_n -modules. In particular $S(Y) \neq 0$ if Y is p -regular. Q.E.D.

In order to describe the Brauer character of some simple S_n -modules, one needs to describe some combinatorics. Let Y be a Young diagram. Its *rim* is the set of boxes of Y of position (i, j) such that there are no boxes in position $(i + 1, j + 1)$. For example the rim of the diagram Y below is the set indexed boxes:

A p -rim of Y is a connected piece Z of the rim of Y of size p such that $Y \setminus Z$ is again a Young diagram. A p -core is a Young diagram where one cannot remove p -rim. In the previous example the p -rim of Y for $p = 2, 3, 5$ are:

There are no 7-rims, therefore Y is a 7-core. Starting with a Young diagram Y , we can remove successively p -rim, until we get a p -core $\bar{Y}$. Although there are usually more than one way to remove p -rim from Y , the p -core $\bar{Y}$ depends only on Y . Therefore $\bar{Y}$ is called the *core of Y* . Let Y be a Young diagram $Y = (m_1, m_2, \dots)$ of height $\leq m$. Denote by $\mathcal{C}_m(Y)$ the set of all Young diagram Y' of height $\leq m$ with the same degree and p -core than Y . Set $\epsilon_m(Y) = (-1)^{l_m(Y)}$, where $l_m(Y) = \sum_{1 \leq i < j \leq m} [(m_i - m_j + j - i)/p]$, and where $[x]$ denotes the integral part of any $x \in \mathbf{Q}$. Assume now $m < p$. The Young diagram Y is called m -small if and only if $m_1 - m_m \leq p - m$. It should be noted that the Specht modules are the reductions modulo p of the simple $\mathbf{C}S_n$ -modules. Thus their Brauer character are well-understood. Therefore the next statement describes the Brauer character of the modular representations $S(Y)$ with Y m -small.

Theorem 14.4: *Let $m < p$ and let Y be a m -small Young diagram Y . Then in $K_0(S_n)$, we have:*

$$[S(Y)] = \sum_{Y' \in \mathcal{C}_m(Y)} \epsilon_m(Y') [Sp(Y')].$$

Proof: The following combinatorial observations are easy:

- (i) As Y is m -small, the weight $\lambda = \lambda(Y)$ is in the fundamental alcove C^0 of G .
- (ii) The Young diagrams $Y \in \mathcal{C}_m(Y)$ correspond exactly to all dominant and polynomial weights of the form $w(\lambda + \rho) - \rho$ for some $w \in W_{aff}$. Moreover $l_m(Y)$ is the length of w and therefore $\epsilon_m(Y) = \epsilon(w)$.

Therefore the theorem follows from Proposition 11.3 (ii). Q.E.D.

It is also possible to use Proposition 11.3 (iii) to write $[S(Y)]$ as a combination of induced modules. However the index set is an affine Weyl group (instead of a Weyl group), therefore it is not possible to express it in terms of a determinant as in characteristic zero. However, one can derive a combinatorial formula for $\dim S(Y)$. Let $\mathcal{Y}$ be the oriented graph whose vertices are the Young diagrams and whose arrows are $Y \rightarrow Y'$ if Y' is

obtained by adding one box to Y . For example, there are three arrows originating in the Young diagram $(3, 2^2)$, as shown below (the cross indicates the added box):

For $m < p$, let $\mathcal{Y}_m$ be the set of all m -small Young diagrams.

Theorem 14.5: ([M3]) *Let $Y \in \mathcal{Y}_m$. Then the dimension of $S(Y)$ is the number of oriented paths from $\emptyset$ to Y entirely contained in $\mathcal{Y}_m$.*

For the proof, see [M3]. In view of the next statement, fix a Young diagram Y of degree n and of height $\leq m$. For any $k \geq 0$, denote by Y_k the Young diagram obtained by adding a rectangle of height m and length k on the left side of Y . Here is an example with $m = 3$ (the added rectangle corresponds with the crossed boxes):

Note that Y_k is a Young diagram (even when $\text{ht } Y < m$) and $\deg Y_k = n + km$. Consider S_n as a subgroup of S_{n+km} as usual; henceforth $S(Y_k)$ can be viewed as S_n -module by restriction. Therefore we can define the formal series $\chi_g^Y(z) = \sum_{k \geq 0} \dim S(Y_k)^g z^k$, for any p -regular element $g \in S_n$, where $S(Y_k)^g$ is the space of g -invariant vectors of $S(Y_k)$. For $g = 1$, the series is simply $\sum_{k \geq 0} \dim S(Y_k) z^k$.

