Isotropization of two-component fluids

Martin Goliath
Department of Physics, Stockholm University, Box 6730, S–113 85 Stockholm, Sweden

Ulf S. Nilsson
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
and
Department of Physics, Stockholm University, Box 6730, S–113 85 Stockholm, Sweden

We consider the problem of late-time isotropization in spatially homogeneous but anisotropic cosmological models when the source of the gravitational field consists of two non-interacting perfect fluids – one tilted and one non-tilted. In particular, we study irrotational Bianchi type V models. By introducing appropriate dimensionless variables, a full global understanding of the state space of the gravitational field equations becomes possible. The issue of isotropization can then be addressed in a simple fashion. We also discuss implications for the cosmic “no-hair” theorem for Bianchi models when part of the source is a tilted fluid.

I. INTRODUCTION

Since the Bianchi models were introduced into cosmology in the 1960s, they have been the most studied generalizations of the spatially homogeneous and isotropic Friedmann-Lemaître (FL) models. The Bianchi models are spatially homogeneous but anisotropic, and thus well suited for studying the effects of anisotropic expansion on the evolution of the universe. The complexity of a Bianchi model is determined by its three-dimensional symmetry group $G_3$, in conjunction with the chosen matter model. Since the present universe seems to be very well described by a FL model, our main interest is models that can be “close to a FL model” at late times. In this paper we will restrict our consideration to models that may approach the open FL model at late times. Within the class of Bianchi models, only models of Bianchi type V and VII$_h$ can possibly have this behavior (see Collins & Hawking [1]). This follows since the open FL metric admits a $G_3$ of these particular Bianchi types. Here we will focus on the type V models. The notion of a model approaching a FL model at late times is often referred to as late-time isotropization.

As regards the matter description, most studies of Bianchi cosmologies use non-tilted barotropic perfect fluids with a linear equation of state. The book by Wainwright & Ellis [2] is basically devoted to such models. In the so-called non-tilted models, the fluid 4-velocity is orthogonal to the orbits of the isometry group $G_3$. Models in which the 4-velocity of the fluid is not orthogonal to the group orbits are referred to as tilted, and were introduced by King & Ellis [3]. These models have been studied by, for example, Collins & Ellis [3], Hewitt & Wainwright [4], and Harnett [5]. Since the non-tilted models are a subset of the tilted models, the latter can be viewed as simple generalizations of the former. Another generalization of non-tilted models is achieved by allowing the source of the gravitational field to be a combination of two non-interacting, non-tilted perfect fluids. Since these models can take into account both a radiation-dominated as well as a matter-dominated epoch of the universe, they may be considered as more physically relevant than single-fluid models. The qualitative behavior of two-fluid models were studied by Coley & Wainwright [6] who showed that the models are dominated at early times by the stiffer fluid, and at late times by the softer fluid. This is the expected behavior of a universe filled with radiation and dust.

The next step is to allow one of the fluids in a two-fluid model to be tilted, and subsequently, to let both fluids be tilted. Since the dynamics of models containing tilted fluids is more difficult to analyze than that of non-tilted models, we will consider the combination of one non-tilted and one tilted fluid. The non-tilted fluid is by definition irrotational, but tilted fluids can rotate. We chose, however, to consider the subset of irrotational tilted fluids$^1$. We are thus focusing on the subclass of irrotational Bianchi type V models. Both fluids are assumed to satisfy linear barotropic equations of state,

$^*$E-mail: goliath@physto.se

$^1$E-mail: ulfn@physto.se

$^1$For a discussion of general single-fluid Bianchi type V models, see Harnett [5].
\[ p_o = (\gamma_o - 1)\mu_o , \]  
\[ p_t = (\gamma_t - 1)\mu_t , \]  

where \( p_{o,t} \) are the pressures of the fluids, \( \mu_{o,t} \) the energy densities, and \( \gamma_{o,t} \) are constants that satisfy \( 0 \leq \gamma_{o,t} \leq 2 \) with \( \gamma_o \neq \gamma_t \). We exclude, however, the specific values 2/3 and 2 since these models behave qualitatively different. The indices \( o \) and \( t \) refer to the orthogonal and the tilted fluid respectively and will be used throughout the paper. We will also assume that the energy densities are non-negative, i.e.,

\[ \mu_o \geq 0 , \]  
\[ \mu_t \geq 0 . \]  

