A Yang–Mills Theory in Loop Space and Generalized Chapline–Manton Coupling

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Abstract

We consider a Yang-Mills theory in loop space with an affine Lie gauge group. The Chapline-Manton coupling, the coupling between Yang-Mills fields and an abelian antisymmetric tensor field of second rank via the Chern-Simons term, is systematically derived within the framework of the Yang-Mills theory. The generalized Chapline-Manton couplings, the couplings among non-abelian tensor fields of second rank, Yang-Mills fields, and an abelian tensor field of third rank, are also derived by applying the non-linear realization method to the Yang-Mills theory. These couplings are accompanied by \( BF \)-like terms.

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1 Introduction

Gauge symmetries have been established as a guiding principle that determines couplings among local fields. The fundamental interactions, electromagnetic, weak and strong interactions, are intermediated by gauge fields, and the couplings between the gauge fields and matter fields and the self-coupling of the gauge fields are determined by the gauge symmetries. In addition, the gauge symmetries lead to the gravitational interaction. Indeed, the gravitational interaction can be formulated within the framework of gauge theory based on a non-compact gauge group.

In supergravity theories, on the other hand, there is a non-trivial interaction which cannot be derived from gauge symmetries alone. In $\mathcal{N}=1$ supergravity theory (with $\mathcal{N}=1$ super Yang-Mills theory) in ten dimensions, the Lagrangian has a coupling between Yang-Mills fields and the abelian antisymmetric tensor field in the supergravity multiplet via the Chern-Simons 3-form. A non-trivial coupling was first introduced by de Wit et al. in the abelian case, and generalized to the non-abelian case by Chapline and Manton. [1] Although this coupling is determined by the local supersymmetry in the supergravity theory, its derivation involves much tedious algebra. [2] In this system, the antisymmetric tensor field must obey a deformed transformation rule including the Yang-Mills fields so as to maintain the invariance of the Lagrangian. This transformation rule can be determined uniquely, but only in a heuristic manner. The transformation rule plays an important role in proving the Green-Schwarz anomaly cancellation in superstring theory. [4] The non-trivial coupling between Yang-Mills fields and the abelian antisymmetric tensor field is referred to as the Chapline-Manton coupling. The derivation of the coupling has also been discussed from the viewpoint of the BRST cohomology. [3]

Recently, the Chapline-Manton coupling has also been derived on the basis of a gauge theory — a Yang-Mills theory in loop space. [10] In this theory, the gauge field on loop space is given by the functional field on space-time, including an infinite number of local component fields. Yang-Mills fields (local Yang–Mills fields) are a part of the local component fields of the Yang-Mills fields on loop space, while an abelian antisymmetric tensor field of second rank is a part of the local component fields of the $U(1)$ gauge field on loop space. The couplings among the local component fields of the Yang-Mills fields are determined by the symmetry of the loop gauge group. [11] These couplings are essentially caused by the non-commutativety of the Lie algebra. In order to derive non-trivial couplings among the local component fields of the Yang-Mills fields and those of the $U(1)$ gauge field, we need an extension of the loop gauge group. The suitable gauge group is the affine Lie group, which is a central extension of the loop group. [13] Owing to the effect of the central extension, the extended gauge symmetry further leads to a new coupling between the local Yang-Mills fields and an abelian antisymmetric tensor field of second rank. This coincides with the Chapline-Manton coupling. In addition, the deformed transformation rule of the abelian antisymmetric tensor field of second rank is also derived from...
the transformation rule of the $U(1)$ gauge field on loop space. [10]

In addition to the abelian antisymmetric tensor field of second rank, for example, abelian (totally) antisymmetric tensor fields of higher rank appear in the Ramond-Ramond sector of superstring theories. [5] These fields also contribute to a cancellation of the anomalies on D-branes under certain conditions. [6] It is not difficult to extend the Chapline-Manton coupling for abelian antisymmetric tensor fields of higher rank. Indeed, such an antisymmetric tensor field can couple with the Chern-Simons $(2n+1)$-form. [6] [15] On the other hand, non-abelian antisymmetric tensor fields appear in the so-called $BF$ term. [7] The $BF$ term is metric independent and takes the form of the product of a non-abelian antisymmetric tensor field $B$ and a field strength $F$ of Yang-Mills fields in the non-abelian case. The $BF$ term is a generalization of the Chern-Simons term and is an important ingredient in $BF$ Yang-Mills theories, [8] topologically massive gauge theories, [9] and so on. However, the extension of the Chapline-Manton coupling for non-abelian antisymmetric tensor fields is not known.

As is mentioned above, the Yang-Mills fields and $U(1)$ gauge field on loop space possess an infinite number of local component fields. Non-abelian antisymmetric and symmetric tensor fields of second rank are the local component fields of the Yang-Mills fields on loop space, [14] while an abelian tensor field of third rank with certain symmetric properties is the local component field of the $U(1)$ gauge field on loop space. [17] The interactions among these non-abelian tensor fields of second rank and local Yang-Mills fields can also be derived using the formalism of a non-linear realization developed for the loop gauge group applied to Yang-Mills theory in loop space. [14] In this paper, we consider the Yang-Mills theory in loop space with the affine Lie gauge group and apply a non-linear realization method to the Yang-Mills theory. As we see below, the non-trivial interactions among the local Yang-Mills fields, non-abelian tensor fields of second rank, and an abelian tensor field of third rank can be systematically determined within the framework of the Yang-Mills theory. These local fields interact via a $BF$-like term.

