ROTH'S THEOREM FOR RULED SURFACES

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Abstract

This paper addresses conjectures of E. Bombieri and P. Vojta in the special case of ruled surfaces not birational to $\mathbb{P}^2$. Apart from this implicit restriction to $\mathbb{P}^1$ bundles S over an elliptic curve, the ultimate question of the arithmetic of pairs (S, D) for a divisor D requires further restrictions on D which turn the proposed conjectures into the study of Roth's theorem on approximation of algebraic numbers α , but for α now parametrised by an elliptic curve. With these restrictions, best possible answers are obtained. The same study may also be carried out for holomorphic maps, and this is done simultaneously.

0. Introduction

0.0. Results and method

Let (S, D) be a projective normal surface with a normal crossing divisor then E. Bombieri and P. Vojta have posed the question to what extent the arithmetic of this pair is determined by K_S and D . Specifically the latter, cf. [V1], has posed the following question, fix a finite set ∂ of places of $\text{Spec } \mathbb{Z}$ which contains the infinite one and $p : Y \rightarrow \text{Spec } \mathbb{Z}$ a finite cover with Y normal then does there exist a proper closed subset Z of S , independent of Y , such that if $f : Y \rightarrow S$ is a map not factoring through Z do we have:

$$m_{D, p^{-1}(\partial)}(f) + K_{S, f} Y \leq_{\text{exc.}} \varepsilon A_{, f} Y$$

where (S, D) now denotes a pair over $\text{Spec } \mathbb{Z}$ whose generic fibre was as before, A is ample, $\leq_{\text{exc.}}$ means $\leq$ for all but finitely many such f , and what are classically termed heights are written as intersection numbers. The defect term $m_{D, \partial}$ measures the closeness of the boundary ∂ to D under f , and indeed should $f : (Y, p^{-1}(\partial)) \rightarrow (S, D)$ be a map of pairs, i.e. an integral point, then $D_{, f} Y = m_{D, p^{-1}(\partial)}(f)$. Whence a corollary of this question would be that if (S, D) has log-general type then all but finitely many integral points are contained in a proper closed subset determined purely by the geometry of the

pair. There is a wholly analogous question for holomorphic maps $f : \mathbb{A}^1 \rightarrow S$ where now the intersection numbers are a function of a positive real number r , i.e. they are defined on discs of radius r , the boundary ∂ is just the boundary of the disc of radius r , and the subscript exc. denotes exclusion from a set of finite Lebesgue measure.

Necessarily the analytic version of this question is somewhat more pliable and a positive answer to this question has been obtained in a substantial number of cases in [M1] and [M2] by a mixture of diophantine approximation and non-commutative analogues of Moris' programme, together with a general programme [M3] for finishing off the rest of the job. The arithmetic version is however wide open, there being only the classical results on hyperplane arrangements of W. Schmidt, [S], and the remarkable work of G. Faltings, [F], which applies in cases of high irregularity. A more general version essentially for S quasi-projective of G. Faltings' results having been obtained by P. Vojta in [V3], and [V4] as well as the second author in [M4]. Whence given the relative paucity of results it seems reasonable to address this question systematically according to the classification of surfaces. As we have noted already in $\mathbb{P}^2$, the diophantine approximation technique, or at least its classical arithmetic manifestation, necessitates a restriction on the divisors that we may study. Consequently in considering the next step in the classification, viz: $\mathbb{P}^1$ -bundles S over a curve X the appearance of a similar restriction should offer no surprises. Needless to say X should be an elliptic curve to avoid a trivial answer, and we restrict our attention to reduced divisors whose components are amiable, i.e. étale over X on the generic fibre, or indeed fibres of S/X . Let us call such a divisor nice (note there are no restrictions on the crossings of components). Our inability to go beyond nice divisors is closely related to the failure of the Thue-Siegel-Roth methodology in attempting to treat simultaneous diophantine approximation of algebraically independent irrationals. In any case our results are as follows,

Theorem 0.0.0. *Let (S, D) be a minimal ruled surface over an elliptic curve X together with a nice divisor D then there is a proper closed subset Z of S such that if $\varepsilon > 0$ and $f : Y \rightarrow S$ a map, enjoying either the arithmetic or analytic flavour explained above, not factoring through Z then,*

$$m_{D, p^{-1}(\partial)}(f) + K_{S/f} Y \leq_{\text{exc.}} \varepsilon A_{\cdot f} Y$$

where A is an ample divisor on S .

To give some sort of feeling for this result consider the case where $S = X \times \mathbb{P}^1$, with say X an elliptic curve over $\mathbb{Q}$ and $\varphi : X \otimes k \rightarrow \mathbb{P}_k^1$, a rational function on X defined over some number field k . For D we take not just the graph of φ but also all its conjugates and conclude,

Corollary 0.0.1. *There is a proper closed subset Z of $X \times \mathbb{P}^1$ such that if $\varepsilon > 0$ and $x \times p/q \in X \times \mathbb{P}^1(\mathbb{Q}) \setminus Z$, then, there is a constant $C(\varepsilon, \varphi)$ depending*

only on φ and ε such that,

$$\left| \varphi(x) - \frac{p}{q} \right| \geq \frac{C(\varepsilon, \varphi)}{q^{2+\varepsilon} H(x)^\varepsilon}$$

where $H(x)$ is the big height of x in any projective embedding.