To discuss isotropization in detail, we need a well-defined notion of when a model is “close to a FL model”. For models with a single non-tilted fluid, a vanishing fluid shear defines the FL models (see, for example, section 2.4 in [2]). It is, however, not sufficient to demand that the fluid shear itself should approach zero at late times, since this will occur in any single-fluid Bianchi model, irrespectively of whether the model isotropizes or not. As realized already by Kristian & Sachs [7], the appropriate quantity to consider is the dimensionless ratio formed by normalizing the fluid shear with the fluid expansion, the so-called dimensionless shear. It measures the dynamical importance of the shear compared to the expansion of the fluid. Since there are now two fluids present, the notion of isotropization at late times needs to be generalized as follows. We say that a two-fluid model isotropizes at late times if the dimensionless shear is smaller than a critical value \( \eta_c \), which we take to be \( 1/2 \). The time variable \( t \) in (2.1) is chosen so that the normal to the symmetry surfaces is \( \frac{\partial}{\partial t} \). The other preferred congruence is that of the tilted fluid. The specific form of the line element, Eq. (2.1), guarantees that this fluid is irrotational. Consequently, the 4-velocity of this fluid, \( u^a_o \), can conveniently be parameterized in terms of the so-called tilt variable \( v \) according to

\[ u^a_o = \Gamma (1, v, 0, 0) , \quad \Gamma = (1 - v^2)^{-1/2} , \]  

where the value \( v = 0 \) corresponds to a non-tilted fluid. The energy-momentum tensor of each fluid is

\[ T^a_{\alpha} = \left[ \gamma_o u^a_o u^b_o + (\gamma_o - 1)\eta^{ab} \right] \mu_o , \]  
\[ T^a_{\beta} = \left[ \gamma_t u^a_t u^b_t + (\gamma_t - 1)\eta^{ab} \right] \mu_t . \]  

The plan of the paper is as follows. In Sec. II, the field equations are rewritten as a first-order system of autonomous ordinary differential equations. Reduced dimensionless variables are then introduced leading to a compact state space. In Sec. III we perform a local analysis of the system of equations. In Sec. IV, Bianchi type I and type V models with two orthogonal fluids are studied, while the locally rotationally symmetric type V models are considered in Sec. V. The question of isotropization is discussed in detail. We end with a discussion in Sec. VI. In App. A, the kinematical properties of both fluids are given. The relationship to the variables used in [6] is discussed in App. B. We use units such that \( c = 8\pi G = 1 \). Orthonormal frame indices are denoted by latin letters \( a, b, \ldots \).

## II. THE GRAVITATIONAL FIELD EQUATIONS

The line element for the irrotational Bianchi type V models can be written

\[ ds^2 = -dt^2 + D_1(t)^2 dx^2 + e^{-2x} \left[ D_2(t)^2 dy^2 + D_3(t)^2 dz^2 \right] , \]  

(2.1)

(see, for example, Ellis & MacCallum [10]). In the models we consider there are two preferred timelike congruences. First we have the congruence associated with the normal to the spatial symmetry surfaces, which is also, by definition, the congruence of the 4-velocity of the non-tilted fluid, \( u^a_o \). The time variable \( t \) in (2.1) is chosen so that the normal to the symmetry surfaces is \( \frac{\partial}{\partial t} \). The other preferred congruence is that of the tilted fluid. The specific form of the line element, Eq. (2.1), guarantees that this fluid is irrotational. Consequently, the 4-velocity of this fluid, \( u^a_t \), can conveniently be parameterized in terms of the so-called tilt variable \( v \) according to

\[ u^a_t = \Gamma (1, v, 0, 0) , \quad \Gamma = (1 - v^2)^{-1/2} , \]  