This paper is organized as follows. In Sec. 2, we construct a Yang-Mills theory in loop space with the affine Lie gauge group. In Sec. 3, it is shown that the local field theory for Yang-Mills fields and an antisymmetric tensor field with the Chapline-Manton coupling is naturally derived on the basis of this theory. In Sec. 4, we apply a non-linear realization method developed for the affine Lie gauge group to the Yang-Mills theory in loop space. In Sec. 5, referring to the previous sections, we derive a local field theory for the non-abelian tensor fields, Yang-Mills fields and abelian tensor fields with non-trivial coupling. Section 6 is devoted to a summary and discussion of the possibilities for future development.
2 The Yang-Mills theory in loop space

We define a loop space $\Omega^D$ as the set of all loops in $D$-dimensional Minkowski space $M^D$. An arbitrary loop $x^\mu = x^\mu(\sigma) \ [0 \leq \sigma \leq 2\pi, x^\mu(0) = x^\mu(2\pi)]$ in $M^D$ is represented as a point in $\Omega^D$ denoted by coordinates $(x^\mu_\sigma)$ with $x^\mu_\sigma \equiv x^\mu(\sigma)$.  

Let us consider a Yang-Mills theory in the loop space $\Omega^D$. We assume that a gauge group is an affine Lie group $\hat{G}_k$ whose generators $T_a(\sigma)$ satisfy the commutation relation

$$[T_a(\rho), T_b(\sigma)] = i f_{abc} T_c(\rho) \delta(\rho - \sigma) + ik \kappa_{ab} \delta(\rho - \sigma) \quad (2.1)$$

and the hermiticity conditions $T_a^*(\sigma) = T_a(\sigma)$. Here the $f_{abc}$ are the structure constants of the semisimple Lie group $G$ and $\kappa_{ab}$ is the Killing metric of $G$. The constant $k$ is called the ‘central charge’ of $\hat{G}_k$ and takes an arbitrary value in this gauge theory. When we set $k = 0$, the commutation relation (2.1) results in that for the loop group $\hat{G}_0$. It is possible to make (non-trivial) central extensions to infinite-dimensional algebras. Hereafter, we refer to the Yang-Mills theory with an affine Lie gauge group as the ‘extended Yang-Mills theory’ (EYMT). It will become clear that the central extension of the gauge group leads to non-trivial couplings among non-abelian gauge fields and abelian gauge fields.

Let $A_{\mu\sigma}[x]$ be a gauge field on $\Omega^D$. Owing to the central extension, the commutator given in (2.1) yields central terms without $T_a(\sigma)$, in addition to a linear combination of $T_a(\sigma)$. Consequently, the gauge field $A_{\mu\sigma}$ needs extra terms without $T_a(\sigma)$ to allow for the consistency of the gauge transformation. We define the gauge field $A_{\mu\sigma}$ as

$$A_{\mu\sigma}[x] = A^{Y}_{\mu\sigma}[x] + \tilde{A}^{U}_{\mu\sigma}[x] \quad (2.2)$$

where $A^{Y}_{\mu\sigma}[x]$ is a Yang-Mills field:

$$A^{Y}_{\mu\sigma}[x] = \int_0^{2\pi} \frac{d\rho}{2\pi} A_{\mu\sigma}^{np}[x] T_a(\rho) \quad (2.3)$$

Here $A_{\mu\sigma}^{np}$ is a vector fields on $\Omega^D$, and $\tilde{A}^{U}_{\mu\sigma}$ is a $U(1)$ gauge field on $\Omega^D$ without $T_a(\sigma)$. As in the ordinary Yang-Mills theory, the infinitesimal gauge transformation for $A_{\mu\sigma}$ is given by

$$\delta A_{\mu\sigma}[x] = \partial_{\mu\sigma}\Lambda[x] + i [A_{\mu\sigma}[x], \Lambda[x]] \quad (2.4)$$

where $\partial_{\mu\sigma} \equiv \partial/\partial x^{\mu\sigma}$. An infinitesimal scalar function $\Lambda$ on $\Omega^D$ is defined by

$$\Lambda[x] = \Lambda^Y[x] + \Lambda^U[x] \quad (2.5)$$

1) In the present paper the indices $\mu, \nu, \kappa, \lambda, \xi$ and $\zeta$ take the values $0, 1, 2, \ldots, D-1$, while the indices $\rho, \sigma, \chi$, and $\omega$ take continuous values from 0 to $2\pi$.

