Abstract

We construct regular and black hole solutions in SU(2) Einstein-Born-Infeld theory. These solutions have many features in common with the corresponding SU(2) Einstein-Yang-Mills solutions. In particular, sequences of neutral non-abelian solutions tend to magnetically charged limiting solutions, related to embedded abelian solutions. Thermodynamic properties of the black hole solutions are addressed.
1 Introduction

Non-linear electrodynamics was proposed in the thirties to remove singularities associated with charged pointlike particles [1]. More recently such non-linear theories were considered in order to remove singularities associated with charged black holes [2, 3, 4].

Among the non-linear theories of electrodynamics Born-Infeld (BI) theory [1] is distinguished, since BI type actions arise in many different contexts in superstring theory [5, 6]. The non-abelian generalization of BI theory yields an ambiguity in defining the Lagrangian from the Lie-algebra valued fields. While the superstring context favors a symmetrized trace [7], the resulting Lagrangian is so far only known in a perturbative expansion [8]. However, the ordinary trace structure has also been suggested [9].

Motivated by this strong renewed interest in BI and non-abelian BI theory, recently non-perturbative non-abelian BI solutions were studied, both in flat and in curved space. In particular, non-abelian BI monopoles and dyons exist in flat space [10] as well as in curved space [11] together with dyonic BI black holes.

Likewise one expects regular and black hole solutions in pure SU(2) Einstein-Born-Infeld (EBI) theory, representing the BI generalizations of the regular Bartnik-McKinnon solutions [12] and their hairy black hole counterparts [13]. This expectation is further nurtured by the observation, that SU(2) BI theory possesses even in flat space a sequence of regular solutions [14].

Here we construct these SU(2) EBI regular and black hole solutions, employing the ordinary trace as in [14]. The set of EBI equations depends essentially on one parameter, $\gamma$, composed of the coupling constants of the theory. The solutions are labelled by the node number $n$ of the gauge field function. We construct solutions up to node number $n = 5$ and discuss the limiting solutions, obtained for $n \to \infty$, for various values of the horizon radius $x_H$ and the parameter $\gamma$. We also address thermodynamic properties of the black hole solutions.

2 SU(2) EBI Equations of Motion

We consider the SU(2) EBI action

$$S = S_G + S_M = \int L_G \sqrt{-g} d^4x + \int L_M \sqrt{-g} d^4x$$

with

$$L_G = \frac{1}{16\pi G} R, \quad L_M = \beta^2 \left( 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu}^a \tilde{F}^{a\mu\nu})^2} \right),$$

field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c,$$
gauge coupling constant $e$, gravitational constant $G$ and BI parameter $\beta$.

To construct static spherically symmetric regular and black hole solutions we employ Schwarzschild-like coordinates and adopt the spherically symmetric metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -A^2 N dt^2 + N^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,$$

with the metric functions $A(r)$ and

$$N(r) = 1 - \frac{2m(r)}{r}.$$  

The static spherically symmetric and purely magnetic Ansatz for the gauge field $A_\mu$ is

$$A_0^a = 0 , \quad A_i^a = \epsilon_{aik} \frac{1 - w(r)}{er^2} .$$

For purely magnetic configurations $F_{\mu\nu} F^{\mu\nu} = 0$.

With the ansatz (4)-(6) we find the static action

$$S = e \int dr \frac{A}{2} \left[ 1 + N \left( 1 + 2r \left( \frac{A'}{A} + \frac{N'}{2N} \right) \right) - \alpha^2 \beta^2 r^2 A \left[ 1 - \sqrt{1 + \frac{2Nw'^2}{\beta^2 e^2 r^2} + \frac{(1 - w^2)^2}{\beta^2 e^2 r^4}} \right] \right] ,$$

where $\alpha^2 = 4\pi G$.