Specifically this is a type of uniform Roth theorem for irrational numbers coming from elliptic curves and should be compared with the more general result of [M5], i.e. given $\varepsilon > 0$ there is a constant $C(\varepsilon, k)$ such that for all $\alpha \in k$ we have,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C(\varepsilon, k)}{q^{2+\varepsilon} H(\alpha)^{b(\varepsilon)(k, \mathbb{Q})}}$$

here $b(\varepsilon)$ is an effectively computable function of ε , but grows like $1/\varepsilon$. In addition we should bear in mind the additional,

Corollary 0.0.2. *Let (S, D) be as in the theorem and suppose further that $S \setminus D$ has log-general type then there is a proper closed subset Z of S such that all but finitely many integral points of $S \setminus D$ (respectively any holomorphic map from the complex line to $S \setminus D$) lie on Z (respectively factors through Z).*

0.1. Notations

It is worth remarking on the slightly non-standard notation being employed. In particular if $X/\text{Spec } \mathbb{Z}$ is a scheme and $\widehat{D}$ a Cartier divisor on X with metrics at infinity then for $p : Y \rightarrow \text{Spec } \mathbb{Z}$ a finite cover and $f : Y \rightarrow X$ a map, we use $\widehat{D}_f$ to denote the Arakelov intersection number, $p_*(f^* \widehat{D}) \in \widehat{CH}_0(\text{Spec } \mathbb{Z}) = \mathbb{R}$. Since variation with metric of these intersection numbers is essentially trivial it will usually be suppressed in the notations, although its existence is implicit. Tensor products of bundles will be understood to have the natural induced metric. In Nevanlinna theory there is a parallel theory of intersections, i.e. one considers metricised Cartier divisors $\overline{D}$, say, on schemes X over $\mathbb{C}$ and for $r \in \mathbb{R}_+$, and $f : \mathbb{A}_{\mathbb{C}}^1 \rightarrow X$ holomorphic, with Y the disc of radius r , $\overline{D}_f$ is just the Nevanlinna characteristic function, classically denoted by, $T_{f, \overline{D}}(r)$. Identical remarks for dependence on the metric of this number, and for tensor products etc. apply; as they do for the defect function which measures the distance of $f(\partial Y)$ to D . Here ∂Y will either be $p^{-1}(\partial)$ where ∂ is a finite set of places in $\text{Spec } \mathbb{Z}$ including infinity or just the topological boundary in the analytic case. Regardless,

$$m_{D, \partial Y}(f) := -\frac{1}{2} \int_{\partial Y} \log \|f^* \mathbb{I}_D\|^2 d\mu$$

where the measure is either counting measure or the usual Lebesgue measure on a circle.

I. Reductions and Trivialities

I.1. The Empty Case

Evidently if we propose to demonstrate Theorem 0.0.0 then certainly we ought to be able to prove the same in the particular case that the divisor in question is empty. As ever we let $\pi : S = \mathbb{P}(F) \rightarrow X$ be a minimal ruled surface with tautological bundle H over an elliptic curve and A an ample divisor then we may observe.

Proposition I.1.1. *Let $\varepsilon > 0$ be rational then the $\mathbb{Q}$ -divisor $\varepsilon A - K_S$ is big. Better still if F is semi-stable then any sufficiently large integral multiple is base point free, while if F is unstable then $-K_S$ is effective.*

From which we obtain without further ado,

Corollary I.1.2. *Theorem 0.0.0 holds for an empty divisor.*

Let us now make the simple remarks that constitute the proof of the proposition,

Proof. If F is semi-stable then essentially by definition $-K_S$ is nef., so there is nothing to prove. Contrary wise let,

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$$

be the destabilizing dévissage and $Y \hookrightarrow S$ the corresponding section, then $-K_S = 2Y + \pi^*(2F' - c_1(F))$. However X is an elliptic curve so $2F' - c_1(F)$ is not just positive, but effective. $\square$

I.2. Amiable Divisors and Étale covers

We begin with the promised definition,

Definition I.2.1. *Let $\pi : S \rightarrow X$ be a ruled surface over an elliptic curve then a geometrically integral divisor D in S is said to be amiable if $\pi|_D$ is étale, and very amiable if D is actually a section of π .*

From which we proceed to the following reduction lemma,

Lemma I.2.2. *Suppose in addition that we are over an algebraically closed field of characteristic zero and D is an amiable divisor on S then there is an étale cover $\rho : Y \rightarrow X$ from an elliptic curve, such that if we form the fibre square,*

$$\begin{array}{ccc} T & \xrightarrow{\rho} & S \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\rho} & X \end{array}$$