2) The indices $a, b, c, d, e$ take the values $1, 2, 3, \ldots, \dim G$.
where $\Lambda^Y[x]$ is written

$$\Lambda^Y[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} \Lambda^{a\sigma}[x] T_a(\sigma), \quad (2.6)$$

with scalar functions $\Lambda^{a\sigma}$ on $\Omega M^D$, and $\Lambda^U$ is a scalar function $\Omega M^D$ without $T_a(\sigma)$. Since there is no relation between the gauge transformation (2.4) and a reparametrization $\sigma \to \bar{\sigma}(\sigma)$, $\Lambda^Y$ and $\Lambda^U$ obey the following reparametrization invariant conditions:

$$x'^\mu(\sigma) \partial_\mu \Lambda^Y[x] = \frac{\partial \Lambda^{a\sigma}[x]}{\partial \sigma} T_a(\sigma), \quad (2.7)$$

$$x'^\mu(\sigma) \partial_\mu \Lambda^U[x] = 0. \quad (2.8)$$

Here the prime denotes differentiation with respect to $\sigma$. Substituting (2.2) and (2.5) into (2.4) and considering the commutation relation (2.1), we obtain (the infinitesimal) gauge transformations for $A^Y_{\mu\sigma}$ and $\tilde{A}^U_{\mu\sigma}$:

$$\delta A^Y_{\mu\sigma} [x] = \partial_\mu \Lambda^Y [x] + i [A^Y_{\mu\sigma}, \Lambda^Y [x]]^Y, \quad (2.9)$$

$$\delta \tilde{A}^U_{\mu\sigma} [x] = \partial_\mu \Lambda^U [x] + i [A^Y_{\mu\sigma}, \Lambda^Y [x]]^U. \quad (2.10)$$

Here, $[ \ , \ ]^Y$ denotes the part of a commutator $[ \ , \ ]$ written as a linear combination of $T_a(\sigma)$, while $[ \ , \ ]^U$ denotes the other part including the central charge $k$. In deriving these gauge transformations, we have used the fact that $\tilde{A}^U_{\mu\sigma}$ and $\Lambda^U$ are commutative. We note that the Yang-Mills fields $A^Y_{\mu\sigma}$ appear in the gauge transformation of the $U(1)$ gauge field $\tilde{A}^U_{\mu\sigma}$. The transformation (2.10) is obviously different from an ordinary $U(1)$ gauge transformation: $\delta A^U_{\mu\sigma} = \partial_\mu \Lambda^U$. The second term on the right-hand side of (2.10) is due to the central extension. Combining the reparametrization invariant condition (2.8) and the ordinary $U(1)$ gauge transformation, we can obtain the condition $x'^\mu(\sigma) \tilde{A}^U_{\mu\sigma} = 0$. [16] In the present case, however, the $U(1)$ gauge field $\tilde{A}^U_{\mu\sigma}$ does not satisfy the condition $x'^\mu(\sigma) \tilde{A}^U_{\mu\sigma} = 0$ unless $k = 0$.

Next, we consider the (naive) field strength of the gauge field $A_{\mu\sigma}$ as

$$F_{\mu\rho,\nu\sigma} = \partial_{\mu\rho} A_{\nu\sigma} - \partial_{\nu\sigma} A_{\mu\rho} + i [A_{\mu\rho}, A_{\nu\sigma}]$$

$$= F^Y_{\mu\rho,\nu\sigma} + F^U_{\mu\rho,\nu\sigma}, \quad (2.11)$$

with

$$F^Y_{\mu\rho,\nu\sigma} \equiv \partial_{\mu\rho} A^Y_{\nu\sigma} - \partial_{\nu\sigma} A^Y_{\mu\rho} + i [A^Y_{\mu\rho}, A^Y_{\nu\sigma}]^Y, \quad (2.12)$$

$$F^U_{\mu\rho,\nu\sigma} \equiv \partial_{\mu\rho} \tilde{A}^U_{\nu\sigma} - \partial_{\nu\sigma} \tilde{A}^U_{\mu\rho} + i [A^Y_{\mu\rho}, A^Y_{\nu\sigma}]^U. \quad (2.13)$$
Note that (2.13) is different from the ordinary field strength of the $U(1)$ gauge field $\tilde{A}^U_{\mu\nu}$. The (naive) field strength (2.11) obeys the ordinary gauge transformation rule: $\delta F_{\mu\rho,\nu\sigma} = i[\mathcal{F}_{\mu\rho,\nu\sigma}, \Lambda]$. Because of the central extension, however, we can immediately find that $\mathcal{F}^Y_{\mu\rho,\nu\sigma}$ obeys the homogeneous gauge-transformation rule $\delta \mathcal{F}^Y_{\mu\rho,\nu\sigma} = i[\mathcal{F}^Y_{\mu\rho,\nu\sigma}, \Lambda^Y]$, while $\mathcal{F}^U_{\mu\rho,\nu\sigma}$ obeys the inhomogeneous gauge-transformation rule $\delta \mathcal{F}^U_{\mu\rho,\nu\sigma} = i[\mathcal{F}^Y_{\mu\rho,\nu\sigma}, \Lambda^Y]$. Therefore, (2.11) is not suitable for the field strength under the affine Lie gauge group. For this reason, we modify (2.11) as

$$\mathcal{H}_{\mu\rho,\nu\sigma} \equiv \mathcal{F}^Y_{\mu\rho,\nu\sigma} + \mathcal{H}^U_{\mu\rho,\nu\sigma}, \quad (2.14)$$

with

$$\mathcal{H}^U_{\mu\rho,\nu\sigma} \equiv \mathcal{F}^U_{\mu\rho,\nu\sigma} + k \int \frac{d\omega}{2\pi} x^\lambda(\omega) \text{Tr} [A^Y_{\mu\sigma} \mathcal{F}^Y_{\mu\rho,\nu\sigma}] . \quad (2.15)$$

