Nonperturbative Gauge Fixing and Perturbation Theory

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Abstract

We compare the gauge-fixing approach proposed by Jona-Lasinio and Parrinello, and by Zwanziger (JPLZ) with the standard Fadeev-Popov procedure, and demonstrate a form of perturbative equivalence. We also show how a set of local, renormalizable Feynman rules can be constructed for the JPLZ procedure.
1. Gauge fixing in Yang–Mills theories is well understood in perturbation theory. But outside of perturbation theory the situation is more complicated, because of the existence of Gribov copies [1]. These complications arise as soon as the theory is nonlinear, which is the case for nonabelian gauge theories, but also, for instance, with a nonlinear gauge-fixing condition in an abelian theory.

This does not mean that Yang–Mills theories do not exist outside perturbation theory. It is well known that they can be defined on the lattice, using the compactness of the group to dispense with gauge fixing altogether. However, nonperturbative gauge fixing is interesting in a variety of contexts, ranging from the need to make contact between lattice and continuum calculations (see for example Ref. [2]) to the definition of chiral gauge theories on the lattice (see for example Ref. [3]).

One can try to extend the standard BRST construction to the lattice. When one does this, one finds, quite generally and rigorously, that the BRST gauge-fixed partition function vanishes [4]. This result can be heuristically explained as the result of pairwise cancellations of lattice Gribov copies, which occur with opposite signs of the Faddeev–Popov determinant. (For some recent work on overcoming this problem, see Refs. [6, 7].)

A different nonperturbative method for gauge fixing has been proposed some time ago by Jona-Lasinio and Parrinello and by Zwanziger (JLPZ) [8, 9] (see also Ref. [10]). Starting from the euclidean ungauged partition function

\[ Z = \int D_H A \exp \left( -S_{\text{inv}}(A) \right), \]  

one inserts one in the form

\[ 1 = \frac{\int D_H h \exp \left( -S_{\text{ni}}(A^h) \right)}{\int D_H h \exp \left( -S_{\text{ni}}(A^h) \right)} \]  

into the functional integral. Here \( D_H A \) and \( D_H h \) denote the invariant measures over the gauge field \( A \) and the group-valued scalar field \( h \), respectively; \( A^h \) is the gauge transform of \( A \) under a gauge transformation \( h \). \( S_{\text{ni}}(A) \) is any gauge noninvariant local functional of \( A \). Because of the invariance of \( D_H A \) and \( S_{\text{inv}}(A) \), we can perform a gauge transformation in the numerator of Eq. (1) with Eq. (2) inserted, and, dropping a trivial factor \( \int D_H h \), we obtain

\[ Z = \int D_H A \frac{\exp \left( -S_{\text{inv}}(A) - S_{\text{ni}}(A) \right)}{\int D_H h \exp \left( -S_{\text{ni}}(A^h) \right)}. \]  

This procedure is completely rigorous on the lattice, and sources coupled to gauge-invariant operators can be added without changing the argument. We also note that the Boltzmann weights for both integrations contained in Eq. (3) are positive.

While this method is conceptually very simple, much less is known about how this nonperturbative method works in perturbation theory, as compared with the standard gauge fixing based on BRST invariance. Note, however, that there are some similarities between the two procedures. A noninvariant term \( S_{\text{ni}}(A) \) is added to the action, and (for a suitable choice) this will make the quadratic part of the
action invertible. The contributions of different orbits are weighted properly because of the group integral in the denominator of Eq. (3), which plays a role similar to that of the Faddeev–Popov (FP) determinant in the usual case. In the FP case, however, the determinant can be expressed as an integral over ghosts, and the gauge-fixed action including the ghost terms is local, whereas here this is not the case. This is an important difference, because locality is a key ingredient in power-counting arguments, and thus at the heart of the usual perturbative analysis of renormalization.