Then $\pi : T \rightarrow Y$ remains a minimal ruled surface, while $\rho^ D$ is a sum of very amiable divisors.*

Proof. The first part of the proposition is just base change for Proj. For the second part we simply induct on the degree of a given amiable divisor D over X . Specifically we consider the étale cover $\rho := \pi|_D: D \rightarrow X$, D certainly being an elliptic curve, and construct the appropriate fibre square as in the statement then $\rho^* D = D' + D''$ where D' is a section of $\pi: T \rightarrow D$. Moreover $\rho^* D \rightarrow D$ is étale so D' and D'' are disjoint. Consequently every component of D'' is an amiable divisor of strictly smaller degree over D , and we are done. $\square$

Given that we are prepared to make étale covers then we may as well utilise a further straightforward reduction, viz:

Lemma I.2.3. *Let $\pi: S \rightarrow X$ be a minimal ruled surface over an elliptic curve over a field of characteristic distinct from 2, then there is an étale map $\rho: X \rightarrow X$ such that if we form the fibre square à la I.2.2 then T is of the form $\mathbb{P}(F)$ for F a bundle of determinant zero.*

Proof. Certainly $X = \mathbb{P}(F)$ for some rank 2-bundle F , so the proposition just amounts to taking covers on order to obtain a square root of $c_1(F)$, which can be done by taking ρ to be no worse than multiplication by 4. $\square$

We can therefore put everything together to obtain the following simplifications, i.e.

Fact I.2.4. In order to prove Theorem 0.0.0 we may without loss of generality make the following assumptions on our ruled surface $\pi: S \rightarrow X$, and divisor D .

- (a) S is a regular proper modification of some $\mathbb{P}(F)$ where F is a rank 2 bundle of determinant 0.
- (b) D is a sum of geometrically integral components, each of which is a section of π , and which are mutually disjoint.

We justify this with some further remarks. In the geometric (i.e. Nevanlinna theory) case this is clear since the complex line is simply connected, and Nevanlinna's second main theorem for elliptic curves allows us to make the implied additional modifications in (b). The corresponding arithmetic analogues are the Chevalley-Weil theorem, and Siegel's theorem for defects of divisors on elliptic curves.

II. Counting

II.1. A suitable bundle

In line with the considerations of I.2.4 we begin by supposing that $\pi: S = \mathbb{P}(F) \rightarrow X$ is a minimal ruled surface of determinant zero and $D \hookrightarrow S$ is a section of π . We wish to construct a line bundle on a m -fold product of S with itself which admits many sections vanishing to high order on D^m .

The reader wishing to understand the *raison d'être* for this curious action should note the definitions, and then pass to III.1. In any case we kick off by introducing the bundles of Faltings-Vojta type on X^m , viz,

Definition II.1.1. *Let $(\varepsilon, \underline{s})$ be a given $(m+1)$ tuple of positive rational numbers and L an ample symmetric bundle on X then denoting pull-back from the i^{th} (resp. $(i, j)^{\text{th}}$) factor $1 \leq i \leq m$ (resp. $1 \leq i, j \leq m$) by appropriate suffices we define the $\mathbb{Q}$ -bundle,*

$$L(\varepsilon, \underline{s}) := \sum_{i=1}^m \varepsilon s_i^2 L_i + \sum_{i < j} (s_i x_i - s_j x_j)^* L$$

on X^m , where $(s_i x_i - s_j x_j)^* L$ is naturally understood as, $s_i^2 L_i + s_j^2 L_j - s_i s_j P_{ij}$, P_{ij} being the Poincaré bundle associated to L .

Naturally enough we continue to use π for the projection from $S^m \rightarrow X^m$, and H the tautological bundle on S , which in the given normalisations is $-\frac{1}{2} K_S$. Regardless we combine the above definition with the rather more classical one of Roth et al, to obtain our desired bundle, i.e.

Definition II.1.2. *Notations and conventions as in II.1.1 then,*

$$M(\varepsilon, \underline{s}) := \pi^* L(\varepsilon, \underline{s}) + \sum_{i=1}^m s_i^2 H_i.$$

We note from [V2] that,

$$L(\varepsilon, \underline{s})^m = m! \varepsilon (m + \varepsilon)^{m-1} (\deg L)^m \prod_{i=1}^m s_i^2$$

so that for any sufficiently large and divisible $d \in \mathbb{N}$ we obtain,

$$\chi(S^m, M(\varepsilon, \underline{s})^{\otimes d}) \sim d^{2m} \varepsilon (m + \varepsilon)^{m-1} (\deg L)^m \prod_{i=1}^m s_i^4.$$

