Scale Invariance in the Causal Approach to Renormalization Theory

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Abstract
The dilation invariance is studied in the framework of Epstein-Glaser approach to renormalization theory. The analogue of the Callan-Symanzik equations are found and they are applied to the scalar field theory and to Yang-Mills models.

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1 Introduction

The causal approach to renormalization theory of by Epstein and Glaser [23], [24] leads to important simplification of the renormalization theory as well as of the computational aspects. This approach works for quantum electrodynamics [41], Yang-Mills theories [11] [12] [14] [15] [1] [2], [9], [10], [32]-[35], [36] [37] [22], gravitation [25], [26], [44], etc.

In this paper we investigate the role of dilation invariance in the causal approach. In the next Section we define the dilation invariance operator for various free fields. Next, we remind the basic facts about renormalization theory. We will emphasize the original Epstein-Glaser approach where one considers a set of (linearly independent) interaction Lagrangian and attaches to each of this Lagrangian a (space-time dependent) coupling constant. Then we are able to prove the basic theorem concerning the arbitrariness of the chronological products for the same set of interaction Lagrangian. This problem was already addressed in [39], but we argue that the natural framework is the multi-valued coupling constant approach of [23].

In Section 4 we obtain consequences about the scale behaviour of the chronological products. As it is well known, these properties are valid only asymptotically, for large momenta. The pioneering works on this subject are [5], [6] and [7]. A mathematical refined analysis was developed in [42] and [43], the main mathematical tool being the so-called quantum action principle [38] (for a review see [40]).

Finally we apply these considerations for Yang-Mills models and obtain a restrictions on the possible form of the anomalies, namely the degree of such a anomalous expression must be 5.
2 Dilation Invariance in Quantum Field Theory

It is well known that the Fock space of the real scalar field of mass $m$ can be defined as:

$$\mathcal{F}_m \equiv \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}_m$$  \hspace{1cm} (2.0.1)

where $\mathcal{F}^{(n)}_m$ is the set of Borel function $\Phi^{(n)}(\mathbb{X}^+)^\otimes n \to \mathbb{C}$ which are square integrable with respect to the Lorentz invariant measure: $\alpha^+_m(p) \equiv \frac{d\alpha^+_m(p)}{\sqrt{p^2+m^2}}$ and completely symmetric in the all variables (see [45] for notations). Then we have:

**Proposition 2.1** Let us define for any $\lambda \in \mathbb{R}^*$ the operators $\mathcal{U}_\lambda : \mathcal{F}^{(n)}_m \rightarrow \mathcal{F}^{(n)}_{\lambda^{-1}m}$ as follows:

$$\left( \mathcal{U}_\lambda \Phi^{(n)} \right)(p_1, \ldots, p_n) = \prod_{i=1}^{n} \lambda^n \Phi^{(n)}(\lambda p_1, \ldots, \lambda p_n).$$  \hspace{1cm} (2.0.2)

Then:

(i) The operators $\mathcal{U}_\lambda$ are unitary;

(ii) The following relations are verified for all $\lambda, \lambda' \in \mathbb{R}^*$:

$$\mathcal{U}_\lambda \mathcal{U}_{\lambda'} = \mathcal{U}_{\lambda \lambda'};$$  \hspace{1cm} (2.0.3)

(iii) If $\mathcal{U}^{[\lambda]}_{a;L}$ is the representation of the Poincaré group in the Fock space $\mathcal{F}^{(n)}_m$, then:

$$\mathcal{U}^{[\lambda]}_{a;L} \mathcal{U}_\lambda = \mathcal{U}_\lambda \mathcal{U}^{[\lambda^{-1}]}_{a;L}$$  \hspace{1cm} (2.0.4)

for all translations $a$ and all Lorentz transformations $L$.

**Proof:** The proof of the first assertion is based on the Lorentz invariance of the measure $d\alpha^+_m(p)$. The next assertions follow from elementary computations. ■

If we use the definition of the annihilation operators

$$(a(q; m) \Phi^{(n)}(p_1, \ldots, p_n) = \sqrt{n+1} \Phi^{(n+1)}(q, p_1, \ldots, p_n)$$  \hspace{1cm} (2.0.5)

then we immediately get the identity:

$$\mathcal{U}_\lambda \ a(q; m) \ \mathcal{U}^{-1}_\lambda = \lambda \ a(\lambda^{-1} q; \lambda^{-1} m).$$  \hspace{1cm} (2.0.6)

By hermitian conjugation we get a similar identity for the creation operators $a^*(q)$.

The expression of the real scalar field of mass $m$ is:

$$\phi(x; m) \equiv \frac{1}{(2\pi)^{3/2}} \int d\alpha^+_m(p) \left[ e^{-i x \cdot p} a(p; m) + e^{i x \cdot p} a^*(p; m) \right]$$  \hspace{1cm} (2.0.7)

so we get from (2.0.6) the following relation:

$$\mathcal{U}_\lambda \ \phi(x; m) \ \mathcal{U}^{-1}_\lambda = \lambda \phi(\lambda x; \lambda^{-1} m).$$  \hspace{1cm} (2.0.8)
Remark 2.2 There is an alternative point of view. One can define the operators $\mathcal{U}_\lambda : \mathcal{F}_m^{(n)} \rightarrow \mathcal{F}_m^{(n)}$ according to

$$\left(\mathcal{U}_\lambda \Phi^{(n)}\right)(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \prod_{i=1}^{n} r_\lambda(p) \Phi^{(n)}(\lambda \mathbf{p}_1, \ldots, \lambda \mathbf{p}_n) \quad (2.0.9)$$

where

$$r_\lambda(p) \equiv \lambda^{3/2} \sqrt{\frac{\omega_m(p)}{\omega_m(\lambda p)}}. \quad (2.0.10)$$

Because we have the cocycle identity

$$r_\lambda(p) r_\lambda'(\lambda p) = r_{\lambda\lambda'}(p) \quad (2.0.11)$$

the map $\lambda \rightarrow \mathcal{U}_\lambda$ defined above is a representation of the multiplicative group $\mathbb{R}^*$ (the dilation) group in the Fock space of the scalar field. Moreover, the relations (2.0.4), (2.0.6) and (2.0.8) are valid only up terms of order $O(m)$ because we have $r_\lambda(p) = \lambda + O(m)$. So, we see that some information is lost in this approach.

It is easy to prove that relations of the same type as (2.0.8) are valid for other types of fields, namely fields of integer spin. This includes the electromagnetic potentials the Yang-Mills fields, the gravitational field and also the ghosts fields used in the process of quantization. For a Dirac field an important difference appears. Instead of (2.0.7) we have:

$$\psi(x; M) \equiv \frac{1}{(2\pi)^{3/2}} \int \mathcal{M}_M^+(p) \left[ e^{-ipx} \sum_{i=1}^{2} u_i(p; M)b_i(p; M) + e^{ipx} \sum_{i=1}^{2} v_i(p; M)b_i^*(p; M) \right] \quad (2.0.12)$$

(see [41]) where $b_i^\#(p; M)$ are the creation (annihilation) operators; the expressions $u_i(p; M)$ and $v_i(p; M)$ are solutions of the free Dirac equation of positive (negative) values. To have Poincaré covariance of the field operator $\psi$ one has to normalize in such a way these spinors such that we have:

$$u_i(\lambda p; \lambda M) = \lambda^{1/2} u_i(p; M), \quad v_i(\lambda p; \lambda M) = \lambda^{1/2} v_i(p; M). \quad (2.0.13)$$

So we get instead of (2.0.8):

$$\mathcal{U}_\lambda \psi(x; M) \mathcal{U}_\lambda^{-1} = \lambda^{3/2} \psi(\lambda x; \lambda^{-1} M). \quad (2.0.14)$$

We can obviously prove that the relations (2.0.4) are valid in the most general case, with fields of various spins.

Let us note that if we apply to the relations (2.0.8) or (2.0.14) a derivation operator $\frac{\partial}{\partial x_i}$ we obtain a supplementary factor $\lambda$ in the right hand side.

Finally, if $W(x; \mathbf{m})$ is a Wick monomial in free fields of various masses $\mathbf{m} = (m_1, \ldots, M_1, \ldots)$ we obtain a generalization of the relations (2.0.8) and (2.0.14), namely:

$$\mathcal{U}_\lambda W(x; \mathbf{m}) \mathcal{U}_\lambda^{-1} = \lambda^{\omega(W)} W(\lambda x; \lambda^{-1} \mathbf{m}) \quad (2.0.15)$$

where the number $\omega(W)$ is called the canonical dimension of the monomial $W$ and is computed according to the well known rule: one attributes to every integer (resp. half-integer) spin field the canonical dimension 1 (resp. 3/2) and to every derivative the canonical dimension 1. Then one postulates that the canonical dimension is an additive function.

One can extend these considerations to Wick monomials in many variables $W(x_1, \ldots, x_n)$. If the interaction Lagrangian of a model verifies a relation of the type (2.0.15) we say that the model is dilation (or scale)-covariant. It also well known that the canonical dimension of fields is an important property in renormalization theory.
3 Renormalization Theory

3.1 Bogoliubov Axioms

We outline here the axioms of a multi-Lagrangian perturbation theory. Following Bogoliubov and Shirkov ideas, in [23] one constructs the $S$-matrix as a formal series of operator valued distributions:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n T_{j_1,\ldots,j_n}(x_1,\ldots,x_n)g_{j_1}(x_1)\cdots g_{j_n}(x_n),$$

(3.1.1)

where $g = (g_j(x))_{j=1,\ldots,p}$ is a multi-valued tempered test function in the Minkowski space $\mathbb{R}^4$ that switches the interaction and $T_{j_1,\ldots,j_n}(x_1,\ldots,x_n)$ are operator-valued distributions acting in the Fock space of some collection of free fields. These operator-valued distributions are called chronological products and verify some properties called in the following Bogoliubov axioms. It is necessary to note that there is a canonical projection $pr$ associating to the point $x_i$ the index $j_i$. One starts from a set of interaction Lagrangians $T_j(x), \ j = 1,\ldots,P$ and tries to construct the whole series $T_{j_1,\ldots,j_n}, \ n \geq 2$.