Introducing dimensionless coordinates and a dimensionless mass function

$$x = \sqrt{e\beta r} , \quad \mu = \sqrt{e\beta m}$$

as well as the dimensionless parameter

$$\gamma = \frac{\alpha^2 \beta}{e} ,$$

we obtain the set of equations of motion

$$\mu' = -\gamma x^2 \left[ 1 - \sqrt{1 + \frac{2Nw'^2}{x^2} + \frac{(1 - w^2)^2}{x^4}} \right] ,$$

$$\frac{A'}{A} = \gamma \frac{2w'^2}{x \sqrt{1 + \frac{2Nw'^2}{x^2} + \frac{(1 - w^2)^2}{x^4}}} ,$$

$$\left( AN \frac{\mu'}{\sqrt{1 + \frac{2Nw'^2}{x^2} + \frac{(1 - w^2)^2}{x^4}}} \right)' = \frac{A\mu(w^2 - 1)}{x^2 \sqrt{1 + \frac{2Nw'^2}{x^2} + \frac{(1 - w^2)^2}{x^4}}} .$$
In eq. (12) the metric function $A$ can be eliminated by means of eq. (11).

We consider only asymptotically flat solutions, where the metric functions $A$ and $\mu$ both approach a constant at infinity. Here we choose

$$A(\infty) = 1.$$  \hspace{1cm} (13)

For magnetically neutral solutions the gauge field configuration approach a vacuum configuration at infinity

$$w(\infty) = \pm 1.$$ \hspace{1cm} (14)

Globally regular solutions satisfy at the origin the boundary conditions

$$\mu(0) = 0, \quad w(0) = 1,$$

whereas black hole solutions with a regular event horizon at $x_H$ satisfy there

$$\mu(x_H) = \frac{x_H}{2}, \quad N' w' \bigg |_{x_H} = \frac{w(w^2 - 1)}{x_H^2} \bigg |_{x_H}.$$  \hspace{1cm} (16)

### 3 Embedded Abelian BI Solutions

Before discussing the non-abelian BI solutions, let us briefly recall the embedded abelian BI solutions [15, 2, 16]. With constant functions $A$ and $w$,

$$A = 1, \quad w = 0,$$ \hspace{1cm} (17)

they carry one unit of magnetic charge, and their mass function satisfies

$$\mu' = \gamma \left( \sqrt{x^4 + 1} - x^2 \right),$$ \hspace{1cm} (18)

or upon integration

$$\mu(x) = \mu(0) + \gamma \int_0^x \left( \sqrt{x'^4 + 1} - x'^2 \right) dx'.$$ \hspace{1cm} (19)

Their mass $\mu_{\infty}$ is thus given by

$$\mu_{\infty} = \mu(0) + \gamma \frac{\pi^{3/2}}{3\Gamma^2(3/4)}.$$ \hspace{1cm} (20)

The solutions are classified according to the integration constant $\mu(0)$ [16]. For $\mu(0) > 0$ black hole solutions with one non-degenerate horizon $x_H$ are obtained. For $\mu(0) = 0$ black hole solutions with one non-degenerate horizon $x_H$ are obtained for $\gamma > 1/2$, otherwise the solutions possess no horizon. For $\mu(0) < 0$ black hole solutions
with two non-degenerate horizons, extremal black hole solutions with one degenerate horizon or solutions with no horizons are obtained, similarly to the Reissner-Nordstrøm (RN) case.

For black hole solutions with event horizon at $x_H = 2\mu(x_H)$ the integration constant $\mu(0)$ is given by

$$\mu(0) = \frac{x_H}{2} - \gamma \int_0^{x_H} \left( \sqrt{x^4 + 1} - x^2 \right) dx . \quad (21)$$

Since extremal black hole solutions satisfy

$$\mu'(x_H) = \frac{\mu(x_H)}{x_H} = \frac{1}{2}, \quad (22)$$

eq (18) yields for the degenerate horizon of extremal black holes

$$x_{ex}^H = \sqrt{\gamma - \frac{1}{4\gamma}} . \quad (23)$$

Thus we find a critical value $\gamma_{cr} = 1/2$, where $x_{H,cr}^{ex} = 0$ and $\mu_{cr}(0) = 0$. Finally, the Hawking temperature $T$ of the black holes is given by [16]

$$T = \frac{1}{4\pi x_H} \left[ 1 - 2\gamma \left( \sqrt{x_H^4 + 1} - x_H^2 \right) \right] . \quad (24)$$

### 4 Regular Solutions

Recently, a sequence of non-abelian regular BI solutions was found in flat space by Gal’tsov and Kerner [14], labelled by the node number $n$ of the gauge field function $w$.