Now if F is semi-stable then $M(\varepsilon, \underline{s})$ is ample, so eventually, i.e. for $d \gg 0$, the Euler characteristic coincides with the space of global sections. In general, however, we have to do a little more work. We begin by considering the divisor C_i , say, on X^m which is the pull back of a point by the i^{th} projection and put $C = C_1 + \dots + C_m$. Certainly for $n \gg 0$, depending on d , $M(\varepsilon, \underline{s})^{\otimes d} \otimes \mathcal{O}_{S^m}(n \pi^* C)$ is acyclic, while for $\deg L \gg 0$, but depending only on S , $M(\varepsilon, \underline{s})^{\otimes d} |_{\pi^* C_i}$ is ample for any $1 \leq i \leq m$, so indeed $M(\varepsilon, \underline{s})^{\otimes d} |_{\pi^* C}$ is ample under the same hypothesis. Better still $\mathcal{O}_{\pi^* C}(\pi^* C)$ is eventually, i.e. after taking a suitable multiple, base point free so that following [Fa] we can show that for all n and all $d \gg 0$, $M(\varepsilon, \underline{s})^{\otimes d}(n \pi^* C) |_{\pi^* C}$ is acyclic. Note that the proof in op.cit. supposes that $\mathcal{O}_{\pi^* C}(\pi^* C)$ is ample, but goes through verbatim with the weaker assumption of base point free. In any case

a straightforward induction on infinitesimal neighbourhoods of $\pi^* C$ allows us to conclude that $H^q(S^m, M(\varepsilon, \underline{s})^{\odot d})$ is zero for $q \geq 2$, and d sufficiently large and divisible. Necessarily we therefore arrive at our goal, i.e.

Lemma II.1.3. *Notations as given, then for $\deg L$ greater than a constant determined by S and, d sufficiently large and divisible,*

$$\lim_{d \rightarrow \infty} h^0 \left(S^m, \frac{M(\varepsilon, \underline{s})^{\odot d}}{d^{2m}} \right) \geq \varepsilon(m + \varepsilon)^{m-1} (\deg L)^m \prod_{i=1}^m s_i^4.$$

II.2. Vanishing conditions

We continue to suppose that $\pi : S \rightarrow X$ is a minimal ruled surface of determinant zero, and now proceed for our section D to count the cost of vanishing along D . We begin with a definition,

Definition II.2.1. *Let $(\delta, \underline{s})$ be a $(m+1)$ tuple of positive rational numbers, and d a sufficiently large and divisible integer then $\mathcal{I}_{\delta, \underline{s}}^{(d)}$ is the coherent sheaf of ideals generated locally by monomials of the form, $h_1^{t_1} \dots h_m^{t_m}$ where h_i is a local generator for $\mathcal{O}_{S^m}(D_i)$ and we impose,*

$$\sum_{i=1}^m t_i s_i^{-2} \geq d\delta.$$

The sheaf of rings $\mathcal{O}_{S^m}/\mathcal{I}_{\delta, \underline{s}}^{(d)}$ admits a filtration F^t , say, by monomials while the regularity of the sequence of Cartier divisors $D_1, \dots, D_m$ permits the deduction that,

$$F^{t+1} \setminus F^t \xrightarrow{\sim} \mathcal{O}_{D^m}(-t_1 D_1 \dots - t_m D_m)$$

identifying F^t with the ideal in S^m which gives rise to it we may observe that if $t_i > ds_i^2$ for some $1 \leq i \leq m$ then the induced map,

$$H^0(S^m, M(\varepsilon, \underline{s})^{\odot d} \otimes F^t) \rightarrow H^0(D^m, M(\varepsilon, \underline{s})^{\odot d} (-t_1 D_1 \dots - t_m D_m))$$

is zero. We may thus deduce,

$$h^0(S^m, M(\varepsilon, \underline{s})^{\odot d} \otimes \mathcal{I}_{\delta, \underline{s}}^{(d)}) \geq h^0(S^m, M(\varepsilon, \underline{s})^{\odot d}) - \sum_{\substack{t_1 s_1^{-2} + \dots + t_m s_m^{-2} < d\delta \\ t_1 s_1^{-2} \leq d}} h^0(D^m, M(\varepsilon, \underline{s})^{\odot d} (-t_1 D_1 \dots - t_m D_m)).$$

Now if the pull-back of H by the section corresponding to D is numerically trivial then the bundles occuring in the summation have the same number of sections as $L(\varepsilon, \underline{s})^{\odot d}$ for $d \gg 0$, while in general,

$$h^0(D^m, M(\varepsilon, \underline{s})^{\odot d} (-t_1 D_1 \dots - t_m D_m)) \leq h^0 \left(X^m, L \left(\varepsilon + \frac{|H \cdot D|}{\deg L}, \underline{s} \right)^{\odot d} \right)$$

provided $t_i \leq d s_i^2$, from which we conclude,

$$\lim_{d \rightarrow \infty} \frac{h^0(S^m, M(\varepsilon, \underline{s})^{\otimes d} \otimes \mathcal{I}_{\delta, \underline{s}}^{(d)})}{h^0(S^m, M(\varepsilon, \underline{s})^{\otimes d})} \geq 1 - \frac{\left(\varepsilon + \frac{|H \cdot D|}{\deg L}\right) \left(m + \varepsilon + \frac{|H \cdot D|}{\deg L}\right)^{m-1}}{\varepsilon(m + \varepsilon)^{m-1}} \int_{\substack{x_1 + \dots + x_m \leq \delta \\ 0 \leq x_i \leq 1}} dx_1 \dots dx_m$$

and so an application of the law of large numbers for the uniform distribution on the unit interval gives,