The interaction Lagrangians must satisfy some requirements such like Poincare invariance, hermiticity and causality. The natural candidates fulfilling these demands is a linearly independent set of Wick polynomials operating in the Fock space (describing a system of weakly interacting particles).

The recursive process of constructing the chronological produces fixes the chronological products almost uniquely. We will study this arbitrariness in detail later.

The physical $S$-matrix is obtained from $S(g)$ taking the adiabatic limit which is, loosely speaking the limit $g_j(x) \to 1, \ \forall j = 1,\ldots,P$.

We give here the set of axioms imposed on the chronological products $T_{j_1,\ldots,j_n}$ following the notations of [23].

- First, it is clear that, without loosing generality, we can consider them completely symmetrical in all variables in the sense:

$$T_{j_{\sigma(1)},\ldots,j_{\sigma(p)}}(x_{\sigma(1)},\cdots,x_{\sigma(p)}) = T_p(x_1,\cdots,x_p), \ \forall \sigma \in \mathcal{P}_p.$$  

(3.1.2)

- Next, we must have Poincaré invariance. Because we will also consider Dirac fields, we suppose that we have an unitary representation $(a,A) \mapsto U_{a,A}$ of the group $inSL(2,\mathbb{C})$ (the universal covering group of the proper orthochronous Poincaré group $\mathcal{P}_+^1$) such that:

$$U_{a,A}T_{j_1,\ldots,j_n}(x_1,\cdots,x_p)U_{a,A}^{-1} = T_{j_1,\ldots,j_n}(\delta(A) \cdot x_1 + a,\cdots,\delta(A) \cdot x_p + a), \ \forall A \in SL(2,\mathbb{C}), \forall a \in \mathbb{R}^4$$

(3.1.3)

where $SL(2,\mathbb{C}) \ni A \mapsto \delta(A) \in \mathcal{P}_+^1$ is the covering map. In particular, translation invariance is essential for implementing Epstein-Glaser scheme of renormalization.

Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available.

- The central axiom seems to be the requirement of causality which can be written compactly as follows. Let us firstly introduce some standard notations. Denote by $V^+=\{x \in \mathbb{R}^4 | \ x^2 > 0, \ x_0 > 0\}$ and $V^- = \{x \in \mathbb{R}^4 | \ x^2 > 0, \ x_0 < 0\}$ the upper (lower) lightcones and by $\overline{V^\pm}$ their closures. If $X \equiv \{x_1,\cdots,x_m\} \in \mathbb{R}^{4m}$ and $Y \equiv \{y_1,\cdots,y_n\} \in \mathbb{R}^{4n}$ are such that
\( x_i - y_j \not\in V \), \( \forall i = 1, \ldots, m, \ j = 1, \ldots, n \) we use the notation \( X \geq Y \). If \( x_i - y_j \not\in V \cup V^c \), \( \forall i = 1, \ldots, m, \ j = 1, \ldots, n \) we use the notations: \( X \sim Y \). We use the compact notation \( T_J(X) \equiv T_{j_1, \ldots, j_n}(x_1, \cdots, x_n) \) with the convention

\[
T_b(\emptyset) = 1
\]  

(3.1.4)

and by \( XY \) we mean the juxtaposition of the elements of \( X \) and \( Y \). Then the causality axiom writes as follows:

\[
T_{J_1J_2}(X_1X_2) = T_{J_1}(X_1)T_{J_2}(X_2), \ \forall X_1 \geq X_2;
\]  

(3.1.5)

here \( J_i \) are the indices corresponding to the the coordinates \( X_i \) i.e \( J_i = pr(X_i), \ i = 1, 2 \). From (3.1.5) one can derive easily:

\[
[T_{J_1}(X_1), T_{J_2}(X_2)] = 0, \ \text{if} \ X_1 \sim X_2.
\]  

(3.1.6)

- The unitarity of the \( S \)-matrix can be expressed if one introduces, the formal series:

\[
\tilde{S}(g) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \tilde{T}_{j_1, \ldots, j_n}(x_1, \cdots, x_n)g_{j_1}(x_1) \cdots g_{j_n}(x_n),
\]  

(3.1.7)

where, by definition:

\[
(-1)^{|X|} \tilde{T}_J(X) \equiv \sum_{r=1}^{|X|} (-1)^r \sum_{X_1, \ldots, X_r \in \text{Part}(X)} T_{J_1}(X_1) \cdots T_{J_r}(X_r);
\]  

(3.1.8)

here \( X_1, \cdots, X_r \) is a partition of \( X \), \( |X| \) is the cardinal of the set \( X \) and the sum runs over all partitions. In the lowest orders we have:

\[
\tilde{T}_j(x) = T_j(x)
\]  

(3.1.9)

and

\[
\tilde{T}_{j_1j_2}(x_1, x_2) = -T_{j_1j_2}(x_1, x_2) + T_{j_1}(x_1)T_{j_2}(x_2) + T_{j_2}(x_2)T_{j_1}(x_1).
\]  

(3.1.10)

One calls the operator-valued distributions \( \tilde{T}_{j_1, \ldots, j_n}(x_1, \ldots, x_n) \) anti-chronological products. The series (3.1.7) is the inverse of the series (3.1.1) i.e. we have:

\[
\tilde{S}(g) = S(g)^{-1}
\]  

(3.1.11)

in the sense of formal series. Then the unitarity axiom is then:

\[
\tilde{T}_J(X) = T_J(X)^\dagger, \ \forall X.
\]  

(3.1.12)

One can show that the following relations are identically verified:

\[
\sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_1|} T_{J_1}(X_1)\tilde{T}_{J_2}(X_2) = \sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_1|} \tilde{T}_{J_1}(X_1)T_{J_2}(X_2) = 0.
\]  

(3.1.13)

Also one has, similarly to (3.1.5):

\[
\tilde{T}_{J_1J_2}(X_1X_2) = \tilde{T}_{J_2}(X_2)\tilde{T}_{J_1}(X_1), \ \forall X_1 \geq X_2.
\]  

(3.1.14)
A renormalization theory is the possibility to construct such a $S$-matrix starting from the first order terms: $T_j(x), \ j = 1, \ldots, P$ which are linearly independent Wick polynomials called interaction Lagrangians which should verify the following axioms:

$$U_{a,A}T_j(x)U_{a,A}^{-1} = T_j(\delta(A) \cdot x + a), \ \forall A \in SL(2, \mathbb{C}), \ \forall j = 1, \ldots, P$$ (3.1.15)

$$[T_j(x), T_k(y)] = 0, \ \forall x, y \in \mathbb{R}^4 \ \text{s.t.} \ x \sim y, \ \forall j, k = 1, \ldots, P$$ (3.1.16)

and

$$T_j(x) = T_j(x), \ \forall j = 1, \ldots, P.$$ (3.1.17)

Usually, these requirements are supplemented by covariance with respect to some discrete symmetries (like spatial and temporal involutions, or PCT), charge conjugations or global invariance with respect to some Lie group of symmetry. Some other restrictions follow from the requirement of the existence of the adiabatic limit, at least in the weak sense.

The case of a single Lagrangian perturbation theory corresponds to $P = 1$. In this case the expression $T_1(x)$ is the interaction Lagrangian and the chronological products are $T(X) \equiv T_{1\ldots1}(X)$. 
3.2 Epstein-Glaser Induction

We summarize the steps of the inductive construction of Epstein and Glaser [23], [31]. Let the interaction Lagrangians $T_j(x), j = 1, \ldots, P$ be some linearly independent Wick polynomials acting in a certain Fock space with $\omega_j, j = 1, \ldots, P$ the corresponding canonical dimensions. We denote by $\omega$ the supremum of all these canonical dimensions. The causality property (3.1.16) is fulfilled, but we must make sure that we also have (3.1.15) and (3.1.17).

We suppose that we have constructed the chronological products $T_{j_1, \ldots, j_p}(x_1, \ldots, x_p)$ for all $p = 1, \ldots, n - 1$ having the following properties: (3.1.2), (3.1.5) and (3.1.12) for $p \leq n - 1$ and (3.1.5) and (3.1.6) for $|X_1| + |X_2| \leq n - 1$. We want to construct the distribution-valued operators $T_j(X), |X| = n$ such that the properties above go from 1 to $n$. We will use the notation

$$\omega_j \equiv \sum_{j \in J} \omega_j \quad (3.2.1)$$

and we call it the canonical dimension of $T_j(X)$. Here are the steps of the construction.

1. One constructs from $T_j(X), |X| \leq n - 1$ the expressions $\hat{T}_j(X), |X| \leq n - 1$ according to (3.1.8) and proves the properties (3.1.14) for $|X_1| + |X_2| \leq n - 1$.

2. Next, we define the expressions:

$$A_{j_1, \ldots, j_n}^\prime(x_1, \ldots, x_{n-1}; x_n) \equiv \sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_2|} T_{j_1}(X_1) \hat{T}_{j_2}(X_2), \quad (3.2.2)$$

$$R_{j_1, \ldots, j_n}^\prime(x_1, \ldots, x_{n-1}; x_n) \equiv \sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_2|} \hat{T}_{j_1}(X_1) T_{j_2}(X_2) \quad (3.2.3)$$

where the sum goes over the partitions of $X = \{x_1, \ldots, x_n\}$ such that $X_2 \neq \emptyset, x_n \in X_1$.