It is therefore of interest to find out whether perturbation theory can be systematically developed for a JLPZ gauge-fixed Yang–Mills theory, and its relation to the usual FP results. This question has been addressed previously by Fachin [11], who analyzed the vacuum polarization for the choice $S_{ni}(A) = \text{tr}(M^2 A_{\mu}^2)$ at one loop. He concluded that for $M \to \infty$ the transverse part of the vacuum polarization agrees with that obtained using the FP method; the longitudinal part vanishes for $M \to \infty$, as in Landau gauge. This equivalence for $M \to \infty$ at fixed cutoff was already derived formally in Ref. [8], and another formal discussion appeared recently in Ref. [12]. Here, we are interested in considering the situation where $M$ is chosen to be of the same order of magnitude as the cutoff. The transverse part of the one-loop vacuum polarization with JLPZ gauge fixing, for example, was found to contain terms proportional to $p^2 \delta_{\mu\nu} - p_\mu p_\nu$ times $p^2/M^2$ or $p^4/M^4$ [11], so, if we choose $M \sim \Lambda$ (with $\Lambda$ the cutoff), such terms are of order $1/\Lambda^2$, and vanish when we take $\Lambda \to \infty$. We present below a general argument that, if we choose $M$ to be fixed in units of the cutoff, perturbation theory for a JLPZ gauge-fixed theory is equivalent to that of the same theory gauge-fixed using the standard FP method. We discuss in some detail what “equivalent” means in this context. We also show that our arguments for the equivalence of JLPZ and FP gauge fixing can be used to construct a set of local Feynman rules for the JLPZ gauge-fixed gauge theory.

2. We begin with the JLPZ gauge-fixed partition function in the presence of a source for the gauge field $A_\mu$ (working in euclidean space-time):

$$Z(J) = \int \mathcal{D}_H A \frac{\exp(-S_{\text{inv}}(A) - S_{ni}(A) + J_\mu A_\mu)}{\int \mathcal{D}_H h \exp(-S_{ni}(A^h))},$$

(4)

where $S_{\text{inv}}(A)$ is the gauge-invariant classical action, and $S_{ni}(A)$ is not invariant. As noted before, $A^h$ is the gauge transform of $A$ under a finite gauge transformation,

$$A^h_\mu = h(A_\mu + \frac{i}{g} \partial_\mu) h^\dagger.$$

(5)

We will take

$$S_{\text{inv}}(A) = \frac{1}{2} \text{tr}(F_{\mu\nu}^2),$$

$$S_{ni}(A) = \text{tr}(M^2 A_{\mu}^2),$$

(6)

where we do not indicate the integration $\int d^4x$ explicitly, and $M$ a parameter with the dimension of a mass. We will assume that $M$ is proportional to the cutoff for the case of regulators such as the lattice, in which case $M$ is proportional to the inverse
lattice spacing. However, it is sufficient for our arguments that \( M \) is chosen large
compared to all physical scales. This assumption applies in the case of dimensional
regularization. Also, \( A_\mu = A_\mu^a T^a \), with \( T^a \) the generators of the gauge group, with

\[
\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab},
\]
\[
[T^a, T^b] = i f^{abc} T^c.
\]

If we use a regulator without power-like divergences, such as dimensional regulariza-
tion, we may extend the range of integration of the variables \( A_\mu^a \) from \(-\infty\) to \( \infty \), and
the invariant measure is just the flat measure

\[
\mathcal{D}_H A = \mathcal{D} A = \prod_{x, \mu, a} dA_\mu^a (x),
\]

and similarly for \( \mathcal{D}_H h \) (after parametrizing \( h \) as in Eq. (11) below).