Lemma II.2.2. *Let $\varepsilon > 0$ and $\delta < 1/2$ be given then for $m \gg 0$ and any m -tuple $\underline{s}$ of positive rational numbers together with any sufficiently large and divisible d we have,*

$$\lim_{d \rightarrow \infty} \frac{h^0(S^m, M(\varepsilon, \underline{s})^{\otimes d} \otimes \mathcal{I}_{\delta, \underline{s}}^{(d)})}{h^0(S^m, M(\varepsilon, \underline{s})^{\otimes d})} \geq 1 - o(m).$$

Remark II.2.3. A rather standard argument gives $o(m) \leq C(\varepsilon) \exp\left(\frac{-m}{\delta} \left(\frac{1}{2} - \delta\right)^2\right)$, where $C(\varepsilon)$ is a constant depending on ε . This is however irrelevant.

Finally we put everything together. We take as in I.2.4 a ruled surface $\pi : S \rightarrow X$ which is a proper modification of the type so far considered in this section and D on S a reduced divisor whose components are sections of π which do not meet. For any component D_j we define à la II.2.1 the corresponding ideal, $\mathcal{I}_{\delta, \underline{s}, j}^{(d)}$ say, and conclude,

Lemma II.2.4. *Let $\pi : S \rightarrow X$ and D be as in I.2.4, with $\varepsilon, \delta, m, \underline{s}, d$ as quantified in II.2.2 then we have,*

$$\lim_{d \rightarrow \infty} \frac{h^0\left(S^m, M(\varepsilon, \underline{s})^{\otimes d} \otimes \prod_j \mathcal{I}_{\delta, \underline{s}, j}^{(d)}\right)}{h^0(S^m, M(\varepsilon, \underline{s})^{\otimes d})} \geq 1 - o(m)$$

where indeed $o(m)$ is at worst the number of components of D times the corresponding error in II.2.2.

III. Nevanlinna Theory

III.1. Dynamic diophantine approximation

Let $\pi : S \rightarrow X$ be a proper regular modification of a minimal ruled surface of determinant 0, and D a reduced divisor whose components D_j , $1 \leq j \leq \nu$, are non-intersecting sections of π . We consider for each $1 \leq j \leq \nu$

the blow up W_j of S^m in D_j^m , and note that if we define a sequence of monoidal transformations,

$$S^m = V^0 \leftarrow V_1 \leftarrow \dots \leftarrow V_\nu = W^{(m)}$$

such that $V_j \rightarrow V_{j-1}$ is the blow up in the pull-back of D_j^m then: $V_j = W_j \times_{S^m} V_{j-1}$ by virtue of our supposition that the components are disjoint. Consequently we let E_j be the restriction to $W^{(m)}$ of the exceptional divisor on W_j , simplify notations by putting $M(\varepsilon) = M(\varepsilon, 1)$ and observe that II.2.4 implies,

Fact III.1.1. Let $\varepsilon > 0$ and $\delta < 1/2$ be given then for $m \gg 0$ there is an effective $\mathbb{Q}$ -divisor E such that,

$$M(\varepsilon)|_{W^{(m)}} = E + \sum_{j=1}^{\nu} m \delta E_j.$$

Now let $f : \mathbb{A}_{\mathbb{C}}^1 \rightarrow S$ be a holomorphic map, and consider the following diagram,

$$\begin{array}{ccccc} & & \mathbb{A}^1 & \xrightarrow{\tilde{f}_\lambda} & S \\ & & \parallel & \nearrow \tilde{\Delta} & \parallel \\ & & W^{(m)} & & \\ & & \parallel & \searrow \tilde{\Delta} & \parallel \\ & & \mathbb{A}^1 & \xrightarrow{f} & S \\ & \nearrow \tilde{f}_\Lambda & & & \\ \mathbb{A}^1 & \xrightarrow{\Delta_\Lambda} & \mathbb{A}^m & \xrightarrow{f^m} & S^m \\ & \nwarrow \Delta & \nwarrow \Delta & \nwarrow \Delta & \\ & & \mathbb{A}^1 & \xrightarrow{f} & S \end{array}$$

$z \longmapsto (\lambda_1 z, \dots, \lambda_m z).$

We identify $\lambda \in \mathbb{C}$ with $(\lambda, \dots, \lambda)$ so that the proper transforms $\tilde{f}_\lambda$ of the compositions $\Delta_\Lambda^* f^m$ should be considered as deformations of $\tilde{f}_\lambda$, which is essentially just f by virtue of the fact that each D_j^m , $1 \leq j \leq \nu$, intersects the diagonal in the Cartier divisor D_j .