Here "Tr" denotes the inner product of two elements of the affine Lie algebra $\hat{G}$: $\text{Tr}[VW] = \sum_{a,b} \int_0^{2\pi} \frac{d\sigma}{2\pi} \kappa^{ab} V^{a\sigma} W^{\sigma}$. We can confirm that $\mathcal{H}^U_{\mu\rho,\nu\sigma}$ is gauge invariant under the reparametrization invariant condition (2.7). Note that the gauge invariance is still maintained without the central extension. In order for the right-hand side of (2.15) to transform in the same manner as $\mathcal{F}^U_{\mu\rho,\nu\sigma}$ under the reparametrization, however, it is necessary to restrict $A_{\mu\sigma}^{a\rho}$ in (2.4) to the form

$$A_{\mu\sigma}^{a\rho}[x] = \delta(\rho - \sigma) A_{\mu}^{a\rho}[x] . \quad (2.16)$$

Here $A_{\mu}^{a\rho}$ are the fields on $\Omega M^D$ that behave as vector functionals on $M^D$.

The action for $A_{\mu\sigma}$ is defined as

$$S_R = \frac{1}{V_R} \int [dx] \left( \mathcal{L}^Y + \mathcal{L}^U \right) \exp \left( -\frac{L}{l^2} \right) , \quad (2.17)$$

with $V_R \equiv f\{dx\} \exp(-L/l^2)$, and the Lagrangians

$$\mathcal{L}^Y = \frac{1}{4} N^Y G^{\kappa\rho,\lambda\sigma} G^{\mu\chi,\nu\omega} \text{Tr} [\mathcal{F}^{Y}_{\kappa\rho,\mu\chi} \mathcal{F}^{Y}_{\lambda\sigma,\nu\omega}] , \quad (2.18)$$

$$\mathcal{L}^U = -\frac{1}{4} N^U G^{\kappa\rho,\lambda\sigma} G^{\mu\chi,\nu\omega} \mathcal{H}^{U}_{\kappa\rho,\mu\chi} \mathcal{H}^{U}_{\lambda\sigma,\nu\omega} , \quad (2.19)$$

where $N^Y$ and $N^U$ are arbitrary constants.

Here, the measures $[dx]$ and $\{dx\}$ are given by $[dx] \equiv \Pi_{\mu=0}^{D-1} \Pi_{n=-\infty}^{\infty} dx^{\mu n}$ and $\{dx\} \equiv \Pi_{\mu=0}^{D-1} \Pi_{n=-\infty}^{n_0} dx^{\mu n}$, where the $x^{\mu n}$ are the coefficients of the Fourier expansion $x^{\mu}(\sigma) = \sum_{n=-\infty}^{\infty} x^{\mu n} e^{i\sigma n}$. These measures are invariant under a reparametrization. [16] The (inverse) metric tensor $G^{\mu\rho,\nu\sigma}$ on $\Omega M^D$ is defined by $G^{\mu\rho,\nu\sigma} \equiv \eta^{\mu\sigma} \delta(\rho - \sigma)$, where $\eta_{\mu\nu}$ is the metric tensor on $M^D$. 3) The

\[3) \text{diag} \eta_{\mu\nu} = (1, -1, -1, \ldots, -1) .\]
damping factor \( \exp(-L/l^2) \) with \( L \equiv -\int_0^{2\pi} \frac{d\sigma}{2\pi} \eta_{\mu\nu} x^\mu(\sigma)x^\nu(\sigma) \) is inserted into the action so that it becomes well defined, where \( l (> 0) \) is a constant with the dimension of length giving the size of loops.

We would like to focus attention on the fact that there is coupling between the Yang-Mills fields \( A_{\mu\sigma}^Y \) and the \( U(1) \) gauge field \( A_{\mu\sigma}^U \) in the Lagrangian (2.19). It is obvious that the coupling is due to the central extension of the gauge group.

The Lagrangians \( L^Y \) and \( L^U \) are gauge invariant, while they are not reparametrization invariant, due to the definition of the inner product “Tr” and the metric tensor \( G^{\mu\rho,\nu\sigma} \). If necessary, we can indeed define an inner product and metric tensor to maintain reparametrization invariance as well as gauge-invariance. [14] The metric \( \eta^{\mu\nu}\delta(\rho - \sigma) \) and the inner product “Tr” that we employ can be shown in a concrete calculation to be forms of the reparametrization invariant inner product and metric tensor in a certain gauge of reparametrization.

3 The Chapline-Manton coupling

In this section, we derive a local field theory with a coupling between local Yang-Mills fields and an abelian antisymmetric tensor field of second rank on the basis of the Yang-Mills theory in loop space. Let us consider the simplest solutions of (2.7) consisting of local functions on \( M^D \),

\[
\Lambda^{Y(0)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} g_0 \Lambda^a(x(\sigma)) T_a(\sigma) ,
\]

(3.1)

where \( \Lambda^a \) is an infinitesimal scalar function on \( M^D \) and \( g_0 \) is a constant of dimension \( [\text{length}]^\frac{D-4}{2} \). On the other hand, the simplest solution of (2.8) consisting of a local function on \( M^D \) is given by

\[
\Lambda^{U(0)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} q_0 x^\mu(\sigma) \lambda_\mu(x(\sigma)) ,
\]

(3.2)

where \( \lambda_\mu \) is a vector function on \( M^D \) and \( q_0 \) is a constant of dimension \( [\text{length}]^\frac{D-6}{2} \).