Next, we construct the expression

$$D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) \equiv A_{j_1, \ldots, j_n}^\prime(x_1, \ldots, x_{n-1}; x_n) - R_{j_1, \ldots, j_n}^\prime(x_1, \ldots, x_{n-1}; x_n). \quad (3.2.4)$$

and prove that it has causal support i.e. $\text{supp}(D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n)) \subseteq \Gamma^+(x_n) \cup \Gamma^-(x_n)$

where we use standard notations:

$$\Gamma^\pm(x_n) \equiv \{(x_1, \ldots, x_n) \in (\mathbb{R}^4)^n | x_i - x_n \in V^\pm, \forall i = 1, \ldots, n - 1\} \quad (3.2.5)$$

3. The distribution $D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n)$ can be written as a sum

$$D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) = \sum_i d_i(x_1, \ldots, x_{n-1}; x_n) W_i(x_1, \ldots, x_n) \quad (3.2.6)$$

where $W_i(x_1, \ldots, x_n)$ are linearly independent Wick monomials and $d_i(x_1, \ldots, x_{n-1}; x_n)$ are numerical distributions with causal support i.e $\text{supp}(d_i(x_1, \ldots, x_{n-1}; x_n)) \subseteq \Gamma^+(x_n) \cup \Gamma^-(x_n)$. The distributions $d_i(x_1, \ldots, x_{n-1}; x_n)$ defined above are $SL(2, \mathbb{C})$-covariant and their degree of singularity is restricted by

$$\omega(d_i) + \omega(W_i) \leq \omega_j - 4(n - 1). \quad (3.2.7)$$
4. There exists a causal splitting

\[ d = a - r, \quad \text{supp}(a) \subset \Gamma^+(x_n), \quad \text{supp}(r) \subset \Gamma^-(x_n) \]  \hfill (3.2.8)

which is also \( SL(2, \mathbb{C}) \)-covariant and such that the order of the singularity is preserved. So, there exists a \( SL(2, \mathbb{C}) \)-covariant causal splitting:

\[ D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) = A_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) - R_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) \]  \hfill (3.2.9)

with \( \text{supp}(A_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n)) \subset \Gamma^+(x_n) \) and \( \text{supp}(R_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n)) \subset \Gamma^-(x_n) \).

The expressions \( A_n \) and \( R_n \) are the advanced (resp. retarded) products.

5. We have the relation

\[ D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n)^\dagger = (-1)^{n-1} D_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n). \]  \hfill (3.2.10)

The causal splitting obtained above can be chosen such that

\[ A_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n)^\dagger = (-1)^{n-1} A_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n). \]  \hfill (3.2.11)

6. Let us define

\[
T_{j_1, \ldots, j_n}(x_1, \ldots, x_n) \equiv A_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) - A_{j_1, \ldots, j_n}'(x_1, \ldots, x_{n-1}; x_n)
\]

\[
\equiv R_{j_1, \ldots, j_n}(x_1, \ldots, x_{n-1}; x_n) - R_{j_1, \ldots, j_n}'(x_1, \ldots, x_{n-1}; x_n).
\]  \hfill (3.2.12)

Then these expressions satisfy the \( SL(2, \mathbb{C}) \)-covariance, causality and unitarity conditions (3.1.3) (3.1.5) (3.1.6) and (3.1.12) for \( p = n \). If we substitute

\[
T_{j_1, \ldots, j_n}(x_1, \ldots, x_n) \rightarrow \frac{1}{n!} \sum_\pi T_{j_{\pi(1)}, \ldots, j_{\pi(n)}}(x_{\pi(1)}, \ldots, x_{\pi(n)})
\]  \hfill (3.2.13)

where the sum runs over all permutations of the numbers \( \{1, \ldots, n\} \) then we also have the symmetry axiom (3.1.2). The generic expression of the chronological product is similar to (3.2.6)

\[
T_{j_1, \ldots, j_n}(x_1, \ldots, x_n) = \sum_i t_i(x_1, \ldots, x_{n-1}; x_n) W_i(x_1, \ldots, x_{n-1}; x_n)
\]  \hfill (3.2.14)

with the same limitation (3.2.7) on the numerical distributions:

\[
\omega(t_i) + \omega(W_i) \leq \omega_f - 4(n - 1), \quad \forall i.
\]  \hfill (3.2.15)
3.3 The Arbitrariness of the Chronological Products

This problem was addressed in [39], as we have already mention in the Introduction. We prefer to give an independent formulation based on the multi-Lagrangian Epstein-Glaser scheme presented above. We consider two solutions of the Bogoliubov axioms with the same "initial conditions" $T_j$, $j = 1, \ldots, P$ chosen as a basis in the space of Wick polynomials of degree $\leq \omega_0$. Usually one takes $\omega_0$ to be the dimension of the Minkowski space. We introduce the following notation: if $X = \{x_1, \ldots, x_n\}$ then

$$\delta(X) \equiv \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n). \quad (3.3.1)$$

We note the identity:

$$\sum_{x_i \in X} \frac{\partial}{\partial x_i^p} \delta(X) = 0. \quad (3.3.2)$$

Now, we have the following result:

**Theorem 3.1** Let $T_j(X)$ and $\tilde{T}_j(X)$ be two solutions of the Bogoliubov axioms such that $T_j(x) = \tilde{T}_j(x)$, $\forall j = 1, \ldots, P$ and both verify the restriction (3.2.15). Then we have the relation:

$$\tilde{T}_j(X) = T_j(X) + \sum_{r=1}^{[X]-1} \frac{1}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} P_{J_1:k_1}(X_1) \cdots P_{J_r:k_r}(X_r) T_{k_1,\ldots,k_r}(x_{i_1}, \ldots, x_{i_r}), \quad \forall |X| \geq 2 \quad (3.3.3)$$

where summation over the indices $k_1, \ldots, k_r = 1, \ldots, P$ is understood, $P_{J,k}(X)$ are distributions of the form

$$P_{J,k}(X) = p_{J,k}(\partial)\delta(X) \quad (3.3.4)$$

with $p_{J,k}(\partial)$ a Lorentz covariant polynomial with constant coefficients in the partial derivatives restricted by:

$$\deg(p_{J,k}) + \omega_k \leq \omega_J - 4(n-1) \quad (3.3.5)$$

and $x_{i_p} \in X_p, \forall p = 1, \ldots, r$. In the preceding equation, the convention

$$P_{J,k}(X) \equiv 0, \quad |X| = 1 \quad (3.3.6)$$

is understood.

**Proof:** We use complete induction. For $n = 2$ one obtains a possible expression $T_{j_1,j_2}(x_1, x_2)$ by causally splitting the distribution $D_{j_1,j_2}(x_1, x_2) = [T_{j_1}(x_1), T_{j_2}(x_2)]$. According to a general result in distribution splitting theory, two such splitting differ by a distribution with support in the set $\{x_1 = x_2\}$ of the type $\sum|p_{j_1,j_2,k}(\partial)\delta(x_1 - x_2)|T_k(x_2)$; the limitation $\deg(p_{j_1,j_2,k}) + \omega_k \leq \omega_{j_1} + \omega_{j_2} - 4$ follows from the restrictions (3.2.15). The Lorentz covariance follows if we make the distribution splitting in a covariant way, which is known to be possible.

We suppose that we have the expressions $P_{J,k}(X), \quad |X| \leq n - 1$ such that the formula from the statement is valid for $|X| \leq n - 1$; we prove the formula for $|X| = n$. Let us consider in this case the expression

$$\Delta_J(X) \equiv \tilde{T}_J(X) - T_J(X) - \sum_{r=2}^{[X]-1} \frac{1}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} P_{J_1:k_1}(X_1) \cdots P_{J_r:k_r}(X_r) T_{k_1,\ldots,k_r}(x_{i_1}, \ldots, x_{i_r}) \quad (3.3.7)$$
and show that it has the support in the set $x_1 = x_2 = \cdots = x_n$. For this, let us suppose that the point $(x_1, \ldots, x_n)$ is outside this set. Then one can find a Cauchy surface separating this set in two non-void subsets $Y$ and $Z$ such that $[Y] \geq [Z]$. Because of the symmetry axiom (3.1.2) we can suppose, without loosing generality, that $Y = \{x_1, \ldots, x_i\}$ and $Z = \{x_{i+1}, \ldots, x_n\}$. In that case, let us notice that in the sum appearing in the preceding formula we can have non-zero contributions only from those partitions $X_1, \ldots, X_r$ such that for every $p = 1, \ldots, r$ we have either $X_p \subset Y$ or $X_p \subset Z$. This means that for such a choice of $(x_1, \ldots, x_n)$ we have:

$$\Delta_f(X) \equiv \tilde{T}_f(X) - T_f(X) - \sum_{2 \leq s+t \leq n-1} \sum_{Y_1, \ldots, Y_s \in \text{Part}(Y)} \sum_{Z_1, \ldots, Z_t \in \text{Part}(Z)} \frac{1}{s!} \frac{1}{t!} P_{j_1:k_1}(Y_1) \cdots P_{j_s:k_s}(Y_s) P_{j_{s+1}:k_{s+1}}(Z_1) \cdots P_{j_{s+t}:k_{s+t}}(Z_t) T_{k_1,\ldots,k_{s+t}}(x_{i_1}, \ldots, x_{i_{s+t}})$$

(3.3.8)

with $x_{ip} \in Y_p$, $\forall p = 1, \ldots, s$ and $x_{is+p} \in Z_p$, $\forall p = 1, \ldots, t$. Now, we can use in the right hand side the causality property (3.1.5) for the chronological products $T_f(X)$ and $\tilde{T}_f(X)$. We have easily get $\Delta_f(X) = 0$. The support property of the distribution $\Delta_f(X)$ is proved. Using Wick theorem and well known facts about the structure of numerical distribution with support included in the set $x_1 = x_2 = \cdots = x_n$ we get the formula (3.3.3) for $|X| = n$. The Lorentz covariance follows like in the case $n = 2$. This finished the induction. ■

It is clear now why do we need the multi-Lagrangian generalisation of Epstein-Glaser formalism. Even if we work in a theory with a single Lagrangian, the best we can do is to choose it among those sets of chronological products $T_f$ such that for every $p = 1, \ldots, r$ we have either $X_p \subset Y$ or $X_p \subset Z$. This means that for such a choice of $(x_1, \ldots, x_n)$ we have:

$$\tilde{T}(X) = T(X) + \sum_{r=1}^{|X|-1} \frac{1}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} P_{k_1}(X_1) \cdots P_{k_r}(X_r) T_{k_1,\ldots,k_r}(x_{i_1}, \ldots, x_{i_r}), \quad \forall |X| \geq 2$$

(3.3.9)

where we have denoted $P_k(X) \equiv P_{\{1,\ldots,1\},k}(X)$ with $|X|$ entries of the figure 1. So, in the difference between two solutions of the problem will certainly appear other chronological products that $T(X)$. 