Theorem 14.5: *Assume $m < p$ and that Y is m -small. Then $\chi_g^Y(z)$ is a rational function with simple and finite poles. We have $\chi_g^Y(\infty) = 0$ and for any pole ζ of $\chi_g^Y(z)$, there are m distinct p -roots of 1: $\zeta_1, \dots, \zeta_m$ such that*

$$\zeta = \left(\sum_{1 \leq k \leq m} \zeta_k \right)^{-m}.$$

Proof Let $\chi^Y(z)$ be the $K_0(KS_n)$ -valued formal series $\sum_{k \geq 0} [S(Y_k)] z^k$. As g is p -regular, its action on any S_n -module X is semi-simple. Therefore the map $X \mapsto \dim X^g$ induces a linear map $L_g : K_0(KS_n) \rightarrow \mathbb{Z}$ and we have $\chi_g^Y(z) = L_g \circ \chi^Y(z)$. Therefore it is enough to prove a similar statement for the series $\chi^Y(z)$.

Set $G' = SL(V)$, $M' = M^{\otimes n}$, $N' = M^{\otimes m}$ and let λ' be the restriction of $\lambda(Y)$ to the group H' of diagonal matrices of G' . The restriction to H' of the weight $\lambda(Y_k)$ are all equal to λ' . Moreover the commutant of the G' -module M' is again $\phi(KS_n)$. Therefore the series $\chi^Y(z)$ is the series $\chi^{\lambda'}(z)$ occurring in Theorem 11.4 and associated to the group G' and its tilting modules M' and N' . Note also that λ' is in the fundamental alcove C^0 of G' . Therefore by Theorem 11.4, the formal series is rational, its poles are finite and simple and of the form $(\text{ch } N'(g))^{-1}$, for some $g \in Z^{1/p}$.

By definition, the elements in $Z^{1/p}$ are the matrices g of determinant 1 with distinct eigenvalues $z_1, \dots, z_m$ such that all z_i^p are equal. For such a matrix g , there are m distinct p -roots of 1: $\zeta_1, \dots, \zeta_m$ such that $z_1 = u\zeta_1, \dots, u\zeta_m$ and u is a m -root of 1. Therefore $\text{ch } N'(g) = (\sum_{1 \leq k \leq m} \zeta_k)^m$ and the poles of $\chi^Y(z)$ are of the form $(\sum_{1 \leq k \leq m} \zeta_k)^{-m}$. Moreover a sum of m distinct p -root of 1 cannot be zero. By Theorem 11.4, $\chi^Y(\infty) = 0$ and therefore $\chi_g^Y(\infty) = 0$. Q.E.D.

Remarks: We can consider similar series $\chi_g^Y(z)$ by using representations over a field of characteristic zero. However, these series are usually not rational. Similar generating functions has been considered by Erdmann for $m = 2$ see [E]. It turns out that for $m = 2$, the series $\chi_g^Y(z)$ are rational for any Y ¹².

Until the end of the section, we will use the following new hypotheses: we fix a p -regular Young diagram Y of degree n and M is a vector space of arbitrary dimension m . A *partition* of n is a sequence $\mathbf{a} = (a_1, \dots, a_k)$ of positive integers with $\sum_{1 \leq j \leq k} a_j = n$. For such a partition $\mathbf{a}$, the group $S_{\mathbf{a}} = \prod_{1 \leq j \leq k} S_{a_j}$ is viewed as a subgroup of S_n as usual. Following Kleshchev's terminology [Kl2], $S(Y)$ is called *completely splittable* if it is semi-simple as a $KS_{\mathbf{a}}$ -module for any partition $\mathbf{a}$ of n .