We wish to show that this gauge-fixed partition function is equivalent (in a way to
be discussed in more detail in the next section), under certain assumptions which are
valid in perturbation theory, to the standard FP gauge-fixed form for the covariant
gauge \( \partial_\mu A_\mu = 0 \). The derivation is based on two ingredients. The first ingredient is
that we insert a constant into the partition function, written in the form

\[
\text{constant} = \det(\Box) \int \mathcal{D} \eta \, \delta(\partial_\mu A_\mu - \frac{1}{M} \Box \eta),
\]

where \( \mathcal{D} \eta \) denotes the (flat) measure for \( \eta \), a new field which takes values in the Lie
algebra of the gauge group. We may choose boundary conditions such as to avoid
the trivial zero mode of \( \Box \). The second ingredient is to make use of the fact that
the physics at scales below \( M \) is not altered by adding or changing terms which are
irrelevant in the sense of Wilson’s renormalization group.

First, let us expand \( S_{ni}(A^h) \) in \( g/M \), writing

\[
h = \exp(i \frac{g}{M} \theta).
\]

We find

\[
S_{ni}(A^h) = \frac{1}{2} M^2 (A_\mu^a)^2 - M \theta^a \partial_\mu A_\mu^a + \frac{1}{2} \partial_\mu \theta^a (D_\mu (A) \theta)^a + O \left( \frac{g}{M} \right),
\]

where

\[
(D_\mu (A) \theta)^a = \partial_\mu \theta^a + g f^{abc} A_\mu^b \theta^c.
\]

Note that \(-\partial_\mu D_\mu (A)\) is just the FP operator for the covariant gauge.

We then shift the vector field

\[
A_\mu \to A_\mu' = A_\mu - \frac{1}{M} \partial_\mu \eta.
\]

This gives

\[
S_{ni}(A^h) = \frac{1}{2} M^2 (A_\mu'^a)^2 + M A_\mu'^a \partial_\mu \eta^a + \frac{1}{2} (\partial_\mu \eta^a)^2
\]
\[
- M \theta^a \partial_\mu A_\mu'^a - \theta^a \Box \eta^a + \frac{1}{2} \partial_\mu \theta^a (D_\mu (A') \theta)^a + O \left( \frac{g}{M} \right),
\]
\[ S_{\text{inv}}(A) + S_{\text{int}}(A) = S_{\text{inv}}(A') + \frac{1}{2} M^2 (A'^a_\mu)^2 + MA'^a_\mu \partial_\mu \eta^a + \frac{1}{2} (\partial_\mu \eta^a)^2 + O \left( \frac{g}{M} \right). \]  

The terms \( \frac{1}{2} M^2 (A'^a_\mu)^2 + MA'^a_\mu \partial_\mu \eta^a + \frac{1}{2} (\partial_\mu \eta^a)^2 \) cancel between the numerator and denominator of the integrand in Eq. (4). The terms \( M \theta^a \partial_\mu A'^a_\mu \) and \( MA'^a_\mu \partial_\mu \eta^a \) (which is equal to \(-M \eta^a \partial_\mu A'^a_\mu \) by partial integration) may be dropped because of the \( \delta \)-function, \( \delta(\partial_\mu A'_\mu) \), in Eq. (10).

All the \( O(g/M) \) terms are irrelevant, and may therefore be omitted without changing the (renormalized) theory. Doing this, we can perform the \( h \)-integral in the denominator of the integrand in Eq. (4), obtaining

\[ Z(J) = \text{det} (\Box) \int \mathcal{D} \eta \mathcal{D} A' \delta(\partial_\mu A'_\mu) \text{det}^{1/2} (-\partial_\mu D_\mu (A')) \times \exp \left( -S_{\text{inv}}(A') + J_\mu A_\mu - \frac{1}{2} \eta \Box (-\partial_\mu D_\mu (A'))^{-1} \Box \eta \right). \]  

Using the fact that physical quantities do not change when we replace the source term \( J_\mu A_\mu \) by a new source term \( J'_\mu A'_\mu \) coupling to \( A'_\mu \) instead of to \( A_\mu \) (\( A_\mu \) and \( A'_\mu \) are equivalent interpolating fields), we can now also perform the \( \eta \) integral. Dropping the primes, we obtain

\[ Z(J) = \int \mathcal{D} A \delta(\partial_\mu A_\mu) \text{det} (-\partial_\mu D_\mu (A)) \exp (-S_{\text{inv}}(A) + J_\mu A_\mu). \]  