(2.2)

where the value \( v = 0 \) corresponds to a non-tilted fluid. The energy-momentum tensor of each fluid is

\[ T^a_{\alpha} = \left[ \gamma_o u^a_o u^b_o + (\gamma_o - 1)\eta^{ab} \right] \mu_o , \]  
\[ T^a_{\beta} = \left[ \gamma_t u^a_t u^b_t + (\gamma_t - 1)\eta^{ab} \right] \mu_t . \]  

(3.3a)

(3.3b)

The assumption that the two fluids are non-interacting leads to equations of motion of the form

\[ \nabla_a T^a_{\alpha} = 0 = \nabla_a T^a_{\beta} . \]  

(4.1)
We choose to parameterize the gravitational field using the expansion, $\theta$, and the non-vanishing components of the shear-tensor $\sigma_{\pm}$, of the normal congruence (and thus of the non-tilted fluid). They are given by

$$\theta = \frac{d}{dt} \ln (D_1 D_2 D_3) \quad , \quad \sigma_+ = -\frac{1}{2} \frac{d}{dt} \ln \left( \frac{D_1^2}{D_2 D_3} \right) \quad , \quad \sigma_- = \frac{\sqrt{3}}{2} \frac{d}{dt} \ln \left( \frac{D_2}{D_3} \right) .$$

The expansion and shear of the tilted fluid can be written as functions of these variables, see App. A. The gravitational field equations,

$$G_{ab} = T_{ab}^o + T_{ab}^t ,$$

and the equations of motion of the fluids, Eq. (2.4), become

**Evolution equations**

$$\dot{\theta} = -\frac{1}{3} \theta^2 - \frac{2}{3} (\sigma_+^2 + \sigma_-^2) - \frac{1}{2} (3 \gamma_0 - 2) \mu_0 - \frac{1}{2} \frac{(3 \gamma_t - 2) + (2 - \gamma_t)v^2}{1 + (\gamma_t - 1)v^2} \mu_n ,$$

$$\dot{\sigma}_+ = - (\theta - 2v B_1) \sigma_+ ,$$

$$\dot{\sigma}_- = - \theta \sigma_- ,$$

$$\dot{B}_1 = \frac{4}{3} (\theta - 2 \sigma_+) B_1 ,$$

$$\dot{\nu} = \frac{v(1 - v^2)}{3 [1 - (\gamma_t - 1)v^2]} [2 \sigma_+ + (3 \gamma_t - 4) \theta - 6(\gamma_t - 1)B_1 v] ,$$

$$\dot{\mu}_o = - \gamma_o \theta \mu_o .$$

**Constraint equation**

$$0 = \gamma_t v \mu_n + 2(1 + (\gamma_t - 1)v^2) B_1 \sigma_+ .$$

**Defining equation for $\mu_n$**

$$3 \mu_n = \theta^2 - \sigma_+^2 - \sigma_-^2 - 9 B_1^2 - 3 \mu_o ,$$

where we have introduced

$$B_1 = D_1^{-1} , \quad \mu_n = \frac{1 + (\gamma_t - 1)v^2}{1 - v^2} \mu_t ,$$

and a dot denotes differentiation with respect to $t$.

From the form of Eq. (2.7h), it is now clear why our choice of variables is a good one. The assumption in Eq. (1.2), in conjunction with Eq. (2.7h) and Eq. (2.8b), implies that $\theta$ cannot change sign. Therefore we can, without loss of generality, assume that $\theta \geq 0$, i.e., we restrict ourselves to expanding models. From Eq. (2.7h) it also follows that $\theta$ is a dominant quantity. We then introduce bounded dimensionless “$\theta$-normalized” variables for which the system of equations (2.7), is reduced as far as possible, and for which the state space is compact. These variables are defined by

$$\Sigma_\pm = \frac{\sigma_\pm}{\theta} , \quad A = \frac{3B_1}{\theta} , \quad \Omega_o = \frac{3 \mu_o}{\theta^2} , \quad \Omega_n = \frac{3 \mu_n}{\theta^2} .$$

