We discover a new mechanism for the cosmological QCD phase transition: inhomogeneous nucleation. The primordial temperature fluctuations, measured to be $\delta T/T \sim 10^{-5}$, are larger than the tiny temperature interval, in which bubbles would form in the standard picture of homogeneous nucleation. Thus the bubbles nucleate at cold spots. We find the typical distance between bubble centers to be a few meters. This exceeds the estimates from homogeneous nucleation by two orders of magnitude. The resulting baryon inhomogeneities may affect primordial nucleosynthesis.

Based on a separation of cosmic phases during a first-order QCD transition [1] Applegate and Hogan [2] suggested that the cosmological QCD transition could give rise to inhomogeneous nucleosynthesis [3–6]. During a thermal first-order phase transition in a homogeneous medium bubbles nucleate due to statistical fluctuations (homogeneous nucleation). Their mean separation at nucleation introduces a scale for isothermal inhomogeneities in the early Universe, which may influence the local neutron-to-proton ratio, providing inhomogeneous initial conditions for nucleosynthesis. The baryon inhomogeneities may survive until the time of neutron freeze-out, if the mean bubble nucleation distance, $d_{nuc}$, exceeds the diffusion length of the proton. Comparing those scales at the time of the QCD transition gives the condition $d_{nuc} > 3 \text{ m}$ [5]. The causal scale is set by the Hubble distance at the QCD transition, $d_H \equiv c/H \sim 10 \text{ km}$.

The order of the QCD transition and the values of its parameters like latent heat or surface tension are still under debate. Nevertheless there are indications from lattice QCD calculations [7–11]. For the physical masses of the quarks the order of the transition is still unclear [7,8]. Quenched QCD (no dynamical quarks) shows a first-order phase transition with a small latent heat, compared to the bag model, and a small surface tension, compared to dimensional arguments [9]. We assume that the QCD transition is of first order and that the values from quenched lattice QCD (scaled appropriately by the number of degrees of freedom) are typical for the physical QCD transition. Based on these values and homogeneous bubble nucleation a small supercooling, $\Delta_{nuc} \equiv 1 - T_i/T_c \sim 10^{-4}$, and a tiny bubble nucleation distance, $d_{nuc} \sim 1 \text{ cm}$, follow [12]. Here, the actual nucleation temperature is denoted by $T_i$ and the thermodynamic transition temperature $T_c$ is $\sim 150 \text{ MeV}$.

We argue that the assumption of homogeneous nucleation is violated in the early Universe by the inevitable density perturbations from inflation [13] or from other seeds for structure formation. Those fluctuations in density and temperature have been measured by COBE [14] to have an amplitude of $\delta T/T \sim 10^{-5}$. The effect of the QCD transition on density perturbations [15,16] and gravitational waves [17] has been studied previously. In this Letter we investigate the effect of the density perturbations on the QCD phase transition. In comparison with a heterogeneous nucleation via ad hoc dirt [18], we do not introduce any new, unknown objects. Based on the COBE measurements and the quenched lattice QCD data we conclude that a first order QCD transition induces an inhomogeneity scale of a few meters. This might have interesting implications for precision measurements of primordial abundances [5,6], e.g. precise deuterium measurements with the FUSE spacecraft [19].

First-order phase transitions normally proceed via nucleation of bubbles of the new phase. When the temperature is spatially uniform and no significant impurities are present, the mechanism is homogeneous nucleation. The nucleation rate of bubbles of the new phase is approximated by

$$\Gamma \approx T^4 e^{-S(T)},$$

where $\Gamma$ is the probability of nucleation per time and volume. The nucleation action $S$ is the free energy difference of the system with and without the nucleating bubble, divided by the temperature.