Now suppose that $f : \mathbb{A}^1 \rightarrow S$ has dense image, then we have the following basic dynamic intersection estimate, viz,

$$\int_{\|\Lambda - \lambda\| = \eta} M(\varepsilon) \tilde{f}_\Lambda^* \mathbb{A}^1(1) \geq 0(\log \eta) + m \delta \sum_{j=1}^{\nu} \int_{\|\Lambda - \lambda\| = \eta} m_{E_j, \partial A^1(1)}(\tilde{f}_\Lambda) d\mu(\eta)$$

where $\mathbb{A}^1(1)$ is the unit disc, and $d\mu(\eta)$ is the usual measure on polydiscs in $\mathbb{A}^{m-1}$ corresponding to the splitting $\mathbb{A}^m = \mathbb{A}^{m-1} \times \Delta$, so that implicitly $\|\cdot\|$ is just the sup-norm.

It follows from the dynamic intersection estimates of [M4] and [M6] that for $\|A - \lambda\|$ bounded by a polynomial in $\{A, \mathbb{A}^1(|\lambda|)\}^{-1}$ and $|\lambda|^{-1}$, where A is ample, that the difference in absolute value of the integrands appearing in this formula from their value at λ is bounded provided λ is excluded from a set of finite measure. Consequently putting $r = |\lambda|$, we obtain,

Proposition III.1.2. *Let $\varepsilon > 0$ and $f : \mathbb{A}^1 \rightarrow S$ be dense then,*

$$m_{D, \partial \mathbb{A}^1}(r)(f) + K_{S, f} \mathbb{A}^1(r) \leq_{\text{exc.}} \varepsilon A, f \mathbb{A}^1(r).$$

III.2. Cleaning up

We are therefore reduced to studying the case where the map f factors through a complete curve C . Consequently we suppose that $\pi : S \rightarrow X$ is a minimal ruled surface, and let Z be the closed subvariety consisting of the union of the curve on which $-K_S$ is negative and any fibres of π that contain the centres of the monoidal transformations implicit in I.2.4. Should C be a fibre of π outside of Z then Theorem 0.0.0 is simply Nevanlinna's second main theorem for $\mathbb{P}^1$ with the divisor $\mathcal{O}_C(D)$. Otherwise C is of non-zero genus and as noted in the reductions pertinent to I.2.4, the theorem again follows from Nevanlinna's 2nd main theorem provided that $-K_S \cdot C \geq 0$, and we are done.

IV. Arithmetic

IV.1. Appropriate models

To proceed further we must make suitable models of the objects that we have introduced so far over a finite regular cover $p : Y \rightarrow \text{Spec } \mathbb{Z}$. These models will now be denoted by the same letters for what we have previously studied, i.e. their generic fibres, and the fibre itself will be specifically so indicated. In particular let X/Y be a flat regular geometrically integral scheme whose generic fibre is an elliptic curve, and following [F], let $B \rightarrow X^m$ be a proper normal modification on which the Poincaré bundles introduced in II.1.1 are defined. Now let F be a coherent rank 2 reflexive sheaf on X whose generic fibre is free and let F' be a line bundle on X which admits a map to F on the generic fibre, then since F is reflexive and X is Cohen-Macaulay (indeed its actually free since X is regular) this map extends across X and defines a Cartier divisor on $\mathbb{P}(F)$ which is a quasi-section (i.e. a section in codimension 2). Consequently given any section on the generic fibre we may find a quasi-section on $\mathbb{P}(F)$ extending it, whence let $D_1, \dots, D_\nu$ denote such quasi-sections whose generic fibres are the sections constituting the components of an amiable divisor à la I.2.4. Finally observe that any point on the generic fibre of $\mathbb{P}(F)$, (respectively a modification of it) extends to a one dimensional subscheme which admits a lifting to the relative fibre product

$\mathbb{P}(F_1) \times_B \dots \times_B \mathbb{P}(F_m)$ (respectively a modification of it). Whence by keeping track of the centres of the monoidal transformations implicit in 1.2.4 we can construct a geometrically integral scheme $\pi : V \rightarrow B$ whose generic fibre is a product of not necessarily minimal ruled surfaces, together with Cartier divisors D_{ij} , $1 \leq i \leq m$, $1 \leq j \leq \nu$ such that for fixed i , the D_{ij} and D_{ik} have empty intersection on the generic fibre, and consequently we have models of everything that has been previously studied.