4 Dilation Covariance of the Chronological Products

We will use the result from the preceding Subsection to study the generic behaviour of the chronological products with respect to the dilation invariance operators which was defined in Section 2. We will emphasize the mass dependence of the chronological products in an obvious way: \( T_j(X; m) \); first we have

**Proposition 4.1** We suppose that the framework from the preceding Section is valid. Then the following relations are valid for all \(|X| \geq 2\):

\[
\mathcal{U}_\lambda T_j(X; m) \mathcal{U}_\lambda^{-1} = \lambda^{\omega_j} T_j(\lambda X; \lambda^{-1})
\]

\[
+ \sum_{r=1}^{[X]-1} \frac{1}{r!} \sum_{x_1, \ldots, x_r \in \text{Part}(X)} P_{j_1; k_1; \lambda}(\lambda x_1) \cdots P_{j_r; k_r; \lambda}(\lambda x_r) T_{k_1, \ldots, k_r}(\lambda x_{i_1}, \ldots, \lambda x_{i_r}; \lambda^{-1} m) \quad (4.0.10)
\]

where the distributions \( P_{j; k; \lambda}(X) \) are of the form \( P_{j; k; \lambda}(X) = p_{j; k; \lambda}(\partial) \delta(X) \) and the following relation is verified:

\[
P_{j; k; 1} = 0. \quad (4.0.11)
\]

**Proof:** Let us consider the following expressions

\[
T_j^\lambda(X) \equiv \lambda^{\omega_j} T_j(X; \lambda^{-1} m), \quad \tilde{T}_j^\lambda(X) \equiv \mathcal{U}_\lambda T_j(\lambda^{-1} X; m) \mathcal{U}_\lambda^{-1}, \quad \forall |X| \geq 2
\]

both acting in the same Fock space: \( \mathcal{F}_{\lambda^{-1} m} \) and having the same “initial conditions”

\[
T_j^\lambda(x) \equiv \lambda^{\omega_j} T_j(x), \quad j = 1, \ldots, P
\]

due to (2.0.15).

Also, these expressions verify the Bogoliubov axioms: the unitarity and the causality are obvious, but for the Poincaré covariance one had to use the relation (2.0.4). We can apply theorem 3.1 and obtain that the difference between the two expressions \( \tilde{T}_j^\lambda(X) \) and \( T_j^\lambda(X) \) is a sum of the type appearing in the right hand side of the relation (3.3.3) but with the polynomials depending on the parameter \( \lambda \). If we make the substitution \( X \rightarrow \lambda X \) we get the relation from the statement.

The central result of this paper describes the explicit \( \lambda \)-dependence of the polynomials appearing in the preceding proposition.

**Theorem 4.2** The polynomials \( p_{j; k; \lambda}(\partial) \) are of the following form:

\[
p_{j; k; \lambda}(\partial) = \lambda^{\omega_j} \ln(\lambda) \sum_{|\alpha| = \omega_j - 4(|X| - 1) - \omega_k} c_\alpha \partial^\alpha \quad (4.0.14)
\]

where \( \alpha \) are multi-indices and \(|\alpha|\) is the corresponding length.

**Proof:** Is done by induction.

(i) First, we consider the case \(|X| = 2\). We start from the relation (4.0.10) from the preceding proposition and apply \( \mathcal{U}_\lambda \cdots \mathcal{U}_\lambda^{-1} \). We easily obtain the cocycle identity

\[
P_{j; k; \lambda'}(X) = \lambda^{\omega_j} P_{j; k; \lambda}(X) + (\lambda')^{\omega_k} P_{j; k; \lambda}(\lambda' X). \quad (4.0.15)
\]
Let us consider a typical term from $P_{J; k; \lambda}(X)$ of the form
\[ c(x) \theta^\alpha \delta(x_1 - x_2) \]  
with $|\alpha| = p$. One finds out immediately from the preceding cocycle identity that we have
\[ c(\lambda x') = \lambda^{\omega_j} c(\lambda') + (\lambda')^{4 + \omega_k + p} c(\lambda) \]  
where the index $\alpha$ was omitted. More conveniently, one defines
\[ d(\lambda) \equiv \lambda^{-\omega_j} c(\lambda) \]  
and has the cocycle identity:
\[ d(\lambda x') = d(\lambda') + (\lambda')^s d(\lambda), \quad s \equiv 4 + \omega_k + p - \omega_j. \]  
From (4.0.11) we have the “initial condition”:
\[ c(1) = 0 \iff d(1) = 0. \]  
The equation (4.0.19) can be analysed elementary if we differentiate with respect to $\lambda'$ and the put $\lambda' = 0$. The following differential equation emerges:
\[ \lambda d'(\lambda) = d(\lambda) + s d(\lambda) \]  
where $d_0 \equiv d'(1)$. We have two cases:
(a) $s \neq 0$

The homogeneous equation $\lambda D'(\lambda) = s D(\lambda)$ has the solution $D(\lambda) = A \lambda^s$. With the methods of variation of constants, we look for a solution of the preceding equation of the form $d(\lambda) = A(\lambda) \lambda^s$ with the initial condition $A(1) = 0$. The function $A(\lambda)$ will verify the equation:
\[ A' = d_0 \lambda^{-s}. \]  
We have two subcases:
(a1) $s \neq 1$

In this case, $A(\lambda) = \text{const.}(1 - \lambda^{1-s})$ and reverting to the initial equation (4.0.19) for the the function $d$ we get $d = 0 \iff c = 0$.

(a2) $s = 1$

In this case we have $A(\lambda) = d_0 \ln(\lambda)$ and substituting in the equation (4.0.19) we obtain the same conclusion as above.

(b) $s = 0$

The equation (4.0.21) becomes:
\[ \lambda d'(\lambda) = d_0 \]  
with the solution $d(\lambda) = d_0 \ln(\lambda)$ which identically verifies the initial equation (4.0.19). This means that the equation (4.0.19) has non-trivial solutions only in this case; the solution is of the form
\[ c(\lambda) = d_0 \lambda^{\omega_j} \ln(\lambda). \]  
This proves the assertion from the statement in the case $|X| = 2$. 

12
(ii) We suppose that the formula from the statement is valid for $2 \leq |X| \leq n - 1$ and we prove it for $|X| = n$. As before, we establish a cocycle identity for $P_{J; k; \lambda}(X)$, $|X| = n$. Instead of (4.0.15) we obtain in the same way:

$$P_{J; k; \lambda}(X) = \lambda^{|X|} P_{J; k; \lambda'}(X) + (\lambda')^{-1} \ln(\lambda') \sum_{r=2}^{|X|-1} c_r \ln^r(\lambda)$$  \hspace{1cm} (4.0.25)

This relation goes into (4.0.15) for $n = 2$ because the sum disappears. We consider a typical term of the form (4.0.16) and use the induction hypothesis. The result is an equation of the type:

$$d(\lambda') = d(\lambda') + (\lambda')^{s} d(\lambda) + (\lambda')^{-\omega k} \ln(\lambda') \sum_{r=2}^{|X|-1} c_r \ln^r(\lambda)$$  \hspace{1cm} (4.0.26)

where, again, the multi-index $\alpha$ was omitted and $c_r$ are some constants; their value will not be needed.

If we define the function $d(\lambda)$ by (4.0.18) we get instead of (4.0.19) the following relation:

$$d(\lambda') = d(\lambda') + (\lambda')^s d(\lambda) + (\lambda')^{-\omega k} \ln(\lambda') \sum_{r=2}^{|X|-1} c_r \ln^r(\lambda)$$

As before we get from this relation the differential equation:

$$\lambda d'(\lambda) = d_0 + s d(\lambda) + \sum_{r=2}^{|X|-1} c_r \ln^r(\lambda)$$  \hspace{1cm} (4.0.28)

We have the same cases as before.

(a) $s \neq 0$

The homogeneous equation is again $\lambda D'(\lambda) = s D(\lambda)$ with the the solution $D(\lambda) = A \lambda^s$. If we look for a solution of the equation (4.0.28) of the form $d(\lambda) = A(\lambda) \lambda^s$ with the initial condition $A(1) = 0$ we get for $A(\lambda)$ the equation:

$$A' = \lambda^{-s} \left[ d_0 + s d(\lambda) + \sum_{r=2}^{|X|-1} c_r \ln^r(\lambda) \right]$$  \hspace{1cm} (4.0.29)

We have two subcases:

(a1) $s \neq 1$

In this case, one can prove by direct computation that $A$ is of the form

$$A(\lambda) = d_0 + \lambda^{-s} \sum_{r=0}^{|X|-1} a_r \ln^r(\lambda) \iff d(\lambda) = d_0 \lambda^s + \sum_{r=0}^{|X|-1} a_r \lambda \ln^r(\lambda)$$  \hspace{1cm} (4.0.30)

with $a_r$ some constants. We substitute in the original equation (4.0.27) for the function $d$ and obtain that $d = 0 \iff c = 0$.

(a2) $s = 1$
In this case we obtain

\[
A(\lambda) = \sum_{r=0}^{|X|} a_r \lambda^r(\lambda) \quad \iff \quad d(\lambda) = \sum_{r=0}^{|X|} a_r \lambda^r(\lambda).
\] (4.0.31)

and substituting in the equation (4.0.19) we obtain the same conclusion as above.