The subsequent introduction of a dimensionless time variable $\tau$, which satisfies

$$\frac{d\tau}{dt} = \frac{\theta}{3} ,$$

leads to a decoupling of the $\theta$-equation, which can be written on the form

$$\theta' = -(1 + q)\theta , \quad q = 2 - 2A^2 - \frac{3(2 - \gamma_0)}{2} \Omega_o - \frac{3(2 - \gamma_t) + (5\gamma_t - 6)v^2}{2 [1 + (\gamma_t - 1)v^2]} \Omega_n ,$$

where the prime denotes differentiation with respect to $\tau$. The parameter $q$ is the deceleration parameter associated with the normal congruence and the non-tilted fluid. The remaining equations can now be written on dimensionless form as follows.
Evolution equations

\[ \Sigma'_+ = -(2 - q - 2Av)\Sigma_+ , \quad (2.12a) \]
\[ \Sigma'_- = -(2 - q)\Sigma_- , \quad (2.12b) \]
\[ A' = (q + 2\Sigma_+)A , \quad (2.12c) \]
\[ v' = \frac{v(1 - v^2)}{1 - (\gamma t - 1)v^2} [2\Sigma_+ + 3\gamma t - 4 - 2(\gamma t - 1)Av] , \quad (2.12d) \]
\[ \Omega_o' = [2q - (3\gamma_0 - 2)] \Omega_o . \quad (2.12e) \]

Constraint equation

\[ 0 = \gamma t v\Omega_n + 2[1 + (\gamma t - 1)v^2]A\Sigma_+ . \quad (2.12f) \]

Defining equation for \( \Omega_n \)

\[ \Omega_n = 1 - \Sigma_+^2 - \Sigma_-^2 - A^2 - \Omega_o . \quad (2.12g) \]

The set of equations (2.12) shows that the irrotational Bianchi type V models with one orthogonal and one tilted fluid is governed by a system of five autonomous ordinary differential equations subject to one constraint. The dimension of the state space is thus four. The set (2.12) is invariant under the discrete transformations

\[ (\Sigma_+, \Sigma_-, A, v, \Omega_o) \rightarrow (\Sigma_+, \Sigma_-, -A, -v, \Omega_o) , \quad (\Sigma_+, \Sigma_-, A, v, \Omega_o) \rightarrow (\Sigma_+, -\Sigma_-, A, v, \Omega_o) , \quad (2.13) \]

so we can, without loss of generality, restrict ourselves to the invariant set defined by \( A \geq 0 \) and \( \Sigma_- \geq 0 \). The variables \( \Sigma_+, \Sigma_-, A, v, \) and \( \Omega_o \) therefore satisfy \( 0 \leq \Sigma_+^2, \Sigma_-^2, A, v^2, \Omega_o \leq 1 \). There is a number of important subsets of Eqs. (2.12):

(i) The three-dimensional subset defined by \( v = 0 \), which describes a Bianchi type V universe with two non-tilted fluids. This case is thus included in the general study of Coley & Wainwright [6]. We will, however, consider these models in Sec. IV for the purpose of comparison with the models where one fluid is tilted. The subset \( v = 0 \) also contains Bianchi type I models and the open FL model as special cases.

(ii) The three-dimensional subset \( \Sigma_- = 0 \), which corresponds to locally rotationally symmetric (LRS) models (see, for example, \[11\]). This subset turns out to be very important for the evolution of the general irrotational type V models and will be considered in detail in Sec. V.

The next step in the analysis is to consider the equilibrium points of Eq. (2.12). This is done in the next section.

### III. QUALITATIVE ANALYSIS

In Table I, we present the equilibrium points of the system (2.12). The corresponding eigenvalues are given in Table II. Since the constraint, Eq. (2.12f), cannot be solved globally in an analytic way, it will be treated locally (see, for example, \[4\]).