Nucleation is a very rapid process, compared with the extremely slow cooling of the Universe. The duration of the nucleation period, $\Delta t_{nuc}$, is found to be [4,20]

$$\Delta t_{nuc} = \frac{\pi^{1/3}}{\text{d}S/\text{d}t} |_{t_f}.$$  

The time $t_f$ is defined as the moment when the fraction of space where nucleations still continue equals $1/e$. The rest of the Universe has been reheated sufficiently to stop further nucleations. Correspondingly, we denote by $v_{\text{heat}}$ the effective speed by which released latent heat propagates in sufficient amounts to shut down nucleations. In general, $v_{\text{def}} < v_{\text{heat}} \leq c$, where $v_{\text{def}}$ is the velocity of the deflagration front. In the unlikely case of detonations $v_{\text{heat}}$ should be replaced by the velocity of the phase boundary.
The mean distance between nucleation centers, measured immediately after the transition completed, is

$$d_{\text{nuc, hom}} = 2v_{\text{heat}} \Delta t_{\text{nuc}}. \quad (3)$$

This nucleation distance sets the spatial scale for baryon number inhomogeneities.

Lattice simulations \([10,11]\) imply that in real-world QCD the energy density must change very rapidly in a narrow temperature interval. This can be seen from the microscopic sound speed in the quark phase, \(c_s\). Lattice QCD indicates that \(3c_s^2(T_c) = \mathcal{O}(0.1)\) \([11]\). Thus, the cosmological time-temperature relation is strongly modified already before the nucleations, due to

$$\frac{dT}{dt} = -3c_s^2 T \frac{T}{t_H}, \quad (4)$$

where \(t_H \equiv 1/H = (3M_{\text{pl}}^2/8\pi\varepsilon_0)^{1/2}\) with \(\varepsilon_0\) being the energy density in the quark phase. This behavior of the sound speed increases the nucleation distance because of the proportionality \(\Delta t_{\text{nuc}} \propto 1/[3c_s^2(T_p)]\) \([12]\).

In the thin-wall approximation the nucleation action has the following explicit expression:

$$S(T) = \frac{C^2}{(1 - T/T_c)^2}, \quad C \equiv 4\sqrt{\frac{\pi}{3}} \frac{\sigma^{3/2}}{\sqrt{T_c}}, \quad (5)$$

for small supercooling. Assuming further that \(c_s\) does not change very much during supercooling, the following relation holds for the supercooling and nucleation scales:

$$\frac{\Delta t_{\text{sc}}}{\Delta t_{\text{nuc}}} = \frac{\Delta_{\text{sc}}}{\Delta_{\text{nuc}}} = \frac{2}{\pi^{1/3}} \tilde{S}. \quad (6)$$

Here we denote by \(\Delta\) a relative (dimensionless) temperature interval and by \(\Delta T\) a dimensionful time interval. \(\tilde{S} \equiv S(T_f)\) is the critical nucleation action, \(\tilde{S} = \mathcal{O}(100)\).

Surface tension and latent heat are provided by lattice simulations with quenched QCD only, giving the values \(\sigma = 0.015T_c^4, \quad l = 1.5T_c^4\) \([9]\). Comparing this latent heat with that of a bag model with gluons only, and assuming that the same ratio would hold for the physical QCD compared with a bag model with 2.5 massless quarks, one arrives at \(l = 3.7T_c^4\) for the physical QCD. We take \(l = 3T_c^4\).

With these values for the latent heat and surface tension, the amount of supercooling is \(\Delta_{\text{sc}} = 2.3 \times 10^{-4}\). From Eq. (6) it follows that \(\Delta_{\text{nuc}} = 1.5 \times 10^{-6}\). Substituting \(3c_s^2 = 0.1\) into Eq. (4), we find \(\Delta t_{\text{nuc}} = 1.5 \times 10^{-5} t_H\) for the duration of the nucleation period. The nucleation distance depends on the unknown velocity \(v_{\text{heat}}\) in Eq. (3). With the value 0.05 for \(v_{\text{heat}}\), the nucleation distance \(d_{\text{nuc, hom}}\) would have the value \(1.5 \times 10^{-6} dh\). One should take these values with caution, due to large uncertainties in \(l\) and \(\sigma\). As our reference set of parameters, we take: \(\Delta_{\text{sc}} = 10^{-4}, \quad \Delta_{\text{nuc}} = 10^{-6}, \quad \Delta t_{\text{nuc}} = 10^{-5} t_H\).