Representing the \( \delta \)-function as

\[ \delta(\partial_\mu A_\mu) \propto \lim_{\xi \to 0} \exp \left( -\frac{\xi}{2} (\partial_\mu A_\mu)^2 \right), \]  

and introducing an algebraic field \( B \) and ghost fields \( c \) and \( \overline{c} \), this can be recast as

\[ Z(J) = \lim_{\xi \to 0} \int \mathcal{D} A \mathcal{D} B \mathcal{D} c \mathcal{D} \overline{c} \times \exp \left( -S_{\text{inv}}(A) - \frac{1}{2} \xi B^2 + i B \partial_\mu A_\mu - \overline{c} (-\partial_\mu D_\mu (A)) c + J_\mu A_\mu \right). \]  

This is the standard BRST-invariant form of the FP gauge-fixed partition function in Landau gauge. Of course, correlation functions of gauge-invariant operators are independent of \( \xi \). We observe that, if we would take the limit \( M \to \infty \) before removing the cutoff, our argument constitutes an alternative derivation of the equivalence of Eq. (4) to Landau gauge given in Ref. [8]. Note that the standard way of gauge fixing employed in lattice QCD computations is formally equivalent to this limit. What is new here, is that we do not take the limit \( M \to \infty \) first, but keep it at the order of the cutoff. Nevertheless, the parameter \( M \) has disappeared from Eq. (20).

The derivation given above is valid only in perturbation theory. We assumed that the \( \theta \) - and \( \eta \)-integrals converge, i.e. that the FP operator \(-\partial_\mu D_\mu (A)\) has only positive eigenvalues. This is not in general the case, but it is true in perturbation theory.
If we use a regulator with a hard cutoff, such as the lattice, additional subtractions will be needed in order to remove power-like divergences, which may appear as a consequence of dropping irrelevant terms. Also note that, even though Eq. (14) is linear, in general the invariant measure written in terms of the Lie-algebra valued fields is nonlinear for such a regulator, and it would therefore change under this transformation. However, this nonlinearity is proportional to the coupling constant $g$, and therefore the effects of this shift are of order $g/M$.

3. In this section, we will discuss in more detail what we mean by “equivalent.” It is clear that, in general, correlation functions of the form $\langle A_\mu^a(x)A_\nu^b(y)\ldots \rangle$ are not the same in the JLPZ and FP versions. A trivial example is the $O(g^0)$ two-point function $\langle A_\mu(p)A_\nu(q)\rangle = \delta(p + q)G_{\mu\nu}(p)$, with $G_{\mu\nu}(p)$ equal to

$$
G_{\mu\nu}^{\text{JLPZ}}(p) = \frac{1}{p^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \frac{1}{M^2} \frac{p_\mu p_\nu}{p^2}, \\
G_{\mu\nu}^{\text{FP}}(p) = \frac{1}{p^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \frac{\xi}{p^2} \frac{p_\mu p_\nu}{p^2}, \ \xi \to 0,
$$

in the theories defined by Eqs. (4) and (20), respectively. This is not in contradiction with the general argument presented above, because of the change of interpolating field, $A_\mu \rightarrow A'_\mu$.

This example also suggests that quantities which are independent of $\xi$ or $M$, such as the transverse part of the two two-point functions, are the same in both theories. This is not entirely correct, because of the fact that we dropped irrelevant terms in going from the JLPZ to the FP version. What is correct is that physical quantities (which are necessarily extracted from correlation functions of gauge-invariant operators) in the JLPZ version can be mapped into those of the FP version by a finite renormalization of the bare coupling constant $g$. In the FP version of the theory the fact that only a coupling constant renormalization is needed follows from BRST invariance. In the JLPZ version, the same follows from the observation that JLPZ gauge fixing can be “undone” by multiplying Eq. (4) (for $J_\mu = 0$) by a constant in the form $\int \mathcal{D}Hg$, and transforming $A_\mu \rightarrow A'_\mu$, removing $S_m$ from the partition function. Wave-function renormalizations are not necessary for physical quantities.