Corresponding to (3.1) and (3.2), we consider the (restricted) Yang-Mills field \( A_{\mu\sigma}^Y \) and the \( U(1) \) gauge field \( A_{\mu\sigma}^U \) written in terms of local fields as

\[
A_{\mu\sigma}^{Y(0)}[x] = g_0 A_{\mu}^a(x(\sigma)) T_a(\sigma) ,
\]

(3.3)

\[
A_{\mu\sigma}^{U(0)}[x] = q_0 x^\mu(\sigma) \left\{ B_{\mu\nu}(x(\sigma)) + C_{\mu\nu}(x(\sigma)) \right\} ,
\]

(3.4)

respectively, where the \( A_{\mu}^a(x) \) are vector fields on \( M^D \), and the \( B_{\mu\nu}(x) \) and \( C_{\mu\nu}(x) \) are antisymmetric and symmetric tensor fields of second rank on \( M^D \). Obviously, the right-hand sides of (3.3) and (3.4) transform in the same manner as the left-hand sides of (3.3) and (3.4) under
the reparametrization. Substituting (3.1) and (3.3) into (2.9), we obtain the transformation rule of Yang-Mills fields for $A_{\mu}^a$ as

$$\delta A_{\mu}^a(x) = D_{\mu}A^a(x) \equiv \partial_{\mu}A^a(x) - q_0 A_{\mu}^b(x)A^c(x)f_{bc}^a, \quad (3.5)$$

by virtue of (2.1). On the other hand, substitution of (3.1), (3.2), (3.3) and (3.4) into (2.10) yields the transformation rules of $B_{\mu\nu}$ and $C_{\mu\nu}$ as

$$\delta B_{\mu\nu}(x) = \partial_{[\mu}A_{\nu]}(x) - \tilde{k}_0 A_{[\mu}^a(x)\partial_{\nu]}\lambda_a(x), \quad (3.6)$$

$$\delta C_{\mu\nu}(x) = -\tilde{k}_0 A_{[\mu}^a(x)\partial_{\nu]}\lambda_a(x), \quad (3.7)$$

where the lowering of the index $a$ has been carried out with $\kappa_{ab}$. Here, $\tilde{k}_0 \equiv kq_0/2q_0$ is a constant of dimension [length]. Owing to the central extension, the local Yang-Mills fields $A_{\mu}^a$ appear in the transformation rules of $B_{\mu\nu}$ and $C_{\mu\nu}$. However, the transformation rules of $C_{\mu\nu}$ and $A_{\mu}^a$ are not independent. Indeed, the following symmetric tensor field of second rank is invariant under the transformation rules (3.5) and (3.7):

$$\tilde{C}_{\mu\nu}(x) \equiv C_{\mu\nu}(x) + \frac{1}{2}\tilde{k}_0 A_{[\mu}^a(x)A_{\nu]}a(x). \quad (3.8)$$

Substituting $C_{\mu\nu} = \tilde{C}_{\mu\nu} - \frac{1}{2}\tilde{k}_0 A_{[\mu}^aA_{\nu]}a$ into (3.4), we obtain $\tilde{A}_{\mu\sigma}^{U(0)}$ written in terms of the local fields $A_{\mu}^a$, $B_{\mu\nu}$ and $\tilde{C}_{\mu\nu}$. As we shall see, the couplings of the local fields are uniquely determined in the Yang-Mills theory in loop space. On examination of the couplings of these local fields, however, we find that the gauge invariant tensor field $\tilde{C}_{\mu\nu}$ is free of $A_{\mu}^a$ and $B_{\mu\nu}$. [10] Since we are interested in the couplings of local fields, we omit $\tilde{C}_{\mu\nu}$ from $\tilde{A}_{\mu\sigma}^{U(0)}$, written in terms of the local fields $A_{\mu}^a$, $B_{\mu\nu}$ and $\tilde{C}_{\mu\nu}$ for simplicity. In other words, we replace $C_{\mu\nu}$ with $-\frac{1}{2}\tilde{k}_0 A_{\mu}^aA_{\nu}a$ in (3.4). Then (3.4) is rewritten as

$$\tilde{A}_{\mu\sigma}^{U(0)}[x] = A_{\mu\sigma}^{U(0)}[x] - \frac{1}{2}\tilde{k}_0 q_0 x^\nu(\sigma) A_{[\mu}^a(x(\sigma))A_{\nu]}a(x(\sigma)), \quad (3.9)$$

with

$$A_{\mu\sigma}^{U(0)}[x] = q_0 x^\nu(\sigma)B_{\mu\nu}(x(\sigma)), \quad (3.10)$$

where $A_{\mu\sigma}^{U(0)}$ is written in terms of the abelian local field only. Since the constant $\tilde{k}_0$ is proportional to the central charge $k$, the $U(1)$ gauge field $\tilde{A}_{\mu\sigma}^{U(0)}$ is reduced to $A_{\mu\sigma}^{U(0)}$ by setting $k = 0$.