We now consider this sequence of BI solutions in the presence of gravity, i.e. for finite $\gamma$ (and thus finite $\alpha$), and compare the non-abelian regular BI solutions to the regular Einstein-Yang-Mills (EYM) solutions [12].

With increasing $\gamma$ the regular BI solutions evolve smoothly from the corresponding flat space BI solutions. Let us first consider the sequence of regular BI solutions for $\gamma_{cr}$. In Fig. 1 we show the gauge field function $w$ of the regular BI solutions with node numbers $n = 1, 3$ and 5.

The metric function $N$ of these BI solutions is shown in Fig. 2 together with the metric function $N$ of the extremal abelian BI solution with $x_{H,cr}^{ex} = 0$ and one unit of magnetic charge. Clearly, with increasing $n$ the metric function of the non-abelian regular BI solutions tends to the metric function of the extremal abelian solution.

In Fig. 3 we show the charge function $P(x)$

$$P^2(x) = \frac{2x}{\gamma} (\mu_\infty - \mu(x)) , \quad (25)$$
obtained from an expansion of the abelian mass function for general charge \( P \), for these regular BI solutions together with the constant charge \( P = 1 \) of the extremal abelian BI solution with \( x_H^{\text{ex}} = 0 \).

Furthermore, for \( \gamma = \gamma_{\text{cr}} \) the masses of the non-abelian regular BI solutions converge exponentially to the mass of the extremal abelian BI solution with \( x_H^{\text{ex}} = 0 \) and unit magnetic charge, which represents the limiting solution of this sequence.

Let us now consider \( \gamma \neq \gamma_{\text{cr}} \). For \( \gamma < \gamma_{\text{cr}} \) the sequence of non-abelian regular BI solutions tends to an abelian BI solution without horizon and unit magnetic charge, and with integration constant \( m(0) = 0 \).

For \( \gamma > \gamma_{\text{cr}} \) we must distinguish two spatial regions, \( x < x_H^{\text{ex}} \) (given in eq. (23)), and \( x > x_H^{\text{ex}} \). Only in the region \( x > x_H^{\text{ex}} \) the metric function \( N \) of the sequence of non-abelian regular BI solutions tends to the metric function of the extremal abelian BI black hole solution with horizon \( x_H^{\text{ex}} \) and unit magnetic charge. For \( x < x_H^{\text{ex}} \) it tends to a non-singular limiting function. This is demonstrated in Fig. 4, where the metric function \( N \) is shown for \( n = 5 \) and \( \gamma = 0.01, \gamma = 1, \) and \( \gamma = 100 \), together with the metric function \( N \) of the corresponding limiting abelian BI solutions.

Thus the non-abelian regular BI solutions are very similar to their EYM counterparts. In particular, there with increasing node number \( n \) the sequence of neutral \( SU(2) \) EYM solutions also tends to a limiting charged solution, which for radius \( x \geq 1 \) is the extremal embedded RN solution with magnetic charge \( P = 1 \) [17, 18].

## 5 BI Black Hole Solutions

Imposing the boundary conditions eq. (16), leads to non-abelian BI black hole solutions with regular horizon. Like their non-abelian EYM counterparts, non-abelian BI black hole solutions exist for arbitrary value of the horizon radius \( x_H \). In both cases the sequences of neutral non-abelian BI black hole solutions tend to limiting solutions with unit magnetic charge.

For \( \gamma < \gamma_{\text{cr}} \) the limiting solution of the non-abelian BI black hole solutions is always the abelian BI black hole solution with the same horizon. The same holds true for \( \gamma = \gamma_{\text{cr}} \).