(b) \( s = 0 \)

The equation (4.0.21) becomes:

\[
\lambda d' (\lambda) = d_0 + \sum_{r=2}^{|X|} c_r \lambda^r(\lambda).
\] (4.0.32)

with the solution

\[
d(\lambda) = d_0 \ln(\lambda) + \sum_{r=2}^{|X|} \frac{c_r}{r+1} \lambda^{r+1}(\lambda).
\] (4.0.33)

We substitute in the initial equation (4.0.19) and obtain that \( c_r = 0 \) so

\[
d(\lambda) = d_0 \ln(\lambda) \quad \iff \quad c(\lambda) = d_0 \lambda^\omega(\lambda).
\] (4.0.34)

It follows that the equation (4.0.27) has non-trivial solutions only in this case and we obtain the
the induction hypothesis for \( |X| = n \).

If we substitute the preceding result into the proposition 4.1 we get the following result:

**Theorem 4.3** The following relations are valid for any \( |X| \geq 2 \):

\[
U^J T J(X; m) U^{-1} = \lambda^\omega J [T J(\lambda X; \lambda^{-1} m) +
\sum_{r=1}^{|X|-1} \frac{\ln^r(\lambda)}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} \lambda^{-(\omega_{k_1} + \cdots + \omega_{k_r})} P_{J; k_1}(X_1) \cdots P_{J; k_r}(X_r) \times
T_{k_1, \ldots, k_r}(\lambda x_{i_1}, \ldots, \lambda x_{i_r}; \lambda^{-1} m)]
\] (4.0.35)

where the distributions \( P_{J; k}(X) \) are of the form

\[
P_{J; k}(X) = \sum_{|\alpha| = \omega_J - 4(|X| - 1) - \omega_k} c_\alpha J^\alpha \delta(X).
\] (4.0.36)

We also have:

\[
U^J \overline{T} J(X; m) U^{-1} = \lambda^\omega J [\overline{T} J(\lambda X; \lambda^{-1} m) +
\sum_{r=1}^{|X|-1} \frac{\ln^r(\lambda)}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} \lambda^{-(\omega_{k_1} + \cdots + \omega_{k_r})} P_{J; k_1}(X_1) \cdots P_{J; k_r}(X_r) \times
\overline{T}_{k_1, \ldots, k_r}(\lambda x_{i_1}, \ldots, \lambda x_{i_r}; \lambda^{-1} m)]
\] (4.0.37)
We translate the preceding result for the renormalized Feynman amplitudes. From this analysis one can obtain the asymptotic behaviour of these amplitudes as it is done in the classic paper of Weinberg [46].

It is known [23] that one can write the expression $T_J(X; m)$ as follows:

$$T_J(X; m) = \sum_{n \leq |X|} t_{J,k_1,\ldots,k_n}(X; m) : T_{k_1}(x_{i_1}; m) \cdots T_{k_n}(x_{i_n}; m) :$$  \hspace{1cm} (4.0.38)

where $x_{i_1}, \ldots, x_{i_n} \in X$. Then we have:

**Theorem 4.4** The following relations are verified:

$$t_{J;K}(X; m) = \lambda^{\omega_J - \omega_K} [T_{J;K}(\lambda X; \lambda^{-1} m) + \sum_{r=|K|}^{[X]-1} \frac{\ln^r(\lambda)}{r!} \sum_{X_1,\ldots,X_r \in \text{Part}(X)} \lambda^{-(\omega_{i_1} + \cdots + \omega_{i_r})} P_{J_{i_1:k_1}}(X_1) \cdots P_{J_{i_r:k_r}}(X_r) \times t_{l_1,\ldots,l_r;R}(\lambda x_{i_1}, \ldots, \lambda x_{i_n}; \lambda^{-1} m)]$$  \hspace{1cm} (4.0.39)

with the convention $t_{\{j\};\{k\}} = \delta_{jk}$.

The proof is done by substituting the expression (4.0.38) into the relation (4.0.35). The preceding theorem elucidates the logarithmic behaviour of the renormalized Feynman amplitudes. Presumably, the terms proportional with $\ln^r$ correspond to graphs with $r$ loops.

One can obtain the infinitesimal form of the preceding relation: we make $\lambda = e^\chi$, differentiate with respect to the variable $\chi$ and put $\chi = 0$. The following relation emerges. If $X = (x_1, \ldots, x_n)$ then:

$$\left( \sum_{l=1}^{n} x_l^{\mu} \frac{\partial}{\partial x_l^\nu} - m \frac{\partial}{\partial m} + \omega_J - \omega_K \right) t_{J;K}(X; m) = 0.$$  \hspace{1cm} (4.0.40)

If we take into account translation invariance, we can express the Feynman amplitudes $t_{J;K}(X; m)$ in terms of translation-invariant variables: $\xi_i \equiv x_i - x_n$, $i = 1, \ldots, n - 1$ and we have:

$$\left( \sum_{l=1}^{n-1} \xi_l^\mu \frac{\partial}{\partial \xi_l^\nu} - m \frac{\partial}{\partial m} + \omega_J - \omega_K \right) t_{J;K}(\Xi; m) = 0.$$  \hspace{1cm} (4.0.41)

or, for the Fourier transform:

$$\left( \sum_{l=1}^{n-1} p_l^\mu \frac{\partial}{\partial p_l^\nu} + m \frac{\partial}{\partial m} - \omega_J + \omega_K \right) \tilde{t}_{J;K}(P; m) = 0.$$  \hspace{1cm} (4.0.42)

Because the relations (4.0.42) follow from scale invariance, they can be called the Callan-Symanzik equation in the framework of Epstein-Glaser perturbative scheme. The usual Callan-Symanzik equation [42], [43] expresses the action of the (infinitesimal) dilation operator on the generating functional of the Green function, but it is natural to suppose that one can prove the two formalisms to be equivalent.
5 Yang-Mills Theories

In this Section we analyse the scale covariance of the Standard Model (SM) and the consequences of this property for the structure of possible anomalies.

5.1 The Fock Space of the Bosons

We give some basic facts about the quantization of a spin 1 Boson of mass \( m > 0 \). One can proceed in a rather close analogy to the case of the photon; for more details see [28] and references quoted there. Let us denote the Hilbert space of the Boson by \( H \).

The Hilbert space of the multi-Boson system should be, as before, the associated symmetric Fock space \( \mathcal{F}_m \equiv \mathcal{F}^+(H_m) \). We construct this Fock space as before in the spirit of algebraic quantum field theory. One considers the Hilbert space \( \mathcal{H}^{\text{th}} \) generated by applying on the vacuum \( \Phi_0 \) the free fields \( A^\mu(x) \), \( u(x) \), \( \bar{u}(x) \), \( \Phi(x) \) which are completely characterize by the following properties:

- Equation of motion:
  \[
  (\Box + m^2)A^\mu(x), \quad (\Box + m^2)u(x) = 0, \quad (\Box + m^2)\bar{u}(x) = 0, \quad (\Box + m^2)\Phi(x) = 0. \tag{5.1.1}
  \]

- Canonical (anti)commutation relations:

  \[
  [A^\mu(x), A^\rho(y)] = -g^{\mu\rho}D_m(x - y) \times 1, \]

  \[
  [A^\mu(x), u(y)] = 0, \quad [A^\nu(x), \bar{u}(y)] = 0, \quad [A^\mu(x), \Phi(y)] = 0, \]

  \[
  \{u(x), u(y)\} = 0, \quad \{\bar{u}(x), \bar{u}(y)\} = 0, \quad \{u(x), \bar{u}(y)\} = D_m(x - y) \times 1, \]

  \[
  [\Phi(x), \Phi(y)] = D_m(x - y) \times 1, \quad [\Phi(x), u(y)] = 0. \tag{5.1.2}
  \]

- \( SL(2, \mathbb{C}) \)-covariance:

  \[
  U_{a,A}A^\mu(x)U_{a,A}^{-1} = \delta(A^{-1})^\mu_\nu A^\nu(\delta(A) \cdot x + a), \]

  \[
  U_{a,A}u(x)U_{a,A}^{-1} = u(\delta(A) \cdot x + a), \quad U_{a,A}\bar{u}(x)U_{a,A}^{-1} = \bar{u}(\delta(A) \cdot x + a) \]

  \[
  U_{a,A}\Phi(x)U_{a,A}^{-1} = \Phi(\delta(A) \cdot x + a). \tag{5.1.3}
  \]

- PCT covariance.

  \[
  U_{PCT}A^\mu(x)U_{PCT}^{-1} = -A^\mu(-x), \quad U_{PCT}\Phi(x)U_{PCT}^{-1} = \Phi(-x) \]

  \[
  U_{PCT}u(x)U_{PCT}^{-1} = -u(-x), \quad U_{PCT}\bar{u}(x)U_{PCT}^{-1} = -\bar{u}(-x), \]

  \[
  U_{PCT}\Phi_0 = \Phi_0. \tag{5.1.4}
  \]

Remark 5.1 Although we could give the expressions for \( U_I, \quad U_t, \quad \text{and} \ U_C \) separately, we prefer to give only the expression of the PCT transform because the interaction Lagrangian of the standard model is not invariant with respect to these three operations but it is PCT-covariant.

We give as before in \( \mathcal{H}^{\text{th}} \) the sesqui-linear form \( \langle \cdot, \cdot \rangle \) which is completely characterize by requiring:

\[
A^\dagger_\mu(x) = A^\mu(x), \quad u^\dagger(x) = u(x), \quad \bar{u}^\dagger(x) = -\bar{u}(x), \quad \Phi^\dagger(x) = \Phi(x). \tag{5.1.5}
\]
Now, the expression of the supercharge gets an extra term:

\[ Q = \int_{\mathbb{R}^3} d^3x \left[ \partial^\mu A_\mu(x) + m\Phi(x) \right] \partial_0^\top u(x) \]  

(5.1.6)

and one can see that we have

\[ [Q, A_\mu] = i\partial_\mu u, \quad \{Q, u\} = -i(\partial_\mu A^\mu + m\Phi), \quad [Q, \Phi] = imu \]  

(5.1.7)

We still have

\[ Q^2 = 0 \implies Im(Q) \subset Ker(Q) \]  

(5.1.8)

and also

\[ U_{a,A}Q = QU_{a,A}, \quad U_{PCT}Q = -QU_{PCT}. \]  

(5.1.9)

Finally:

**Theorem 5.2** The sesqui-linear form \( \langle \cdot, \cdot \rangle \) factorizes to a well-defined scalar product on the completion of the factor space \( Ker(Q)/Im(Q) \). Then there exists the following Hilbert spaces isomorphism:

\[ \overline{Ker(Q)/Im(Q)} \cong \mathcal{F}_m; \]  

(5.1.10)

The representation of the Poincaré group and the PCT operator are factorizing to \( Ker(Q)/Im(Q) \) and are producing unitary operators (resp. an anti-unitary operator).