Lemma 14.6: *Let Assume that $\dim T_M(Y)$ is not divisible by p . Then $S(Y)$ is completely splittable.*

Proof: Note that $\dim M \geq \text{ht } Y$, otherwise $T_M(Y)$ would be zero. Let $\mathbf{a}$ be any partition of n . Set $M_j = M^{\otimes a_j}$ for any j . By Theorem 14.1, S_{a_j} generates the commutant of the tilting G -module M_j . By definition, $S(Y) = T_Y^M(\otimes_{1 \leq j \leq k} M_j)$. Therefore by Proposition 11.2, the $S_{\mathbf{a}}$ -module $S(Y)$ is semi-simple. Q.E.D.

Whenever Y is m -small for some $m < p$, $\dim T_M(Y)$ is not divisible by p by Lemma 10.4. Thus Lemma 14.6 provides a simple proof of the following Kleshchev's theorem:

Theorem 14.7: (Kleshchev [Kl1][Kl2])

If Y is m -small for some $m < p$, then $S(Y)$ is completely splittable.

Remark: The m -small Young diagrams are considered in [We] in the context of Hecke representations.

15. Comparison with the quantum case. Let G be a reductive group. Denote by U_η be the corresponding quantum group at a p -root of unity η . Tilting modules are defined as well for quantum groups. Denote by $T_\eta(\lambda)$ the tilting U_η -module with highest weight λ . Although $\text{ch } T(\lambda)$ is still unknown, the character of tilting modules $T_\eta(\lambda)$ has been determined by Soergel [So2][So3]. Therefore one should try to compare the tilting module $T(\lambda)$ with its quantum analog. It follows from Theorem A4 that $\text{Ext}_G^k(T(\lambda), T(\lambda)) = 0$ for $k = 1, 2$. By deformation theory, the obstruction of a lifting lies in the Ext^2 -group and its unicity in the Ext^1 -group. Therefore $T(\lambda)$ can be uniquely lifted to a representation of U_η , and $\text{ch } T(\lambda) - \text{ch } T_\eta(\lambda)$ is a non-negative linear combination of $\text{ch } T_\eta(\mu)$ for some $\mu < \lambda$ (see [J]). The following two conjectures are closely related:

Conjecture 15.1: (Andersen [A7]) *If $(\lambda + \rho)(h_0) < p^2$, then $\text{ch } T(\lambda) = \text{ch } T_\eta(\lambda)$.*

Conjecture 15.2: ($G = GL(V)$) *Let Y be a Young diagram with $\deg Y < p^2$. Then $\text{ch } T(\lambda) = \text{ch } T_\eta(\lambda)$, where $\lambda = \lambda(Y)$.*

Let $\mathcal{H}_n(\eta)$ be the Hecke $\mathbb{C}$ -algebra of S_n evaluated at η . The simple modules $\mathcal{H}_n(\eta)$ -modules are denoted by $S_\eta(Y)$, where Y is a p -regular Young diagram of degree n , see [DJ]. The character of $S_\eta(Y)$ are determined [KL][Ar][Gr]. Therefore it is interesting

¹² E.g. this follows from the fact that any tilting module for $SL(2)$ is outside a cofinite tilting ideal. It seems unlikely that the series $\chi_g^Y(z)$ is rational for any Y when $m > 3$

to know when the characters of $S(Y)$ and $S_\eta(Y)$ are equal, or equivalently when their dimensions are the same.

Conjecture 15.3: (James) *Let Y be a Young diagram with $\deg Y < p^2$. Then $\dim S(Y) = \dim S_\eta(Y)$.*

Andersen Conjecture implies Lustig Conjecture [A7]. Using the methods of [M3] it is easy to show that Conjecture 15.2 is indeed equivalent to James Conjecture: they are equivalent to the fact that $M^{\otimes n}$ decomposes in the same way that its quantum analog, whenever $n < p^2$, for any vector space M . James conjecture cannot hold for $n \geq p^2$. It should be noted that the condition $n < p^2$ is exactly the validity domain of Broué's conjecture: for $n < p^2$, the p -Sylow subgroups of S_n are abelian. These conjectures are unstable, i.e. for a given p they concern only Young diagrams of bounded size. Based on the clever $SL(3)$ -computations of [Je], we try the following stable conjecture:

Conjecture 15.4 *Let m be an integer with $3 \leq m \leq p$. Let $Y = (m_1, \dots, m_m)$ be a Young diagram of height $\leq m$ such that for any positive integer $i \leq m$, $m_1 - m_i + (i-1) < p$ or $m_i - m_m + (m-i) < p$. Then $\dim S(Y) = \dim S_\eta(Y)$.*