For \( \gamma > \gamma_{\text{cr}} \) and \( x_H < x_H^{\text{ex}} \) the limiting solution is the extremal abelian BI black hole solution with horizon \( x_H^{\text{ex}} \)

only in the region \( x > x_H^{\text{ex}} \), but differs from it in the region \( x_H < x < x_H^{\text{ex}} \). For \( x_H > x_H^{\text{ex}} \) the limiting solution is always the abelian BI black hole solution with the same horizon. This is demonstrated in Fig. 5 for \( \gamma = 1 \) and horizon radii \( x_H = 0.1, 0.2, 0.5, \) and \( 10 \).

The temperature of the non-abelian BI black holes is obtained from [19]

\[
T = \frac{1}{4\pi} AN'.
\] (26)
In Fig. 6 we show the inverse temperature of the non-abelian BI black hole solutions with node numbers $n = 1$, 3 and 5 as a function of their mass for $\gamma = 0.01$ and $\gamma = 1$. Also shown is the inverse temperature of the corresponding limiting abelian BI solutions. Again, with increasing $n$ rapid convergence towards the limiting values is observed, analogously to the EYM and EYM-dilaton case [20].

Further details will be given elsewhere [21].

6 Conclusions

We have constructed sequences of regular and black hole solutions in SU(2) EBI theory. The solutions are labelled by the node number $n$. With increasing node number these sequences of non-abelian neutral solutions tend to limiting solutions, corresponding (at least in the outer spatial region) to abelian BI solutions with unit magnetic charge. These features are similar to those observed for non-abelian EYM and EYM-dilaton regular and black hole solutions [13, 17, 20, 18]. This similarity is also observed for the Hawking temperature.

By generalizing the framework of isolated horizons to non-abelian gauge theories [22], recently new results were obtained for EYM black hole solutions. In particular nontrivial relations between the masses of EYM black holes and regular EYM solutions were found, and a general testing bed for the instability of non-abelian black holes was suggested [22]. Application of such considerations to non-abelian BI black holes appears to be interesting.

References


Figure 1: The gauge field function $w$ is shown as a function of the dimensionless coordinate $x$ for the regular BI solutions with node numbers $n = 1, 3, 5$ for $\gamma = \gamma_{cr}$. 
Figure 2: The metric function $N$ is shown as a function of the dimensionless coordinate $x$ for the regular BI solutions with node numbers $n = 1, 3, 5$ for $\gamma = \gamma_{cr}$. Also shown is the metric function $N$ for the extremal abelian BI solution with $x_{H}^{\text{ex}} = 0$ and unit magnetic charge.
Figure 3: The charge function $P^2$ is shown as a function of the dimensionless coordinate $x$ for the regular BI solutions with node numbers $n = 1, 3, 5$ for $\gamma = \gamma_{cr}$. Also shown is the constant function $P = 1$ of the abelian BI solution.
Figure 4: The metric function $N$ is shown as a function of the dimensionless coordinate $x$ for the regular BI solutions with node number $n = 5$ for $\gamma = 0.01$, $\gamma = 1$, and $\gamma = 100$. Also shown are the metric functions $N$ of the corresponding limiting abelian BI solutions.
Figure 5: The metric function $N$ is shown as a function of the dimensionless coordinate $x$ for the BI black hole solutions with node number $n = 5$ for $\gamma = 1$ and horizon radii $x_\text{H} = 0.1, 0.2, 0.5,$ and $10$. Also shown are the metric functions $N$ of the corresponding limiting abelian BI solutions.
Figure 6: The inverse temperature $1/T$ is shown as a function of the dimensionless mass $\mu$ for the non-abelian BI black hole solutions with node numbers $n = 1, 3$ and 5 for $\gamma = 0.01$ and $\gamma = 1$. Also shown is the inverse temperature for the corresponding limiting abelian BI solutions. (The $\gamma = 0.01$ curves cannot be graphically resolved in the figure.)