If \( W \) the linear space of all Wick monomials in the fields \( A_\mu, u, \bar{u} \) and \( \Phi \) acting in the Fock space \( \mathcal{H}^{gh} \) then the expression of the BRST operator is determined by

\[ d_Q u = 0, \quad d_Q \bar{u} = -i(\partial^\mu A_\mu + m\Phi), \quad d_Q A_\mu = i\partial_\mu u, \quad d_Q \Phi = imu. \]  

(5.1.11)

and, as a consequence we have

\[ d_Q^2 = 0. \]  

(5.1.12)

If one adds matter fields we proceed as before. In particular, this will mean that the BRST operator acts trivially on the matter fields.

Now we can define the Yang-Mills field. We must consider the case when we have \( r \) fields of spin 1 and some of them will have zero mass and the others will be considered of non-zero mass. Apparently, we need the scalar ghosts only in the last case. However it can be shown that with this assumption, there are no non-trivial models. To avoid this situation, we make the following amendment. All the fields considered above will carry an additional index \( a = 1, \ldots, r \) i.e. we have the following set of fields: \( A_{a\mu}, u_a, \bar{u}_a, \Phi_a, \quad a = 1, \ldots, r \). If one of the fields \( A_{a\mu} \) has zero mass we postulate that the corresponding scalar fields \( \Phi_a \) are physical fields and they will be called Higgs fields. Moreover, we do not have to assume that they are massless i.e. if some Boson field \( A_\mu^a \) has zero mass \( m_a = 0 \), we can suppose that the corresponding Higgs field \( \Phi_a \) has a non-zero mass: \( m_a^H \). It is convenient to use the compact notation

\[ m_a^* = \begin{cases} 
    m_a & \text{for } m_a \neq 0 \\
    m_a^H & \text{for } m_a = 0
\end{cases} \]  

(5.1.13)

These fields verify the following equations of motion:

\[ (\Box + m_a^2)A_\mu^a(x) = 0, \quad (\Box + m_a^2)u_a(x) = 0, \quad (\Box + m_a^2)\bar{u}_a(x) = 0, \quad (\Box + (m_a^*)^2)\Phi_a(x) = 0 \]  

(5.1.14)
The rest of the formalism stays unchanged. The canonical (anti)commutation relations are:

\[
\{A_a(x), A_b(y)\} = -\delta_{ab} g_{\mu\nu} D_m(x-y) \times 1, \\
\{u_a(x), \bar{u}_b(y)\} = \delta_{ab} D_m(x-y) \times 1, \\
\{\Phi_a(x), \Phi_b(y)\} = \delta_{ab} D^*_m(x-y) \times 1;
\]

(5.1.15)

and all other (anti)commutators are null. The supercharge is given by

\[
Q = \sum_{a=1}^r \int_{\mathbb{R}^3} d^3 x \left[ \partial^\mu A^\mu_a(x) + m_a \Phi_a(x) \right] \partial_0 u_a(x)
\]

(5.1.16)

and verifies all the expected properties.

The Krein operator is determined by:

\[
A^\dagger_a(x) = A_a(x), \quad u^\dagger_a(x) = u_a(x), \quad \bar{u}^\dagger_a(x) = -\bar{u}_a(x), \quad \Phi^\dagger_a(x) = \Phi_a(x).
\]

(5.1.17)

The ghost degree is defined in an obvious way and the expression of the BRST operator is similar to the previous one. In particular we have:

\[
d_Q u_a = 0, \quad d_Q \bar{u}_a = -i(\partial_A A^\mu_a + m_a \Phi_a), \quad d_Q A^\mu_a = i\partial^\mu u_a, \quad d_Q \Phi_a = i m_a u_a, \quad \forall a = 1, \ldots, r.
\]

(5.1.18)

Finally, the condition of gauge invariance is (see [12]):

\[
d_Q T_n(x_1, \ldots, x_n) = i \sum_{l=1}^n \frac{\partial}{\partial x^l} T^\mu_l(x_1, \ldots, x_n)
\]

(5.1.19)

for some Wick polynomials \( T^\mu_l(x_1, \ldots, x_n), \quad l = 1, \ldots, n. \)

5.2 Matter Fields and the Interaction Lagrangian of the SM

In this case the matter field is a set of Dirac fields of mass \( M_A, \quad A = 1, \ldots, N \) denoted by \( \psi_A(x) \). These fields are characterized by the following relations [29]: here \( A, B = 1, \ldots, N \):

- **Equation of motion:**
  \[
  (i\gamma \cdot \partial + M_A)\psi_A(x) = 0.
  \]

(5.2.1)

- **Canonical (anti)commutation relations:**
  \[
  [\psi_A(x), A^\mu_B(y)] = 0, \quad [\psi_A(x), u_a(y)] = 0, \quad [\psi_A(x), \bar{u}_a(y)] = 0, \quad [\psi_A(x), \Phi_a(y)] = 0
  \]

  \[
  \{\psi_A(x), \psi_B(y)\} = 0, \quad \{\bar{\psi}_A(x), \bar{\psi}_B(y)\} = \delta_{AB} S_{MA}(x-y) \times 1.
  \]

(5.2.2)

- **Covariance properties with respect to the Poincaré group:**
  \[
  U_{a,A} \psi_A(x) U^\dagger_{a,A} = S(A^{-1}) \psi_A(\delta(A) \cdot x + a).
  \]

(5.2.3)

- **PCT-covariance:**
  \[
  U_{PCT} \psi_A(x) U^\dagger_{PCT} = \gamma_1 \gamma_2 \gamma_3 \psi_A(-x)^\dagger.
  \]

(5.2.4)
The condition of gauge invariance remains the same (5.1.19) and one can prove [28] that this condition for \(n = 1, 2\) determines quite drastically the interaction Lagrangian of canonical dimension \(\omega(T_1) = 4\):

\[
T_1(x) \equiv f_{abc} [\partial_\mu A_{ab}(x) A^\mu_c(x) - :A_0^\mu A_0^c(x) : + : A_0^\mu u_b(\partial_\alpha u_c(x) : ] , \\
+ f_{abc}' [\Phi_0(x) \partial_\mu \Phi_0(x) A_0^\mu(x) - :m_b : \Phi_0(x) A_{ab}(x) A_0^\mu(x) - :m_b : \Phi_0(x) u_b(x) u_c(x) : ] , \\
+ f_{abc}'' : \Phi_0(x) \Phi_0(x) \Phi_0(x) + j_a(x) A_{ab}(x) + j_a(x) \Phi_0(x)
\]

where:

\[
j_a(x) = \bar{\psi}_A(x)(t_a)_{AB} \gamma^\mu \psi_B(x) + \bar{\psi}_A(x)(t_a')_{AB} \gamma^\mu \gamma^5 \psi_B(x)
\]

and

\[
j_a(x) = \bar{\psi}_A(x)(s_a)_{AB} \psi_B(x) + \bar{\psi}_A(x)(s_a')_{AB} \gamma^5 \psi_B(x)
\]

are the so-called currents. The conditions of \(SL(2, \mathbb{C})\) and PCT-covariance of the interaction Lagrangian are easy to prove as well as the causality condition. The hermiticity conditions are equivalent to the fact that the complex matrices are hermitian and verify the Jacobi identity so they generate a compact semi-simple Lie group quite naturally. There are other conditions on the rest of the constants as well, but because we do not need these properties in the subsequent analysis, we refer to the literature [28], [29] and references quoted there.

Moreover, it can be proved that the condition of gauge invariance (5.1.19) is valid for \(n = 1, 2\) and we can take \(T_1^\mu\) to be of canonical dimension \(\omega(T_1) = 4\) with the explicit form:

\[
T_1^\mu \equiv f_{abc} (u_a A_{ab} F^{\mu}_{bc} - \frac{1}{2} u_a u_b \partial_\mu u_c , \\
+ f_{abc}' (m_a : A_0^\mu \Phi_0 u_c + : \Phi_0 \partial_\mu \Phi_0 u_c : ) + u_a (x) j_a^\mu(x).
\]

The following relations are verified:

- **SL(2, \mathbb{C})-covariance:** for any \(A \in SL(2, \mathbb{C})\) we have

\[
U_{a,A} T_1(x) U_{a,A}^{-1} = T_1(\delta(A) \cdot x + a), \quad U_{a,A} T_1^\mu U_{a,A}^{-1} = \delta (A^{-1})_\rho^\mu T_{1/1}^\rho (\delta(A) \cdot x + a).
\]

- **PCT-covariance:**

\[
U_{PCT} T_1(x) U_{PCT}^{-1} = T_1(-x), \quad U_{PCT} T_1^\mu U_{PCT}^{-1} = T_1^\mu(-x).
\]

- **Causality:**

\[
[T_1(x), T_1(y)] = 0, \quad [T_{1/1}^\mu(x), T_{1/1}^\rho(y)] = 0, \quad [T_{1/1}^\mu(x), T_1(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \text{ s.t. } x \sim y.
\]

- **Unitarity:**

\[
T_1(x)^\dagger = T_1(x), \quad T_{1/1}^\mu(x)^\dagger = T_{1/1}^\mu(x).
\]

- **Ghost content:**

\[
gh(T_1) = 0, \quad gh(T_{1/1}^\mu) = 0.
\]

We mention that in [27]-[29], the condition of gauge invariance is analysed up to the order 3.
5.3 Dilation Covariance of the Standard Model