In other words, renormalized perturbation theory for physical quantities is the same in both versions for a suitable definition of the renormalized coupling constant, but the relation between the renormalized and bare coupling constants is, in general, different in the FP and JLPZ versions of the theory.

4. In the case of noninvariant correlation functions, it is well known that, in the FP version of the theory, only a universal coupling-constant renormalization and multiplicative wave-function renormalizations are necessary as a consequence of BRST invariance. The situation is less clear in the JLPZ version of the theory. Some progress can be made however, by observing that a set of local and renormalizable Feynman rules can be constructed for the JLPZ partition function, Eq. (4). Start with diagonalizing the quadratic form in $\theta$ and $\eta$ in Eq. (15) by a shift $\theta = \theta' - \eta$. 

5
After this shift, Eq. (15) can be written as the sum of two parts,

\[ S_{ni}(A^b) = S_{ni}^q(A', \eta) + S_{ni}^\theta(A', \eta, \theta') , \]  

with \( S_{ni}^\theta(A', \eta, \theta' = 0) = 0 \). To order \( g/M \) we obtain

\[ \begin{align*}
S_{ni}^q(A', \eta) &= \frac{1}{2} M^2 (A'_\mu)^2 + \frac{1}{2} g f^{abc} \partial_\mu \eta^a A^b \eta^c + O \left( \frac{g}{M} \right) , \\
S_{ni}^\theta(A', \eta, \theta') &= \frac{1}{2} \partial_\mu \theta^a (D_\mu (A') \theta')^a - g f^{abc} \partial_\mu \theta^a A^b \eta^c + O \left( \frac{g}{M} \right) .
\end{align*} \]

upon using \( \partial_\mu A'_\mu = 0 \). Next, we define the following action (dropping the primes on \( A_\mu \) and \( \theta \)):

\[ S_{pert}^{JLPZ} = S_{inv}(A + \partial \eta/M) + S_{ni}(A + \partial \eta/M) - S_{ni}^q(A, \eta) + S_{ni}^\theta(A, \eta, \theta) \quad (24) \]

\[ \begin{align*}
&= \frac{1}{4} (F^a_{\mu\nu})^2 + \frac{1}{2} (\partial_\mu \eta^a)^2 - \frac{1}{2} g f^{abc} \partial_\mu \eta^a A^b \eta^c \\
&\quad + \frac{1}{2} \partial_\mu \theta^a (D_\mu (A) \theta)^a - g f^{abc} \partial_\mu \theta^a A^b \eta^c + O \left( \frac{g}{M} \right) .
\end{align*} \]

Here “\( O(g/M) \)” indicates all higher order terms in the expansion in \( g \), including those coming from the shift Eq. (14), and they should be kept (up to the order in \( g \) of interest). This action will give rise to a correct set of Feynman rules if we add the rule that a factor \(-1\) be applied for each connected \( \theta \)-subdiagram without external \( \theta \) lines. This is reminiscent of the minus sign for ghost loops in the FP case: the integral over \( \theta \) in the denominator of Eq. (4) gives rise to an effective action \( S_{eff}(A, \eta) \),

\[ \exp(S_{eff}(A, \eta)) = \int \mathcal{D} \theta \exp \left( -S_{ni}^\theta(A, \eta, \theta) \right) , \]

the vertices of which correspond precisely to these connected \( \theta \)-subdiagrams. The additional minus sign corresponds to the fact that this integral appears in the denominator of the integrand in Eq. (4). If we work with a regulator in which it is important to keep the nonlinear terms in the measure, this can be taken into account by treating these nonlinear terms as part of the action.