Some of the equilibrium points correspond to exact solutions of the field equations. For example, the equilibrium points \( F_0^0 \) and \( F_0^0 \) correspond to flat FL models in which the tilted fluid and the non-tilted fluid is dominant, respectively\(^2\). The point \( M^0 \) is the Milne model, while the equilibrium set \( K^0 \) corresponds to Kasner-like models. The point \( M \) corresponds to flat space and coincides with \( M^0 \) for \( \gamma_t = 4/3 \). We also note the appearance of the equilibrium set \( F_o^\pm \) for the specific value \( \gamma_t = \frac{4}{3} \). It is associated with a line bifurcation that transfers stability between the equilibrium points \( F_o^\pm \) and \( F_o^0 \). The points \( F_o^\pm \) correspond to particular Kasner solutions. There is also a number of equilibrium points for which the tilt is extreme \((v^2 = 1)\). Whether these equilibrium points correspond to exact Bianchi solutions or not seems to be an open question (see comment on p. 4245 in \[4\]). We note that the constraint, Eq. (2.12f), is degenerate \((\nabla G = 0)\) at the point \( F_o^0 \), allowing all five eigenvector directions to be physical at this point.

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\(^2\)The “tilted” fluid is in fact non-tilted at \( F_o^0 \) since \( v = 0 \), but we refer to it as the tilted fluid for simplicity.
TABLE I. Equilibrium points for the irrotational Bianchi type V models with one orthogonal and one tilted perfect fluid. Unless indicated, all equilibrium points and sets are in the physical part of state space for $0 < \gamma_o < 2$, $\gamma_o \neq 2/3$.

<table>
<thead>
<tr>
<th></th>
<th>$\Sigma_+$</th>
<th>$\Sigma_-$</th>
<th>$A$</th>
<th>$v$</th>
<th>$\Omega_o$</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_o^o$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$F_o^{o}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$M^o$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$K^o$</td>
<td>$\Sigma_+^2 + \Sigma_-^2 = 1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$M$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$v = \frac{3\gamma_o-1}{2\gamma_o-4}$</td>
<td>0</td>
<td>$\frac{4}{3} &lt; \gamma_t &lt; 2$</td>
</tr>
<tr>
<td>$C^\pm$</td>
<td>$\Sigma_+ = -\frac{3}{2}(3\gamma_t - 4)$</td>
<td>$\Sigma_- = \pm \frac{3}{2}\sqrt{3(2 - \gamma_t)(3\gamma_t - 2)}$</td>
<td>0</td>
<td>$v$</td>
<td>0</td>
<td>$\gamma_t = \frac{4}{3}$</td>
</tr>
<tr>
<td>$F_o^o$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$v$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$F_o^{o}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\pm 1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$K^\pm$</td>
<td>$\Sigma_+^2 + \Sigma_-^2 = 1$</td>
<td>0</td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>$\Sigma_+ = -1 + A$</td>
<td>0</td>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

TABLE II. Eigenvalues for the equilibrium points of the irrotational Bianchi V models, with one orthogonal and one tilted perfect fluid.

<table>
<thead>
<tr>
<th></th>
<th>Eigenvalues</th>
<th>Elim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_o^o$</td>
<td>$-\frac{3}{2}(2 - \gamma_t)$</td>
<td>$-\frac{3}{2}(2 - \gamma_t)$</td>
</tr>
<tr>
<td>$F_o^{o}$</td>
<td>$-\frac{3}{2}(2 - \gamma_o)$</td>
<td>$-\frac{3}{2}(2 - \gamma_o)$</td>
</tr>
<tr>
<td>$M^o$</td>
<td>$-2$</td>
<td>$-3(\gamma_t - 2)$</td>
</tr>
<tr>
<td>$K^o$</td>
<td>0</td>
<td>$3(2 - \gamma_t)$</td>
</tr>
<tr>
<td>$M$</td>
<td>$-2$</td>
<td>$-\frac{2\gamma_t}{\gamma_t - 1}$</td>
</tr>
<tr>
<td>$C^\pm$</td>
<td>$3(2 - \gamma_t)$</td>
<td>0</td>
</tr>
<tr>
<td>$F_o^o$</td>
<td>$-\frac{3}{2}(2 - \gamma_t)$</td>
<td>$-\frac{3}{2}(2 - \gamma_t)$</td>
</tr>
<tr>
<td>$F_o^{o}$</td>
<td>$-\frac{3}{2}(2 - \gamma_o)$</td>
<td>$-\frac{3}{2}(2 - \gamma_o)$</td>
</tr>
<tr>
<td>$M^o$</td>
<td>$-2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$K^\pm$</td>
<td>$2(1 + \Sigma_+)$</td>
<td>0</td>
</tr>
<tr>
<td>$H$</td>
<td>$-2(1 + \Sigma_+)$</td>
<td>$-\frac{2(2\gamma_t - 1)}{2\gamma_t - 4}$</td>
</tr>
</tbody>
</table>
IV. TWO ORTHOGONAL FLUIDS