In this Subsection we generalize the arguments from the Sections 2.1 for the standard model. We denote the set of all masses by \( m \equiv (m_a, m_r^*, M_A)_{a=1, \ldots, r; A=1, \ldots, N} \) and the Fock space of all particles (physical or ghosts) by \( \mathcal{H}_m^\text{gh} \). This Hilbert space is generated from the vacuum by applying the operators: \( A_a^0(x; m_a), u_a(x; m_a), \bar{u}_a(x; m_a), \Phi_a(x; m_a^*) \) and \( \psi_A(x; M_A) \). We define the dilation operators in the total Hilbert space in analogy to (2.0.2) and the result from the proposition 2.1 stays true; we also have the commutations relations with the Poincaré (2.0.4). Finally, we have from (2.0.8) and (2.0.14):

\[
\begin{align*}
U_\lambda A_a^0(x; m_a)U_\lambda^{-1} &= \lambda A_a^0(\lambda x; \lambda^{-1} m_a), \\
U_\lambda \Phi_a(x; m_a)U_\lambda^{-1} &= \lambda \Phi_a(\lambda x; \lambda^{-1} m_a), \\
U_\lambda u_a(x; m_a)U_\lambda^{-1} &= \lambda u_a(\lambda x; \lambda^{-1} m_a), \\
U_\lambda \bar{u}_a(x; m_a)U_\lambda^{-1} &= \lambda \bar{u}(\lambda x; \lambda^{-1} m_a), \\
U_\lambda \psi_A(x; M_A)U_\lambda^{-1} &= \lambda^{3/2} \psi_A(\lambda x; \lambda^{-1} M_A), \quad \forall a = 1, \ldots, r, \quad \forall A = 1, \ldots, N. 
\end{align*}
\] (5.3.1)

From these relations and from the expressions (5.2.5) and (5.2.8) we obtain particular cases of the relation (2.0.15):

\[
\begin{align*}
U_\lambda T_1(x; m)U_\lambda^{-1} &= \lambda^4 T_1(\lambda x; \lambda^{-1} m), \\
U_\lambda T_1^0(x; m)U_\lambda^{-1} &= \lambda^4 T_1^0(\lambda x; \lambda^{-1} m);
\end{align*}
\] (5.3.2)

this means that both expressions have canonical dimension equal to 4 which is also the dimension of the Minkowski space-time. We have from (4.0.35):

**Proposition 5.3** For all \(|X| \geq 1\): we have the following formula:

\[
\begin{align*}
U_\lambda T(X; m)U_\lambda^{-1} &= \lambda^4|X|[T(\lambda X; \lambda^{-1} m) \\
&+ \sum_{r=1}^{|X|-1} \frac{\ln^r(\lambda)}{r!} \sum_{\text{Part}(X)} \lambda^{-\left(\omega_1 + \cdots + \omega_r\right)} P_k(X_1) \cdots P_k(X_r) \times \\
&\quad T_{k_1, \ldots, k_r}(\lambda x_{i_1}, \ldots, \lambda x_{i_r}; \lambda^{-1} m)]
\end{align*}
\] (5.3.3)

where the distributions \( P_k(X) \) are Lorentz covariant and of the form

\[
P_k(X) = \sum_{|\alpha| = 4 - \omega_k} c_\alpha \theta^\alpha \delta(X). 
\] (5.3.4)

We also mention the following result which easily follows from the definitions:

**Lemma 5.4** The following relations is valid for every Wick monomial:

\[
U_\lambda [d_Q W(X; m)]U_\lambda^{-1} = \lambda^\omega(W) + 1 W(\lambda X; \lambda^{-1} m).
\] (5.3.5)

**Proof:** If the expression \( W \) is one of the fields \( A_a^0(x; m_a), u_a(x; m_a), \bar{u}_a(x; m_a), \Phi_a(x; m_a^*) \) or \( \psi_A(x; M_A), \bar{\psi}_A(x; M_A) \) the formula from the statement follows elementary; then we extend to any Wick monomial by induction, using the derivative properties of the BRST operator. 

\[\square\]
5.4 The Structure of the Anomalies in the Standard Model

We consider the standard model as defined by the Lagrangian (5.2.5) and suppose that there are no anomalies up to the order \( n - 1 \), i.e., we have (5.1.19) up to this order. The purpose of this Subsection is to find if possible anomalous terms can appear in this relation in order \( n \) and what limitation are imposed by scale covariance. The analysis will be extremely similar to the case of the quantum electrodynamics [31].

(i) Suppose that we have constructed the chronological products \( T(X) \) and the associated chronological products \( T^\mu_i(X) \) for all cases \( |X| \leq n - 1 \) with the following conventions:

\[ T(\emptyset) = 1, \quad T^\mu_i(\emptyset) = 0, \quad T^\mu_i(X) = 0, \quad \text{for} \quad x_i \notin X. \]  

Moreover, we suppose that the following conditions have are true for any \( X \) of cardinal \( |X| \leq n - 1 \) (we use obvious compact notations):

- Symmetry:
  \[ T(\pi(X)) = T(X), \quad T_{\pi(i)}(\pi(X)) = T_i(X); \]  

- Covariance with respect to \( SL(2, \mathbb{C}) \):
  \[ U_{a,A}T(X)U_{a,A}^{-1} = T(\delta(A) \cdot X + a), \quad U_{a,A}T^\mu_i(X)U_{a,A}^{-1} = \delta(A^{-1})^\mu_p T^\mu_p(\delta(A) \cdot X + a); \]  

- PCT covariance
  \[ U_{PCT}T(X)U^{-1}_{PCT} = T(-X), \quad U_{PCT}T^\mu_i(X)U^{-1}_{PCT} = T^\mu_i(-X); \]  

- Ghost number content
  \[ gh(T(X)) = 0, \quad gh(T^\mu_i(X)) = 1; \]  

- Causality
  \[ T^\mu_i(X_1X_2) = T^\mu_i(X_1X_2) + T(X_1)T^\mu_i(X_2) \quad \forall X_1 \leq X_2 \]  
  and
  \[ [T^\mu_1(X_1), T^\mu_2(X_2)] = 0, \quad [T^\mu_i(X_1), T(X_2)] = 0 \quad \text{if} \quad X_1 \sim X_2 \]  
  for \( |X_1| + |X_2| \leq n - 1 \).

- Unitarity; we introduce, in analogy to (3.1.8):
  \[ (-1)^{|X|}T^\mu_i(X) \equiv \sum_{r=1}^{|X|} (-1)^r \sum [T^\mu_i(X_1)T(X_2) \cdots T(X_r) + \cdots + T(X_1) \cdots T(X_{r-1})T^\mu_i(X_r)] \]  
  where \( X_1, \cdots, X_r \) is a partition of \( X \) and we use in an essential way the convention (5.4.1). We require
  \[ T^\mu_i(X) = T^\mu_i(X)^\dagger; \]  

- Gauge invariance:
  \[ d_Q T(X) = i \sum_\mu \frac{\partial}{\partial x^\mu_i} T^\mu_i(X). \]
Now we can construct the expressions $T(X)$ and $T^\mu_l(X)$ for $|X| = n$ as in [31] such that we have all the relations enumerated above except the last one which is replaced by a weaker form:

$$d_Q T(X) = i\sum_l \frac{\partial}{\partial x_l^\mu} T^\mu_l(X) + P(X), \quad |X| = n$$

(5.4.11)

where $P(X)$ is a Wick polynomial (called anomaly) of the following structure:

$$P(X) = \sum_i p_i(\partial)\delta(X)W_i(X);$$

(5.4.12)

here $p_i$ are polynomials in the derivatives with the maximal degree restricted by

$$\deg(p_i) + \omega(W_i) \leq 5.$$  

(5.4.13)

Moreover, we can prove following [31] the following properties:

- **Symmetry**
  $$P(\pi(X)) = P(X)$$

  (5.4.14)

  for any permutation $\pi \in \mathcal{P}_n$.

- **$SL(2, \mathbb{C})$-covariance:**
  $$U_{a,A} P(X) U_{a,A}^{-1} = P(\delta(A) \cdot X + a), \quad \forall (a, A) \in \text{in}SL(2, \mathbb{C}).$$

  (5.4.15)

- **PCT-covariance:**
  $$U_{\text{PCT}} P(X) U_{\text{PCT}}^{-1} = (-1)^{|X|} P(-X).$$

  (5.4.16)

- **Unitarity:**
  $$P(X)^\dagger \equiv (-1)^{|X|} P(X).$$

  (5.4.17)

- **Ghost numbers restrictions:**
  $$\text{gh}(P(X)) = 1.$$ \hspace{1cm} (5.4.18)

- **Gauge invariance:**
  $$d_Q P(X) = i\sum_l \frac{\partial}{\partial x_l^\mu} P^\mu_l(X)$$

  (5.4.19)

  for some operators $P^\mu_l(X)$.

By “integrations by parts” (see [31]) we can exhibit the anomaly as follows:

$$P(X) = i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} N_l^\mu(X) + P'(X)$$

(5.4.20)

where $P'(X)$ is of the following form:

$$P'(X) = \delta(X)\mathcal{P}(x_n)$$

(5.4.21)

with $\mathcal{P}(x)$ a Wick polynomial. So, by redefining the expressions $T^\mu_l(X)$ we can take the anomaly of the form

$$P(X) = \delta(X)\mathcal{P}(x_n).$$

(5.4.22)

It is obvious that the Wick polynomial $\mathcal{P}(x)$ will verify the following restrictions:
\[ U_{a,A} \mathcal{P}(x) U_{a,A}^{-1} = \mathcal{P}(\delta(A) \cdot x + a), \quad \forall (a,A) \in \text{in} SL(2, \mathbb{C}). \] (5.4.23)

- **PCT-covariance:**
  \[ U_{\text{PCT}} \mathcal{P}(x) U_{\text{PCT}}^{-1} = (-1)^n \mathcal{P}(-x). \] (5.4.24)

- **Unitarity:**
  \[ \mathcal{P}(x)^\dagger \equiv (-1)^n \mathcal{P}(x). \] (5.4.25)

- **Ghost numbers restrictions:**
  \[ \text{gh}(\mathcal{P}(x)) = 1. \] (5.4.26)

- **Canonical dimension restriction:**
  \[ \omega(\mathcal{P}) \leq 5. \] (5.4.27)

(ii) Further restrictions come from (asymptotic) scale invariance. Here are the details.