Finally, there is still the \( \delta \)-function of Eq. (10), through which the field \( \eta \) was introduced. After the shift Eq. (14), this is just \( \delta(\partial_\mu A_\mu) \), which again can be represented as in Eq. (19). This leads to a set of Feynman rules for a local theory, Eq. (24), which is renormalizable by power counting for any value of \( \xi \), with the original JLPZ version (Eq. (4)) corresponding to the limit \( \xi \to 0 \) (at fixed cutoff).

We conclude this section by noting that other field redefinitions of the field \( \theta \) can be used, for instance

\[ \exp \left( i \frac{g}{M} \theta \right) = \exp \left( i \frac{g}{M} \theta' \right) \exp \left( -i \frac{g}{M} \eta \right) , \quad (26) \]

or

\[ \exp \left( i \frac{g}{M} \theta \right) = \exp \left( i \frac{g}{M} \theta' \right) \left( 1 - i \frac{g}{M} \eta \right) . \quad (27) \]

These examples differ only by terms of order \( g/M \) from the one employed above, and thus lead to a different specific form of the \( O(g/M) \) terms in Eq. (24).
5. We argued that the gauge-fixing procedure proposed in Refs. [8, 9], with the choice of Eq. (6) for $S_{\text{nl}}(A)$, is perturbatively equivalent to the standard gauge-fixing procedure with Fadeev–Popov ghosts. In our derivation, we chose $M$ to be of the order of the cutoff, thus extending earlier arguments in which the (formal) limit $M \to \infty$ was considered.

“Equivalent” here means that perturbatively calculated relations between physical quantities will be the same in both versions of the theory. Since, in addition, the JLPZ method leads to a weighting of the integration over orbits which takes Gribov copies correctly into account, this method may be the preferred one for nonperturbative calculations in gauge-fixed Yang–Mills theories.

We also derived local Feynman rules for the JLPZ version of the theory, Eq. (4), from which it can be seen that the theory is renormalizable by power counting. The choice of $M$ at the order of the cutoff is a key ingredient here. By construction, correlation functions of gauge-invariant operators are the same when calculated perturbatively from either Eq. (4) or Eq. (24), after the field redefinition of Eq. (14) is taken into account. In order to renormalize correlation functions of gauge noninvariant operators, counterterms may have to be added to Eq. (24), in addition to those needed to renormalize gauge-invariant quantities. The locality guarantees that all counterterms necessary for renormalization are local, and the renormalizability guarantees that only a finite number, all with mass dimension less than or equal to four, will be needed. It is also clear that this can be done in such a way that the invariance of gauge-invariant correlation functions is maintained, because of the gauge invariance of the original formulation of Eq. (4). Hence, we believe that no problems will be encountered in carrying out this program order by order in perturbation theory, for the theory of Eq. (24). But it is not clear how this would then “translate back” to a JLPZ-like formulation as in Eq. (4). What is lacking is a tool similar to BRST symmetry in the FP version of the theory, which could be used to further control the form of the counterterms. It would be interesting and useful if such a mechanism could be found.

We end with a comment on our use of the specific form of the action $S_{\text{nl}}(A)$ in Eq. (6). This choice is the “most” (and only) relevant local operator in the sense of the renormalization group. Our analysis does not work when $S_{\text{nl}}(A)$ is chosen to be a marginal operator. We expect that they will work if we would add a marginal operator to $S_{\text{nl}}(A)$ of Eq. (6). For instance, a term of the form $c \Sigma_{\mu} \text{tr}(A_{\mu}^4)$, with $c$ a constant of order $g^2$ (which is natural on the lattice), can be removed by a field redefinition of the form $A_\mu \to A_\mu + (c/2)A_\mu^3/M^2$. Since the nonlinear term of this field redefinition is of order $1/M^2$, this will just remove the term $\Sigma_{\mu} \text{tr}(A_{\mu}^4)$, without introducing any other marginal terms.

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