For $m = 3$, the conjecture holds [JM]:

Theorem 15.4: ([JM]) *Assume p odd. Let $Y = (m_1, m_2, m_3)$ be a Young diagram such that $m_1 - m_2 \leq p-2$ or $m_2 - m_3 \leq p-2$. Then $\dim S(Y) = \dim S_\eta(Y)$.*

Appendix: Cohomological criterion for good filtrations. In section 4, we try to provide the most elementary approach of good filtrations. Especially, we only use the trivial part of the vanishing theorem of Cline, Parshall, Scott and van der Kallen (Theorem A4). In this appendix, we will connect the approach of Section 4 with the usual cohomological description of good filtrations [FP]. For a weight $\mu \in Q^+$, we set $\text{ht}(\mu) = \sum_{i \in I} m_i$ if $\mu = \sum_{i \in I} m_i \alpha_i$. This is sometimes called the *height* of μ , but this terminology should not be confused with the height of Young diagrams.

Lemma A.1; *Let M be a B -module and $k \geq 0$. If ν is a weight of $H^k(U, M)$ we have $\nu \leq \mu$ and $\text{ht}(\mu - \nu) \geq k$ for some weight μ of M .*

Proof: Any B -module M admits an injective envelope $I(M)$: we have $M \subset I(M)$, $H^0(U, M) = H^0(U, I(M))$ and $I(M)$ is injective. Set $Z = I(M)/M$. For any weight ν of Z , we have $\nu < \mu$ for some weight μ of M . Choose an injective resolution of M : $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$, such that $I_0 = I(M)$, $I_1 = I(Z)$ and I_k is the injective envelope of the cokernel of $I_{k-2} \rightarrow I_k$ for any $k \geq 2$. By induction, one proves that for any weight ν of I_k we have $\nu \leq \mu$ and $\text{ht}(\mu - \nu) \geq k$, for some weight μ of M . Any weight ν of $H^k(U, M)$ is a weight of I_k and the lemma follows. Q.E.D.

Recall that $D : \text{Mod}(B) \rightarrow \text{Mod}(G)$ is the induction functor from B to G . The functor D is left exact and we denote by D^* its derived functor. By definition, we have $DM = D^0 M = H^0(G/B, \mathcal{L}(M))$. The next two lemmas are well-known and they fit in the framework of Zuckerman's functors.

Lemma A.2: *For any B -module M , we have $D^k M = H^k(G/B, \mathcal{L}(M))$, for all $k \geq 0$.*

Proof: As the functor $\mathcal{L}$ is exact, it is enough to prove that $H^k(G/B, \mathcal{L}(M)) = 0$, for all $k > 0$ and any injective B -module M . As any indecomposable injective module is a direct

summand of $K[B]$, we only have to prove the claim for $M = K[B]$. Let $\pi : G \rightarrow G/B$ be the natural projection. We have $\mathcal{L}(K[B]) = \pi_* \mathcal{O}_G$. As the variety G and the morphism π are affine, we have $R^i \pi_* \mathcal{O}_G = 0$ and $H^i(G, \mathcal{O}_G) = 0$ for $i > 0$ by Serre's vanishing theorem. Thus the vanishing of $H^i(G/B, \mathcal{L}(K[B]))$ follows from Leray's spectral sequence. Q.E.D.

Lemma A.3: *Let M be a G -module. We have: $\text{Ext}_G^i(\Delta(\lambda), M) = H^i(U, M)_\lambda$, for any $\lambda \in P^+$.*

Proof: Let N be a B -module. We have $H^0(G, DN) = H^0(B, N)$, thus the functor $H^0(B, -)$ is the composite of the functors D and $H^0(G, -)$. Clearly D maps injective B -modules to injective G -modules. So there is a spectral sequence converging to $H^*(B, N)$ whose E_2^{**} -term is $H^*(B, D^* N)$.