Setting \( v = 0 \) in the constraint Eq. (2.12f), implies either \( A = 0, \Sigma_+ \neq 0 \) (Bianchi type I models) or \( \Sigma_+ = 0, A \neq 0 \) (Bianchi type V models). Without loss of generality we can assume that \( \gamma_t > \gamma_o \). The boundary subsets of the state space for these two classes of models are given by the two invariant submanifolds \( \Omega_o = 0 \) and \( \Omega_n = 0 \), which describe the corresponding one-fluid models. The dynamics of the Bianchi I state space is shown in Fig. 1, while the dynamics of the type V models is shown in Fig. 2.

For type I models there are two sources. The equilibrium point \( F^0_o \) gives rise to a single orbit ending at \( F^0_o \). It corresponds to a flat Friedmann model with two orthogonal fluids. The other source is the equilibrium set \( K^0_o \), of which each point is associated with a one-parameter set of orbits. Therefore this equilibrium set describes the generic behavior at early times. The future attractor of these orbits is the point \( F^0_o \). Thus, from Eqs. (A3) and (A9), all orthogonal two-fluid Bianchi type I models isotropize.

For type V models there are three sources. The equilibrium point \( F^0_o \) is associated with a one-parameter set of orbits (characterized by \( - = 0 \)), which corresponds to open Friedmann models with two orthogonal fluids. They all end at the equilibrium point \( M^0 \). The other two sources are the two equilibrium points belonging to the set \( K^0 \) with \( \Sigma_- = \pm 1 \). Both of these are associated with two-parameter sets of orbits, and thus describe the generic early-time behavior. They are future attracted to \( M^0 \). Consequently, from Eqs. (A3) and (A9), all orthogonal two-fluid Bianchi type V models isotropize.

From the analysis of the orthogonal two-fluid models, it is clear that the generic behavior is described by equilibrium points associated with two-parameter sets. Thus, in what follows, we will focus on such equilibrium points. Note that the variable \( \chi \) of Coley & Wainwright [6] (see App. B) is a monotone function when both the fluids are orthogonal. This is no longer the case when one of the fluids is tilted.

V. BIANCHI TYPE V LRS MODELS WITH ONE TILTED AND ONE ORTHOGONAL FLUID

As Bianchi type I models do not allow the combination of one non-tilted and one tilted fluid as a source due to the constraint Eq. (2.12f), we focus on the Bianchi type V models. Since \( \Omega_o, \Omega_n > 0 \) imply \( q < 2 \) by Eq. (2.11), the evolution equation for \( \Sigma_- \) implies that \( \Sigma_- \) is a monotone decreasing function along all orbits with \( \Sigma_- \geq 0 \) and \( \Omega_o, \Omega_n > 0 \). This fact significantly restricts the evolution at late times. It implies that \( \lim_{\tau \to \infty} \Sigma_- = 0 \), for all orbits with \( \Omega_o, \Omega_n > 0 \). This can be proven along the lines used to prove the similar statement for tilted single-fluid models, see [4]. The asymptotic behavior as \( \tau \to \infty \) is thus contained in the three-dimensional invariant set \( \Sigma_- = 0 \) corresponding to the LRS models.