Now let us pause for a moment to consider the case where the generic fibre of V is simply a product of minimal ruled surfaces and for an arbitrary tuple $\underline{s}$ of positive rational numbers put $d_i = 4ds_i^2$, and let M_i be the bundle obtained by pull-back via the i^{th} projection of $H^{\frac{1}{2}} \otimes L$, where without loss of generality the latter may be supposed ample. Once more following Faltings we may observe that the difference of $\bigotimes_{i=1}^m M_i^{d_i}$ and $M(\varepsilon, \underline{s})^{\otimes d}$ is generated by sections with poles at the finite primes of order no worse than $\exp\left(0 \left(\sum_i d_i\right)\right)$, where the implied constant depends only on m and the choice of models, and so form a Koszul like complex,

$$0 \rightarrow M(\varepsilon, \underline{s})^{\otimes d} \rightarrow \bigoplus_{k=1}^N \bigotimes_{i=1}^m M_i^{d_i} \rightarrow \bigoplus_{k=1}^{N^3} \bigotimes_{i=1}^m M_i^{3d_i}$$

where N is the number of said sections, the poles being no worse than the aforesaid bound, and better still the same bound applies to the norms at the infinite places.

Next we consider the vanishing conditions with respect to a single section D_j , say. The divisors $D_{1j}, \dots, D_{mj}$ continue to form a regular sequence and the filtration F_j^k introduced in II.1 continues to be well defined. Moreover this filtration induces a filtration on $H^0(V, M(\varepsilon, \underline{s})^{\otimes d})$ whose successive quotients, considered as vector bundles over Y , have in the induced metric slopes bounded by $0 \left(\sum_i d_i\right)$, with the implied constant as before. Whence on taking global sections of the above Faltings' complex, and inductively filtering we obtain,

Lemma IV.1. *Let $\varepsilon > 0$ and $\delta < 1/2$ be given then for $m \gg 0$ there is a scheme V/Y whose generic fibre is a m -fold product of minimal ruled surfaces such that $M(\varepsilon, \underline{s})^{\otimes d} \otimes \bigcap_{j=1}^{\nu} \mathcal{I}_{D_j, m\delta}^{(d)}$ has a global section γ of logarithmic norm bounded by $0 \left(\sum_i d_i\right)$ where the implied constant depends only on m, ε, δ and the said model. The said norm being that of γ considered as a section of $M(\varepsilon, \underline{s})^{\otimes d}$.*

Unfortunately our work is not yet done since we require to replace the intersection of ideals by a product, and the said norm by something more delicate. To this end let $\pi : V \rightarrow B$ now denote the rather more complicated

model discussed previously, such that on the generic fibre the divisors D_{ij} , for fixed i , do not intersect. We may observe then that necessarily the section γ of IV.1 pulls back to a global section of $M(\varepsilon, \underline{s})^{\otimes d} \otimes \prod_{j=1}^{\nu} \mathcal{I}_{\tilde{D}_j, m\delta}^{(d)}$ with possibly finite poles, where $\mathcal{I}_{\tilde{D}_j, m\delta}^{(d)}$ is constructed via the same recipe as in II.2.1, but this time using the divisors $\tilde{D}_{ij}, \dots, \tilde{D}_{mj}$ which by construction are Cartier. Furthermore the poles which may appear are governed via induction on j , cf. [M5], by the global sections of,

$$\text{Tors}_1^{\mathcal{O}_V} \left(\mathcal{O}_V / \mathcal{I}_{\tilde{D}_{j+1}, m\delta}^{(d)}, \mathcal{O}_V / \prod_{k=1}^j \mathcal{I}_{\tilde{D}_k, m\delta}^{(d)} \right)$$

which in turn may be bounded in size by $\exp(0(\sum d_i))$ using the aforesaid filtration, to obtain a global section of the required sheaf with an appropriately bounded norm.

Finally we turn our attention to the correct norm for our problem, and so consider the sequence of monoidal transformations,

$$V = V_0 \leftarrow V_1 \leftarrow \dots \leftarrow V_\nu = W_{\underline{s}}^{(m)}$$

where $V_j \rightarrow V_{j-1}$ is the blow up in the pull-back of $\mathcal{I}_{\tilde{D}_j, m\delta}^{(d)}$, and we denote by E_j the corresponding exceptional divisor on $W_{\underline{s}}^{(m)}$. Naturally for E_j we consider the norm such that off the exception locus,

$$\|\mathbb{I}_{E_j}\|^2 = \sum_{t_1/s_1^2 + \dots + t_m/s_m^2 \geq d\delta m} \|\mathbb{I}_{D_{1j}}\|^{2t_1} \dots \|\mathbb{I}_{D_{mj}}\|^{2t_m}.$$

The sum being truncated for $t_i > m d \delta s_i^2$, and the implied norm on D_{ij} being induced from any a priori choice of norm on D_j . We thus choose a norm on $M(\varepsilon, \underline{s})^{\otimes d} \left(-m \delta \sum_j E_j \right)$ which is induced from this norm and the norm on $M(\varepsilon, \underline{s})^{\otimes d}$ apply the Schwartz lemma, or a suitable variant of it, cf. op.cit., to obtain,

Proposition IV.2. *Let $\varepsilon > 0$ and $\delta < 1/2$ be given then for $m \gg 0$ there is a global section over $W_{\underline{s}}^{(m)}$ of $M(-\varepsilon, \underline{s})^{\otimes d} \left(-m \delta \sum_j E_j \right)$ of norm at the infinite places bounded in logarithm by $0 \left(\sum_i d_i \right)$, where the implied constant depends only on ε , δ , m and V .*