Note that $\tilde{A}_{\mu\sigma}^{U(0)}$ does not satisfy the condition $x^\mu(\sigma)\tilde{A}_{\mu\sigma}^{U(0)} = 0$ for $k \neq 0$, while $A_{\mu\sigma}^{U(0)}$ satisfies the condition $x^\mu(\sigma)A_{\mu\sigma}^{U(0)} = 0$. As we have discussed in Sec. 2, this fact is consistent with the fact that the $U(1)$ gauge field $\tilde{A}_{\mu\sigma}^{U(0)}$ does not satisfy the condition $x^\mu(\sigma)\tilde{A}_{\mu\sigma}^{U(0)} = 0$ unless $k = 0$.

4) $X_{[\mu}Y_{\nu]} \equiv X_{\mu}Y_{\nu} - X_{\nu}Y_{\mu}, \quad X_{(\mu}Y_{\nu)} \equiv X_{\mu}Y_{\nu} + X_{\nu}Y_{\mu}$. 

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Next, let us consider the field strength. Substituting (3.3) into (2.12), by virtue of (2.1) we obtain $\mathcal{F}^{Y(0)}_{\mu\rho,\nu\sigma}$ written in terms of the field strength of $A^a_\mu$:

$$\mathcal{F}^{Y(0)}_{\mu\rho,\nu\sigma} = g_0 F^{a}_\mu(x(\sigma)) T_a(\sigma) \delta(\rho - \sigma) ,$$

with

$$F^{a}_\mu(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - A^b_\mu(x) A^c_\nu(x) f_{bc}^a .$$

Also the substitution of (3.3), (3.4) and (3.11) into (2.15) yields

$$H^{U(0)}_{\mu\rho,\nu\sigma} = q_0 x^\lambda(\sigma) H_{\mu\nu\lambda}(x(\sigma))$$

with

$$H_{\mu\nu\lambda}(x) = F_{\mu\nu\lambda}(x) + \tilde{k}_0 \Omega_{\mu\nu\lambda}(x) ,$$

and

$$F_{\mu\nu\lambda}(x) \equiv \partial_\mu B_{\nu\lambda}(x) + \partial_\nu B_{\lambda\mu}(x) + \partial_\lambda B_{\mu\nu}(x) ,$$

$$\Omega_{\mu\nu\lambda}(x) = A^{[\mu}_{\rho}(x) \partial_\nu A^{\lambda]}_{\sigma}(x) - \frac{g_0}{3} A^{[\mu}_\nu A^\rho_\nu A^\lambda_\sigma f_{\rho\sigma}^\lambda .$$

Here, $\Omega_{\mu\nu\lambda}$ occurring in $H_{\mu\nu\lambda}$ is a Chern-Simons 3-form. Reflecting the fact that $\tilde{F}^{U(0)}_{\mu\rho,\nu\sigma}$ is gauge invariant, $H_{\mu\nu\lambda}$ becomes also invariant under the transformation rules of (3.5) and (3.6).

Finally, let us derive the action $S_R^{(0)}$ of the local fields $A^a_\mu$ and $B_{\mu\nu}$. Substituting (3.11) and (3.13) into (2.18) and (2.19), respectively, and integrating them over $\rho, \chi$ and $\omega$, we obtain the Lagrangians $\mathcal{L}^{Y(0)}$ and $\mathcal{L}^{U(0)}$ expressed as integrals over $\sigma$:

$$\mathcal{L}^{Y(0)} = -\frac{1}{4} N^Y g_0^2 \delta(0)^2 \int_0^{2\pi} d\sigma \frac{d\sigma}{2\pi} F_{\mu\nu,a}(x(\sigma)) F^{<\mu,a}(x(\sigma)) ,$$

$$\mathcal{L}^{U(0)} = -\frac{1}{4} N^U g_0^2 \delta(0)^2 \int_0^{2\pi} d\sigma \frac{d\sigma}{2\pi} x^\nu(\sigma) x^\lambda(\sigma) H_{\kappa\mu\nu}(x(\sigma)) H^{\nu\lambda}(x(\sigma)) .$$

We next insert (3.17) and (3.18) into (2.17) and expand the functions of $x^\mu(\sigma)$, $F_{\mu\nu,a}(x(\sigma)) F^{\mu,a}(x(\sigma))$ and $H_{\mu\nu\lambda}(x(\sigma)) H^{\mu\nu\lambda}(x(\sigma))$, about $x^\mu_0$. Then, all the differential coefficients at $x^\mu_0$ in each Taylor series become total derivatives with respect to $x^\mu_0$ and vanish under the boundary conditions with $|x^\mu_0| \to \infty$. As a result, we obtain an action in which the argument $x^\mu(\sigma)$ of the functions is replaced with $x^\mu_0$.