In the real Universe the local temperature of the radiation fluid fluctuates. We decompose the local temperature \(T(t, x)\) into the mean temperature \(\bar{T}(t)\) and the perturbation \(\delta T(t, x)\). The temperature contrast is denoted by \(\Delta = \delta T/T\). On subhorizon scales in the radiation dominated epoch, each Fourier coefficient \(\Delta(t, k)\) oscillates with constant amplitude, which we denote by \(\Delta_T(k)\). Inflation predicts a Gaussian distribution,

$$p(\Delta) d\Delta = \frac{1}{\sqrt{2\pi \Delta_T^{\text{rms}}}} \exp\left(-\frac{1}{2} \frac{\Delta^2}{(\Delta_T^{\text{rms}})^2}\right) d\Delta. \quad (7)$$

Allowing for a tilt in the power spectrum of density fluctuations, the COBE normalized \([14]\) rms temperature fluctuation reads

$$\Delta_T^{\text{rms}} \approx 10^{-5} \left(\frac{k}{k_0}\right)^{(n-1)/2}, \quad (8)$$

where \(k_0 = (aH)_0\). The case \(n = 1\) gives the Harrison-Zel’dovich spectrum. For \(k_{\text{QCD}} \equiv (aH)_{QCD}\) a moderate tilt of \(|n - 1| < 0.2\) gives \(\Delta_T^{\text{rms}}(k_{\text{QCD}}) \in (10^{-4}, 10^{-6})\).

A small scale cut-off in the spectrum of primordial temperature fluctuations comes from collisional damping by neutrinos \([21,16]\). The interaction rate of neutrinos is \(\sim G^2 T^3\). This has to be compared with the angular frequency \(c_s k_{\text{ph}}\) of the acoustic oscillations. At the QCD transition neutrinos travel freely on scales \(l \approx 4 \times 10^{-9} d_H\). Fluctuations below the diffusion scale of neutrinos are washed out,

$$l_{\text{diff}} = \left[\int_0^{l_c} l_{\nu}(i) di\right]^\frac{1}{2} \approx 7 \times 10^{-4} d_H. \quad (9)$$

In Ref. \([16]\] the damping scale from collisional damping by neutrinos has been calculated to be \(k_{\text{ph}}^3 = 10^4 H\) at \(T = 150\) MeV. The estimate (9) is consistent with this damping scale. We assume \(l_{\text{smooth}} = 10^{-4} d_H\). The compression timescale for a homogeneous volume \(\sim l^3_{\text{smooth}}/c_s \sim 10^{-3} H\). Since \(\Delta t_{\text{nuc}} < l_{\text{smooth}}\) the temperature fluctuations are frozen with respect to the timescale of nucleations. Therefore homogeneous bubble nucleation applies within these small homogeneous volumes.

Let us now investigate bubble nucleation in a Universe with spatially inhomogeneous temperature distribution. Bubble nucleation effectively takes place while the temperature drops by the tiny amount \(\Delta_{\text{nuc}}\). To determine the mechanism of nucleation, we compare \(\Delta_{\text{nuc}}\) with the rms temperature fluctuation \(\Delta_T^{\text{rms}}\):

1. If \(\Delta_T^{\text{rms}} < \Delta_{\text{nuc}}\), the probability to nucleate a bubble at a given time is homogeneous in space. This is the case of homogeneous nucleation.
2. If \(\Delta_T^{\text{rms}} > \Delta_{\text{nuc}}\), the probability to nucleate a bubble at a given time is inhomogeneous in space. We call this inhomogeneous nucleation.
By these definitions heterogeneous nucleation, in which impurities act as seeds for nucleation, is a subclass of inhomogeneous nucleation, because the probability to nucleate a bubble is inhomogeneous. However, we do not assume any ad hoc impurities.

The quenched lattice QCD data for latent heat and surface tension and the assumption of a COBE normalized Harrison-Zel’dovich spectrum provide the values $\Delta_{\text{nuc}} \sim 10^{-6}$ and $\Delta_T^{\text{rms}} \sim 10^{-5}$. We conclude that the cosmological QCD transition may proceed via inhomogeneous nucleation. A sketch of inhomogeneous nucleation is shown in Fig. 1. The basic idea is that temperature inhomogeneities determine the location of bubble nucleation. Bubbles nucleate first in the cold regions.