The three Kasner circles \( K^0 \) and \( K^\pm \) each reduce to two equilibrium points in the LRS submanifold, namely points for which \( \Sigma_+ = \pm 1 \). We denote these equilibrium points \( K^0_\pm, K^+_\pm \), and \( K^-_\pm \), where the subscript distinguishes between the two signs of \( \Sigma_+ \). The equilibrium points for which \( v \) and \( \Sigma_+ \) have the same sign (collectively denoted \( K^+_\pm \)) are not located on the boundary of the interior of the LRS submanifold. Consequently, we do not need to consider them when studying the dynamics.

The state space for the LRS submanifold with \( 2/3 < \gamma_o < \gamma_t < 2 \) is presented in Figs. 3–6. The effect of changing the equation-of-state parameters so that \( \gamma_o > \gamma_t \) is that the flow along the orbit between \( F^0_o \) and \( F^0_t \) is reversed.
Thus, the analysis of isotropization for the LRS submanifold holds for the general class of models as well.

2 if this is indeed the case for the equilibrium points in question, as can be seen from Eq. (A8), noting that behavior for models with $\gamma_0 < 2/3$, all models isotropize. For $\gamma_0 > 2/3$, $6/5 < \gamma_1 < 4/3$ there is a class of solutions of non-zero measure that does not isotropize, namely the class of solutions associated with $M^-$. For $\gamma_0 > 2/3$, $\gamma_1 = 4/3$, all models isotropize. For $\gamma_1 > 4/3$ no models isotropize. Note that the physically interesting combination of one dust fluid and one radiation fluid always isotropizes, regardless of which fluid is tilted.

When $\gamma_0 < 2/3$, $\gamma_1 > 4/3$, the future attractors are $F^\pm$. Solutions corresponding to orbits approaching these equilibrium points do not isotropize, see Table III, even though the final states are inflationary in the sense that the deceleration parameter associated with the normal congruence $q = \frac{1}{2}(3\gamma_0 - 2)$ is negative. Thus, in some sense it seems to be misleading to refer to solutions corresponding to orbits approaching these equilibrium points as “asymptotically Friedmann models”, although the solutions corresponding to the points themselves are Friedmann models. In particular, for the case of a cosmological constant $\gamma_0 = 0$, the solutions corresponding to $F^\pm_0$ are de Sitter models [8]. This is consistent with the cosmic “no-hair” theorem for Bianchi models [12]. However, the theorem does not guarantee that the tilt tends to zero, as pointed out by Raychaudhuri & Modak [13]. From our analysis, it is clear that this cautionary note is crucial for these models. As the tilted fluid does not become orthogonal at late times, the expansion-normalized shear of the tilted fluid does not vanish at late times. This stresses the point made in [8] that one should be cautious about the isotropization of tilted models. As seen in Table III, this is in fact the generic behavior for models with $0 \leq \gamma_0 < 2/3$ and $\gamma_1 > 4/3$.

For general irrotational Bianchi type V models, there is no a priori reason that $\Sigma_{i-} \to 0$ when $\Sigma_{i+} \to 0$. However, this is indeed the case for the equilibrium points in question, as can be seen from Eq. (A8), noting that $\Sigma_{-} \to 0$. Thus, the analysis of isotropization for the LRS submanifold holds for the general class of models as well.

VI. DISCUSSION

In this paper we have continued the study of irrotational Bianchi type V cosmologies, using the dynamical systems approach initiated by Hewitt & Wainwright [4] and Coley & Wainwright [6]. The source of the gravitational field has been taken to be two non-interacting fluids, one orthogonal and one tilted. Such models can describe a universe
where one of the fluids models the contribution of radiation to the energy density of the universe, and the other the matter content.

We have found that, although the orthogonal two-fluid models isotropize, this is not necessarily the case when one of the fluids is tilted. Thus, depending on the equation-of-state parameters $\gamma_o$ and $\gamma_t$, it is possible to find cases for which all or a subset of the solutions are anisotropic to the future. In particular, there are models which are inflationary, but do not isotropize. However, we emphasize that the cases of dust plus radiation always isotropize.