Assume now that $N = M \otimes K(-\lambda)$. Then $D^k N = M \otimes D^k K(-\lambda) = 0$ for $k > 0$ by Lemma A.2 and Kempf's vanishing theorem 3.2. Thus the previous spectral sequence degenerates, and we have $H^k(G, \nabla(w_0 \lambda) \otimes M) = H^k(B, N)$ for all k . Thus we get:

$$H^k(U, M)_\lambda = H^k(B, M \otimes K(-\lambda)) = H^k(G, \nabla(w_0 \lambda) \otimes M) = \text{Ext}_G^k(\Delta(\lambda), M). \text{ Q.E.D.}$$

Theorem A.4: (Cline-Parshall-Scott-van der Kallen vanishing Theorem [CPSV]) *Let $\lambda, \mu \in P^+$. Then $\text{Ext}_G^k(\Delta(\lambda), \nabla(\mu)) = 0$, for all $k > 0$.*

Proof: Let $\lambda, \mu \in P^+$. We claim that $H^k(G, \nabla(\lambda) \otimes \nabla(\mu)) = 0$ for any $k > 0$. By symmetry of the roles of λ and μ , we can assume $-w_0 \mu \not\leq \lambda$. By the Lemma A.3, $H^k(G, \nabla(\lambda) \otimes \nabla(\mu)) = H^k(U, \nabla(\lambda))_{w_0 \mu}$ and this last group is 0 by Lemma A.1. Therefore $\text{Ext}_G^k(\Delta(\lambda), \nabla(\mu)) = H^k(G, \nabla(-w_0 \lambda) \otimes \nabla(\mu)) = 0$. Q.E.D.

For any G -module M , $\lambda \in P^+$ and $k \geq 0$, set $h^k(M, \lambda) = \dim H^k(U, M)_\lambda$. For $k = 0, 1$, these numbers have been defined in Section 4, and by Lemma A.3 the two definitions agree. For a given G -module M , almost all numbers $h^k(M, \lambda)$ are zero (see Lemma A.1) and all of them are $< \infty$.

Theorem A.5: *Let M be a G -module, and let $n \geq 0$.*

- (i) *If n is even, we have $\text{ch } M \leq \sum_{\lambda \in P^+} \sum_{k \leq n} (-1)^k h^k(M, \lambda) \text{ch } \nabla(\lambda)$,*
- (ii) *if n is odd, we have $\text{ch } M \geq \sum_{\lambda \in P^+} \sum_{k \leq n} (-1)^k h^k(M, \lambda) \text{ch } \nabla(\lambda)$.*

Proof: By induction on n . It follows from the proof of Proposition 4.5 that there exists a short exact sequence $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$, where X has a good filtration. From the vanishing theorem A.4, we get:

$$\begin{aligned} h^0(M, \lambda) - h^1(M, \lambda) &= h^0(X, \lambda) - h^0(N, \lambda), \\ h^k(M, \lambda) &= h^{k-1}(N, \lambda), \text{ for } k \geq 2. \end{aligned}$$

By Lemma 4.2, we have $\text{ch } X = \sum_{\lambda \in P^+} h^0(X, \lambda) \text{ch } \nabla(\lambda)$, and by induction hypothesis $\text{ch } N$ can be compared with $\sum_{\lambda \in P^+} \sum_{k \leq n} (-1)^k h^k(N, \lambda) \text{ch } \nabla(\lambda)$. The inequality involving M follows.

Following Friedlander and Parshall [FP], we say that a G -module M has *good dimension* $\leq m$ if there exists a resolution $0 \rightarrow M \rightarrow X_0 \rightarrow \dots \rightarrow X_m \rightarrow 0$, where all X_i are (finite dimensional) G -modules with a good filtration.

Corollary A.6: *Let $m \geq 0$ and let M be a G -module. The following assertions are equivalent:*

- (i) $\text{ch } M = \sum_{\lambda \in P^+} \sum_{k \leq m} (-1)^k h^k(M, \lambda) \text{ch } \nabla(\lambda)$,

- (ii) M has good dimension $\leq m$,
- (iii) $H^{m+1}(U, M)_\lambda = 0$, for any $\lambda \in P^+$.

Proof: The equivalence (i) $\iff$ (iii) follows from Theorem A5 (apply it for $n = m$ and for $n = m + 1$). Using a short exact sequence $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$, where X has a good filtration, the equivalence (i) $\iff$ (ii) follows also by induction over m . Q.E.D.

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