Carrying out the integrations with respect to $x^{\mu n}$ after the Wick rotations $x^{\mu n} \to -ix^{\mu n}$ ($n \neq 0$), we obtain the action $S_R^{(0)}$ as

$$S_R^{(0)} = \int d^D x \left\{ -\frac{1}{4} F^{a}_\mu(x) F^{a}_\mu(x) + \frac{1}{12} H_{\mu\nu\lambda}(x) H^{\mu\nu\lambda}(x) \right\} .$$
Here, we set the normalization conditions as \( N^2 q_0 \delta(0) = 1 \) and \( 3N^2 q_0^2 \delta^2(0)/2 = 1 \). Thus, we obtain the action describing the system with the coupling between the local Yang-Mills fields \( A_\mu \) and the abelian antisymmetric tensor field \( B_{\mu\nu} \) via the Chern-Simons 3-form. Setting the structure constants \( f_{abc} \) to 0 in (2.1), we can obtain the action (3.19) in the abelian version. It describes the system with the coupling between the \( U(1) \) gauge field \( A_\mu \) and the abelian antisymmetric tensor field \( B_{\mu\nu} \) via the abelian Chern-Simons 3-form. Such couplings have been introduced by de Wit et al. in the abelian case and Chapline and Manton in the non-abelian case. [1] [2] They assign the transformation rule (3.6) to the antisymmetric tensor field introduced by de Wit et al. in the abelian case and Chapline and Manton in the non-abelian case. In contrast, we can naturally derive (3.6) in the framework of the gauge theory.

### 4 Application of the non-linear realization to the EYMT

In this section, we discuss the local field theories for higher rank tensor fields based on the EYMT.

Let us consider the solutions of (2.7) consisting of local functions on \( M^D \):

\[
\Lambda^{(p)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} g_p Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) \lambda_{\mu_1\mu_2\ldots\mu_p}(x(\sigma)) T_\mu(\sigma) .
\]

Here, \( Q_\mu(\sigma) \equiv x'^\mu(\sigma)/\sqrt{-x'^2(\sigma)} \) where the \( \lambda_{\mu_1\mu_2\ldots\mu_p}(x) \) are infinitesimal tensor function of rank \( p \) \( (p = 0, 1, 2, \ldots) \) on \( M^D \) and \( g_p \) is a constant of dimension \([\text{length}]^{-1} \). Similarly, the solutions of (2.8) are given by

\[
\Lambda^{(p)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} e_p Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) \kappa_{\mu_1\mu_2\ldots\mu_p}(x(\sigma)),
\]

where \( \lambda_{\mu_1\mu_2\ldots\mu_p}(x) \) and \( \kappa_{\mu_1\mu_2\ldots\mu_p}(x) \) are infinitesimal tensor functions of rank \( (p + 1) \) and \( p \) on \( M^D \), and \( e_p \) and \( e_p \) are constants of dimension \([\text{length}]^{-1} \) and \([\text{length}]^{-4} \), respectively. Setting \( p = 0 \), the infinitesimal functions (4.1) and (4.2) correspond to (3.1) and (3.2), respectively.

Any general solution of (2.7) is given as a linear combination of \( \Lambda^{(p)} \), while any general solution of (2.8) is given as a linear combination of \( \Lambda^{(p)} \). Explicitly, \( \Lambda^{(p)}[x] \) and \( \Lambda^{(p)}[x] \) can be expressed as \( \Lambda^{(p)}[x] \equiv \sum_{p=0}^{\infty} \Lambda^{(p)}[x] \) and \( \Lambda^{(p)}[x] \equiv \sum_{p=0}^{\infty} \Lambda^{(p)}[x] \). (The coefficients of the terms in these sums are absorbed into \( g_p \) and \( q_p \).) Consequently, any infinitesimal function \( \Lambda \) consisting of local functions on \( M^D \) is given in the form of a linear combination:

\[
\Lambda[x] = \Lambda^{(p)}[x] + \Lambda^{(p)}[x] \equiv \sum_{p=0}^{\infty} \Lambda^{(p)}[x] ,
\]

\[ \text{5) } x'^2(\sigma) \equiv x'_\mu(\sigma)x'^\mu(\sigma) . \]
with
\[ \Lambda^{(p)}[x] \equiv \Lambda^Y(p)[x] + \Lambda^U(p)[x] . \] (4.4)

Corresponding to (4.3), we express the gauge field \( \mathcal{A}_{\mu\sigma} \) as a linear combination of \( \mathcal{A}^{(p)}_{\mu\sigma} \) associated with \( \Lambda(p) \):
\[ \mathcal{A}_{\mu\sigma}[x] = \mathcal{A}^Y_{\mu\sigma}[x] + \mathcal{A}^U_{\mu\sigma}[x] \equiv \sum_{p=0}^{\infty} \mathcal{A}^{(p)}_{\mu\sigma}[x] , \] (4.5)
with
\[ \mathcal{A}^{(p)}_{\mu\sigma}[x] \equiv \mathcal{A}^Y_{\mu\sigma}[x] + \tilde{\mathcal{A}}^U_{\mu\sigma}[x] , \] (4.6)

where \( \mathcal{A}^{(p)}_{\mu\sigma} \) are Yang-Mills fields consisting of the local tensor fields of rank \( p \) and \( (p+1) \) on \( M^D \), and \( \tilde{\mathcal{A}}^{U(p)}_{\mu\sigma} \) are \( U(1) \) gauge fields consisting of the local tensor fields of rank \( p \), \( (p+1) \) and \( (p+2) \) on \( M^D \). (As an exceptional case, \( \mathcal{A}^{(0)}_{\mu\sigma} \) consists of local vector fields only, and \( \tilde{\mathcal{A}}^{U(0)}_{\mu\sigma}[x] \) consists of local fields of second rank only.)