The temperature change at a given point is governed by the Hubble expansion and by the temperature fluctuations. For the fastest changing fluctuations, with angular frequency $c_s / \ell_{\text{smooth}}$, we find

$$\frac{dT(t,x)}{dt} = \frac{T}{t_H} \left[ -3c_s^2 + \mathcal{O} \left( \frac{\Delta_T t_H}{\delta t} \right) \right]. \quad (10)$$

The contribution of temperature fluctuations is estimated for the COBE normalized rms fluctuations with $n = 1$ to be $\Delta_T^{\text{rms}} t_H / \delta t \sim 10^{-3}$. For $3c_s^2 = 0.1$, as indicated from quenched lattice QCD, the Hubble expansion is the dominant contribution. This holds true even if either $\Delta_T$ or $c_s$ is different by two or four orders of magnitude, respectively. This means that the local temperature can never increase, except by the released latent heat during bubble growth.

A rough estimate for the minimum distance in inhomogeneous nucleation is provided by the following argument. Consider nucleation in a cold spot (having uniform temperature). The time it takes for bubbles within this cold spot to merge is given by

$$\Delta t_{\text{merge}} = \frac{d_{\text{nuc, hom}}}{2v_{\text{def}}} = \frac{v_{\text{heat}}}{v_{\text{def}}} \Delta t_{\text{nuc}}, \quad (11)$$

where the homogeneous nucleation distance can be used because of Eq. (10). The rms temperature difference for two randomly chosen spots is $\sqrt{2}T l \Delta_H^{\text{rms}}$. Spots which are hotter than the cold spot by this typical amount, need the time

$$\Delta t_{\text{cool}} = \frac{\sqrt{2} \Delta_{\text{rms}}}{3c_s^2} t_H \quad (12)$$

to cool to the nucleation temperature $T_H$. Bubbles merge within cold spots before nucleations can take place in hotter regions if

$$\Delta t_{\text{merge}} < \Delta t_{\text{cool}}. \quad (13)$$

For the temperature intervals this condition reads $\Delta_{\text{nuc}} < \sqrt{2} (v_{\text{def}} / v_{\text{heat}}) \Delta_H^{\text{rms}}$, which can be compared with the definition for inhomogeneous nucleation, $\Delta_{\text{nuc}} < \Delta_T^{\text{rms}}$. For our reference set of parameters the condition (13) should be clearly fulfilled. Therefore

$$d_{\text{nuc, ihn}} \geq \ell_{\text{smooth}} \quad (14)$$

follows, because hadronic bubbles of scale $\ell_{\text{smooth}}$ have formed before any bubbles in hotter regions have nucleated.

Above we did not specify how cold and rare those spots are, which drive the inhomogeneous transition. If the released latent heat may quench the nucleation of bubbles in the intervening space, once nucleation has started in the very cold spots, the effective nucleation distance may be much larger than $\ell_{\text{smooth}}$. The fraction of space that is not reheated at time $t$ is given by

$$f(t) \approx 1 - \int_0^t \Gamma_{\text{ihn}}(t') V(t, t') dt', \quad (15)$$

where we neglect overlap and merging of heat fronts. At time $t$ heat, coming from a cold spot which was transformed into hadron phase at time $t'$, occupies the volume $V(t, t') = (4\pi^2 / 3) [\ell_{\text{smooth}} / 2 + v_{\text{heat}} (t - t')].$ Based on earlier discussion, the merging of tiny bubbles within a cold spot can be treated as an instantaneous process as far as inhomogeneous nucleation is concerned. The other factor in Eq. (15), $\Gamma_{\text{ihn}}$, is the volume fraction converted into the new phase, per physical time and volume as a function of the mean temperature $T = \bar{T}(t)$. $\Gamma_{\text{ihn}}$ is proportional to the fraction of space for which temperature is in the interval $[T_1, T_F(1 + d\Delta)]$. This fraction of space is given by Eq. (7) with $\Delta = T_1 / T - 1$. Finally, rewriting $d\Delta$ by means of Eq. (4) leads to the expression

3
\[ \Gamma_{\text{ihn}} = 3c_s^2 \frac{T_f}{T} \frac{1}{t_H} \nu_{\text{smooth}} p(\Delta = \frac{T_f}{T} - 1), \]  
\[ (16) \]

where the relevant physical volume is \( \nu_{\text{smooth}} = (4\pi/3)(\nu_{\text{smooth}}/2)^3 \).