It should be stressed that the relevant quantities when determining whether the shear dies away are shear quantities normalized with respect to the expansion associated with each fluid.

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**APPENDIX A: FLUID PROPERTIES**

Here we present the kinematical properties of the two fluids in terms of the variables used to parameterize the gravitational field. For the non-tilted fluid we have

**Fluid expansion**

$$\theta_o = \theta$$ \hspace{1cm} (A1)

**Fluid shear**

$$\sigma_o^2 = \sigma_+^2 + \sigma_-^2$$ \hspace{1cm} (A2)

The dimensionless shear $\Sigma_o$ of the non-tilted fluid is thus

$$\Sigma_o^2 = \frac{\sigma_o^2}{\theta_o^2} = \Sigma_+^2 + \Sigma_-^2.$$ \hspace{1cm} (A3)

The fluid properties of the tilted fluid are as follows.

**Fluid expansion**

$$\theta_t = \frac{1}{\sqrt{1-v^2}} \left[ \frac{3\theta - 6\nu B_1 + v^2(2\sigma_+ - \theta)}{3[1 - (\gamma_t - 1)v^2]} \right] = \frac{1}{\sqrt{1-v^2}} \left[ \frac{3 - 2\nu A + v^2(2\Sigma_+ - 1)}{3[1 - (\gamma_t - 1)v^2]} \right] \theta,$$ \hspace{1cm} (A4)

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$^3$Except for $v = 0$. 

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TABLE III. Sources and sinks for the Bianchi type V LRS models with two fluids.
Fluid shear
\[ \sigma_t^2 = \frac{1}{1-v^2} \left[ \sigma_+^2 + \sigma_-^2 - B_1 v(2\sigma_+ - B_1 v) - \frac{2v\dot{v}}{1-v^2}(\sigma_+ - B_1 v) + \frac{v^2\dot{v}^2}{(1-v^2)^2} \right] . \] (A5)

The dimensionless shear \( \Sigma_t \) of the tilted fluid can be written
\[ \Sigma_t^2 = \frac{\sigma_t^2}{\theta_t^2} = \Sigma_t^2 + \Sigma_t^2 , \] (A6)

where
\[ \Sigma_t^+ = -1 + \frac{3(1 + \Sigma_+ - vA) \left[ 1 - (\gamma - 1)v^2 \right]}{3 - 2vA + (2\Sigma_+ - 1)v^2} , \] (A7)
\[ \Sigma_t^- = \frac{3 \left[ 1 - (\gamma - 1)v^2 \right] \Sigma_-}{3 - 2vA + (2\Sigma_+ - 1)v^2} , \] (A8)

(see Eqs. (A3), (A4) and (A6) in [4]). Note that when both fluids are orthogonal \( (v = 0) \), Eq. (A6) reduces to
\[ \Sigma_t^2 = \Sigma_t^2 + \Sigma_t^2 . \] (A9)

**APPENDIX B: THE VARIABLE \( \chi \)**

In their study of two orthogonal fluids, Coley & Wainwright [6] introduced the following variable instead of \( \Omega_0 \):
\[ \chi := \frac{\mu_o - \mu_t}{\mu_o + \mu_t} = \frac{\Omega_o - \Omega_t}{\Omega_o + \Omega_t} . \] (B1)

The evolution equation for \( \chi \) becomes
\[ \chi' = -\frac{(1 - \chi^2)}{1 + (\gamma - 1)v^2} \left\{ 3\gamma_o \left[ 1 + (\gamma - 1)v^2 \right] - \gamma_t \left[ 3 + v^2 - 2v(A + v\Sigma) \right] \right\} . \] (B2)

For the submanifold \( v = 0 \) corresponding to two orthogonal fluids, the above equation simplifies to
\[ \chi' = -\frac{1}{2}(1 - \chi^2)(\gamma_o - \gamma_t) . \] (B3)