First, we have a generalization of the formula (3.3.9). Let us consider two solutions \( T^\mu_i(X) \) and \( \hat{T}^\mu_i(X) \) of the axioms (5.4.2), (5.4.3), (5.4.4), (5.4.6), (5.4.7) and (5.4.9) for \( p = 1, \ldots, n \). Then we have for all \( 2 \leq |X| \leq n \):

\[
\hat{T}^\mu_i(X) = T^\mu_i(X)
\]

\[ + \sum_{r=1}^{|X|-1} \frac{1}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} [P^\mu_{t;k_1}(\lambda X_1)P_{k_2}(\lambda X_2) \cdots P_{k_r}(\lambda X_r) + \cdots
\]

\[ + P_{k_1}(\lambda X_1) \cdots P_{k_{r-1}}(\lambda X_{r-1})P^\mu_{t;k_r}(\lambda X_r)]T_{k_1, \ldots, k_r}(x_{i_1}, \ldots, x_{i_r}) \] (5.4.28)

where \( P_k(X) \) and \( P^\mu_{t;k}(X) \) are distributions of the type (3.3.4) and we use, as before, the conventions \( P_k(X) = P^\mu_{t;k}(X) = 0, \ |X| = 1 \).

The proof of this result follows the lines of theorem 3.1. From this result we obtain the analogue of the proposition 4.1. We have:

\[
\mathcal{U}_\lambda T^\mu_i(X; m) \mathcal{U}_\lambda^{-1} = \lambda^{4|X|} T^\mu_i(\lambda X, \lambda^{-1} m)
\]

\[ + \sum_{r=1}^{|X|-1} \frac{1}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} [P^\mu_{t;k_1,\lambda}(\lambda X_1)P_{k_2,\lambda}(\lambda X_2) \cdots P_{k_r,\lambda}(\lambda X_r) + \cdots
\]

\[ + P_{k_1,\lambda}(\lambda X_1) \cdots P_{k_{r-1},\lambda}(\lambda X_{r-1})P^\mu_{t;k_r,\lambda}(\lambda X_r)]T_{k_1, \ldots, k_r}(\lambda x_{i_1}, \ldots, \lambda x_{i_r}, \lambda^{-1} m). \] (5.4.29)

The generic structure distributions \( P_{k,\lambda}(X) \) and \( P^\mu_{t;k,\lambda}(X) \) is given by (3.3.4):

\[
P_{k,\lambda}(X) = p_{k,\lambda}(\partial)\delta(X), \quad P^\mu_{t;k,\lambda}(X) = p^\mu_{t;k,\lambda}(\partial)\delta(X)
\] (5.4.30)

where the expressions \( p_{k,\lambda}(\partial) \) and \( p^\mu_{t;k,\lambda}(\partial) \) are polynomials in the partial derivatives. In fact, we have a more precise result in analogy to theorem 4.2; namely both expressions are of the type:

\[
p_{\lambda}(\partial) = \lambda^{4|X|} ln(\lambda) \sum_{|\alpha| = 4 - \omega_k} c^\alpha \partial_\alpha.
\] (5.4.31)
As a result, we have an analogue of the proposition 5.3, namely the following formula is valid:

\[ U \mathcal{T}^\mu_t(X; \mathbf{m})U^{-1}_\lambda = \lambda^{4|X|}\{T^\mu_t(\lambda X; \lambda^{-1}\mathbf{m}) + \sum_{r=1}^{|X|-1} \frac{\ln^r(\lambda)}{r!} \sum_{X_1, \ldots, X_r \in \text{Part}(X)} \lambda^{-\omega_{k_1} + \cdots + \omega_{k_r}} \}
\]

\[ \left[ P^\mu_{l;k_1}(X_1)P_{k_1}(X_1) \cdots P_{k_r}(X_r) + \cdots + P_{k_1}(X_1) \cdots P_{k_{r-1}}(X_{r-1})P^\mu_{l;k_r}(X_r) \right] \times \]

\[ T_{k_1, \ldots, k_r}(\lambda x_{i_1}, \ldots, \lambda x_{i_r}; \lambda^{-1}\mathbf{m}) \} \quad (5.4.32) \]

where the distributions \( P_k(X) \) and \( P^\mu_{l;k}(X) \) are Lorentz covariant and of the form

\[ P(X) = \sum_{|\alpha|=1-\omega_k} c_\alpha \partial^\alpha \delta(X). \quad (5.4.33) \]

The main result can be obtained combining gauge invariance and scale invariance. We start from the equation (5.4.11) and apply to it \( U \mathcal{T}^\mu_t(X; \mathbf{m})U^{-1}_\lambda \). Then we use the lemma 5.3.5 and the formulæ (5.4.32) and (5.4.32). The result is the following identity verified by the anomaly:

\[ U \mathcal{A} P(X; \mathbf{m})U^{-1}_\lambda - \lambda^{4|X|+1} P(\lambda X; \lambda^{-1}\mathbf{m}) = dQ N_\lambda(X) + i \sum_{t \in X} \frac{\partial}{\partial x^\mu_t} N^\mu_{l;\lambda}(X) \quad (5.4.34) \]

where the explicit expressions for \( N(X) \) and \( N^\mu_{l;\lambda}(X) \) are not necessary and we have emphasized the mass-dependence of the anomaly \( P \) from the relation (5.4.11).

We introduce here the generic structure of the anomaly given by (5.4.22); if

\[ \mathcal{P}(X; \mathbf{m}) = \sum_j c_j(\mathbf{m}) \quad T_j(x; \mathbf{m}) \]

with \( c_j(\mathbf{m}) \) some mass-dependent constants, then we get an equation of the form:

\[ \delta(X) \sum_j \left[ \lambda^{\omega_j} c_j(\mathbf{m}) - \lambda^{\omega_j} c_j(\lambda^{-1}\mathbf{m}) \right] \quad T_j(\lambda x; \lambda^{-1}\mathbf{m}) = dQ N_\lambda(X) + i \sum_{t \in X} \frac{\partial}{\partial x^\mu_t} N^\mu_{l;\lambda}(X). \quad (5.4.36) \]

We see immediately that all the contributions for which we have

\[ c_j(\lambda \mathbf{m}) = \lambda^{5-\omega_j} c_j(\mathbf{m}) \]

disappear and all other contribution are of the form

\[ dQ N_\lambda(X) + i \sum_{t \in X} \frac{\partial}{\partial x^\mu_t} N^\mu_{l;\lambda}(X) \]

so they can be eliminated by redefining the expressions \( T(X) \) and \( T^\mu_t(X) \).

It follows that scale invariance can be used to impose the additional condition (5.4.37). One cannot conclude from this relation that the constants are polynomials of degree \( 5 - \omega_j \); logarithms of the type \( \ln(m/m') \) are, in principle possible. However, it is plausible that a polynomial structure in the masses can be proved.
(iii) The list of possible anomalies can be written now as in [31]. We only remark that the restrictions imposed above do not lead to the conclusion that there are no anomalies in order \( n \). In fact, a number of hard anomalies remain such as:

\[
P_1 = c_{abcde}^1 \sum_{m_a=m_b=m_c=m_d=m_e=0} u_a : \Phi_b \Phi_c \Phi_d \Phi_e : \\
P_2 = c_{abc}^2 \sum_{m_a=m_b=m_c=0} u_a : \partial^\mu \Phi_b \partial_\mu \Phi_c : \\
P_3 = c_{abc}^3 \varepsilon_{\mu\nu\rho\sigma} u_a : F_b^{\mu\nu} F_c^{\sigma\rho} : \\
\]

where

\[
P_4 = \sum_{m_a=m_b=0} \left[ : \bar{\psi}_A(K_{ab})_{AB} \psi_B : + : \bar{\psi}_A(K'_{ab})_{AB} \gamma_5 \psi_B : \right] u_a \Phi_b.
\]

One can show that from unitarity (or PCT-covariance) that we have

\[
c_{-} = (-1)^n c_{-}, \quad K_{ab}^* = (-1)^n K_{ab}, \quad (K'_{ab})^* = (-1)^n K'_{ab}.
\]

The list of hard anomalies is larger: all the anomalies appearing in the second and in the third order of perturbation theory (see [28] and [29]) should appear.

6 Conclusions

The expression (5.4.40) is the famous Adler-Bardeen-Bell-Jackiw anomaly (ABBJ). So, we see that the various symmetries of the standard model (including scale covariance) are not sufficient to prove the anomalies are absent in higher orders of the perturbation theory if they are absent in orders \( n = 1, 2, 3 \) (at least in Epstein-Glaser approach). In fact, if a certain type of anomaly is present in low orders of perturbation theory, this means that the corresponding expression is not in conflict with the various symmetries of the model. Then it is hard to imagine why such a conflict would appear in higher orders of perturbation theory. Such a result would be possible in our formalism only if in the parenthesis in left hand side of the equation (5.4.36) the order \( n \) of the perturbation theory would survive.

To obtain the cancelation of anomalies in all orders in our formalism a more refined formula for the distribution splitting seems to be needed.

Our result seems to be at odds with the analysis from [3] (see also [4] and [40]) where it is showed that the ABBJ anomaly can appear only in the order \( n = 3 \). However the proof uses Slavnov and Callan-Symanzik equations for the generating functional of the Green distributions; but these equations are expressing gauge invariance (resp. scale covariance) of the model in this formalism. The origin of the discrepancy between the two results is still to be investigated.
References


[34] T. Hurth, “A Note on Slavnov-Taylor Identities in the Causal Epstein-Glaser Approach”, Zürich-University-Preprint ZU-TH-21/95, hep-th/9511176