The transition is defined to be completed when \( f(t_{\text{ihn}}) = 0 \). We introduce the variables \( N \equiv (1 - T/T_f) / \Delta_{\text{ms}}^3 \) and \( \mathcal{N} \equiv N(t_{\text{ihn}}) \). Since \( c_s \) may be assumed to be constant during the tiny temperature interval where nucleations actually take place, we find from Eq. (4):

\[ 1 - t/t_{\text{ihn}} \approx 2/(3c_s^2) \Delta_{\text{ms}}^3 (N - \mathcal{N}). \]

Putting everything together we determine \( \mathcal{N} \) from

\[ A^3 \frac{1}{2\pi} \int_{N}^\infty dN e^{-N^2} \left( x + N - \frac{1}{3}N^3 \right)^3 = 1, \]

\[ (17) \]

where \( A \equiv (2v_{\text{heat}}/3c_s^2) \Delta_{\text{ms}}^3 (dt/dt_{\text{ihn}}) \). The COBE normalized spectrum with \( n = 1(1.2) \) gives \( A \approx 0.1(1) \times (2v_{\text{heat}}/3c_s^2) \). If \( A \) is below unity typical fluctuations \( (N \sim 1) \) will dominate the nucleation process, whereas for \( A > 1 \) the rare cold spots are dominant. In the former case the approximate character of Eq. (15) makes the quantitative analysis less accurate. Let us note that the latter case \( A > 1 \) is quite possible, since \( 2v_{\text{heat}}/3c_s^2 \) may be clearly larger than unity. For the values \( A = 1, 2, 5, 10 \) we find \( \mathcal{N} \approx 0.8, 1.4, 2.1, 2.6 \), respectively.

The effective nucleation distance in inhomogeneous nucleation is found from the approximate relation

\[ d_{\text{nuc},\text{ihn}} \equiv n_{\text{bubbles}}^{-1/3} \approx \left( \int_0^{t_{\text{ihn}}} \Gamma_{\text{ihn}}(t) dt \right)^{-1/3}. \]

\[ (18) \]

Writing this as a function of \( \mathcal{N} \) we find:

\[ d_{\text{nuc},\text{ihn}} = \left( \frac{2}{\pi} (1 - \text{erf}(\sqrt{\mathcal{N}})) \right)^{-1/3} l_{\text{smooth}}. \]

\[ (19) \]

With the above values \( A = 1, 2, 5, 10 \) we get \( d_{\text{nuc},\text{ihn}} = 1.4, 1.8, 3.0, 4.8 \times l_{\text{smooth}} \), where \( l_{\text{smooth}} \approx 1 \). In the case of a COBE normalised spectrum without tilt and \( 2v_{\text{heat}}/3c_s^2 < 10 \) we are in the region \( A < 1 \), leading to a nucleation distance \( d_{\text{nuc},\text{ihn}} = \mathcal{O}(1 \text{ m}) \).

According to recent studies \([5,6]\) inhomogeneous nucleosynthesis is consistent with observations from primordial abundances, with an inhomogeneous QCD scale of \( \mathcal{O}(10 \text{ m}) \). We find that this requirement might be met in the inhomogeneous QCD transition for \( A > 1 \).

In conclusion, we found that inhomogeneous nucleation leads to nucleation distances that exceed by two orders of magnitude estimates based on homogeneous nucleation. We emphasise that this new effect appears for the (today) most probable range of cosmological and QCD parameters. Inhomogeneous nucleation might be the generic mechanism of many phase transitions in Nature.

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