IV.2. Vanishing of a section

Let us begin this section by supposing that the theorem as stated is false and let $\varepsilon > 0$ be given then as in I.1.1 we have a big line bundle $G_\varepsilon = \varepsilon A + H$, thus should the theorem be false with the exceptional set Z being the union of

the possibly bad divisor in I.1.1 and any fibres containing the centres implicit in I.2.4 then we may find an infinite sequence of rational points $\{f_i\} \subseteq S(Y)$ and positive rational numbers s_i such that,

(a) $s_i^2 G_{\varepsilon, f_i} Y$ converges to 1, and,

$$s_1^2 \gg s_2^2 \gg s_3^2 \gg \dots, \text{ etc.}$$

(b) Considered as elements of the Mordell-Weil group $X(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, the points $s_i^2 \pi(f_i)$ form a Cauchy sequence.

(c) There are constants $c_{v,j}$, $v \in \partial$, $1 \leq j \leq v$ determined by the sequence $\{f_i\}$ such that $\sum_{v,j} c_{v,j} > 2$ and for every v and j , $-s_i^2 \log \|\mathbb{I}_{D_j}(f_i)\|_v$ converges to $c_{v,j}$.

Consequently by taking δ sufficiently close to $1/2$, a correspondingly sufficiently large m to guarantee IV.1.2 and s_1^{-2} big enough to cancel any extraneous factors we obtain that the degree of $M(\varepsilon, \underline{s})^{\otimes d} \left(-m \delta \sum_j E_j^{(d)} \right)$ along the lifting to $W_{\underline{s}}^{(m)}$ of the arithmetic curve corresponding to $f_1 \times \dots \times f_m$ is at most $-m d \eta$ for some $\eta > 0$, and whence the section γ of IV.1.2 must vanish at the corresponding point. Better still following [F] once more we can construct outside of a closed product subvariety enjoying the bounds in IV.1.2 below differential operators $D_{\underline{t}}$ on V with poles and norms no worse than $\exp(0(\sum d_i))$ which on the generic fibre are of the form $\prod_{i=1}^m \frac{1}{t_i!} \partial_i^{t_i}$, ∂_i a derivation in a neighbourhood of f_i and $m d \sigma = \sum t_i s_i^{-2} \ll m d \eta$ which give sections $D_{\underline{t}} \gamma$ of $M(\varepsilon + (\text{const})\sigma, \underline{s})^{\otimes d} \left(-m(\delta - \sigma) \sum_j E_j^{(d)} \right)$ of similar norm which by an identical argument must themselves vanish at the point. Whence on combining this with (a) we may apply Faltings' product theorem to conclude,

Fact IV.1.2. There is a product subvariety $S_1 \times \dots \times S_m$ of S^m containing $f_1 \times \dots \times f_m$ defined over $k(Y)$ which satisfies,

(a) The degrees of the S_i are bounded by a constant depending only on m , ε , and δ .

(b) The heights h_i of the S_i in some suitable projective embedding satisfy,

$$\sum_{i=1}^m d_i h_i \leq 0 \left(\sum_{i=1}^m d_i \right)$$

where the dependence of the implied constant is as in (a).

IV.3. End of demonstration

Normally one hopes when presented with the information of IV.1.2 to proceed inductively however the fact that components D_j of our divisor may

restrict to the same divisor on some S_i requires us to exercise a little more care. Certainly if all the S_i are fibres of π none of which is equal to the bad fibres implicit in I.2.4 then such an induction is straightforward, and we arrive to conclude that the height of any f_i is arbitrarily smaller (i.e. according to how small we take s_1^2) than itself, which is clearly absurd. Otherwise some S_i is not a fibre and we appeal to the following uniformisation, cf. [M4], of the results of Faltings and Vojta across Hilbert schemes, viz,

Fact IV.3.1. Let T be a quasi-projective scheme of finite type parametrising pairs (X, D) where X is a curve and D a divisor whose degrees are appropriately bounded, and let $(\mathcal{X}, \mathcal{D}) \rightarrow T$ be the universal object. Suppose further that everything is defined over $\overline{\mathbb{Q}}$ and that $(\mathcal{X}, \mathcal{D}) \rightarrow T$ admits a map to the Hilbert scheme of divisors on some elliptic curve then for any sufficiently large number field k and any $\varepsilon > 0$ there is a constant $b(\varepsilon)$ such that if f is a k -rational point on X then,

$$m_{D, \partial}(f) \leq_{\text{exc.}} \varepsilon h(f) + b(\varepsilon) h(X)$$

where the heights are understood according to projective embeddings of the elliptic curve and T respectively.

Necessarily if S_i is not a fibre of π nor the possibly exceptional section of I.1.1 then a combination of IV.1.2 (b) and IV.3.1 actually yield a conclusion stronger than the theorem; from which we conclude.

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