Substituting (4.3) and (4.5) into (2.2), and comparing the two sides of the resulting equation, we conclude that
\[ \delta \mathcal{A}^{(p)}_{\mu\sigma} = \partial_{\mu\sigma} \Lambda^{(p)} + i \sum_{k=0}^{p} \left[ \mathcal{A}^{(k)}_{\mu\sigma}, \Lambda^{(p-k)} \right] . \] (4.7)

We note that \( \mathcal{A}^{(0)}_{\mu\sigma} \) obeys the same gauge transformation as (2.2), while the \( \mathcal{A}^{(p)}_{\mu\sigma} \) \( (p = 1, 2, 3, \ldots) \) do not obey the gauge transformation as (2.2). Indeed, the gauge transformation of \( \mathcal{A}^{(p)}_{\mu\sigma} \) \( (p = 1, 2, 3, \ldots) \) depends on other gauge fields \( \mathcal{A}^{(k)}_{\mu\sigma} \) \( [k(<p) = 0, 1, 2, 3, \ldots] \).

Next, we consider the (naive) field strength of \( \mathcal{A}^{(p)}_{\mu\sigma} \). We substitute (4.5) into (2.11). Then we can decompose \( \mathcal{F}_{\mu\rho,\nu\sigma} \) as
\[ \mathcal{F}_{\mu\rho,\nu\sigma} = \mathcal{F}^Y_{\mu\rho,\nu\sigma} + \mathcal{F}^U_{\mu\rho,\nu\sigma} = \sum_{p=0}^{\infty} \mathcal{F}^{(p)}_{\mu\rho,\nu\sigma} , \] (4.8)
with
\[ \mathcal{F}^{(p)}_{\mu\rho,\nu\sigma} \equiv \partial_{\mu\rho} \mathcal{A}^{(p)}_{\nu\sigma} - \partial_{\nu\sigma} \mathcal{A}^{(p)}_{\mu\rho} + \sum_{k=0}^{p} \left[ \mathcal{A}^{(k)}_{\mu\rho}, \mathcal{A}^{(p-k)}_{\nu\sigma} \right] . \] (4.9)

Under the gauge transformation (4.7), the \( \mathcal{F}^{(p)}_{\mu\rho,\nu\sigma} \) transform as
\[ \delta \mathcal{F}^{(p)}_{\mu\rho,\nu\sigma} = i \sum_{k=0}^{p} \left[ \mathcal{F}^{(k)}_{\mu\rho,\nu\sigma}, \Lambda^{(p-k)} \right] . \] (4.10)

From (4.10), we see that the \( \mathcal{F}^{(p)}_{\mu\rho,\nu\sigma} \) \( (p = 1, 2, 3, \ldots) \) do not obey the same transformation rule \( \delta \mathcal{F}_{\mu\rho,\nu\sigma} = i[\mathcal{F}_{\mu\rho,\nu\sigma}, \Lambda] \), except \( \mathcal{F}^{(0)}_{\mu\rho,\nu\sigma} \). Accordingly, we cannot construct the “modified”
field strengths corresponding to (2.14) for $A^{(p)}_{\mu\nu} (p = 1, 2, 3, \ldots)$ in the manner discussed in Sec. 2. To begin with, we must find the suitable field strengths of $A^{(p)}_{\mu\nu} (p = 1, 2, 3, \ldots)$ that obey the same transformation rule as $F_{\mu\nu,\sigma}$. Such field strengths can be systematically derived by using a non-linear realization method.

Let us consider the linear subspace $\hat{G}^{(p)}_k$ all of whose elements have the form

$$\Xi^{(p)}[x] = \Xi^{(p)}_Y[x] + \Xi^{(p)}_U[x]$$

with

$$\Xi^{(p)}_Y[x] = \int \frac{d\sigma}{2\pi} g_p Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) \xi_{\mu_1,\mu_2,\ldots,\mu_p}(x(\sigma)) T_a(\sigma) ,$$

$$\Xi^{(p)}_U[x] = \int \frac{d\sigma}{2\pi} g_p \sqrt{-x''(\sigma)} Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) Q^{\mu_{p+1}}(\sigma) \xi_{\mu_1,\mu_2,\ldots,\mu_p,\mu_{p+1}}(x(\sigma))$$

$$+ \int \frac{d\sigma}{2\pi} e_p Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) \xi_{\mu_1,\mu_2,\ldots,\mu_p}(x(\sigma)).$$

Here, $\xi_{\mu_1,\mu_2,\ldots,\mu_p}(x)$ and $\xi_{\mu_1,\mu_2,\ldots,\mu_p}(x)$ are arbitrary tensor functions of rank $p$ and $\xi_{\mu_1,\mu_2,\ldots,\mu_{p-1}}(x)$ is an arbitrary tensor function of rank $(p+1)$ on $M^D$. Using (2.1), we obtain a commutation relation for every $\Xi^{(p)}[x] \in \hat{G}^{(p)}_k$ and $\Xi^{(q)}[x] \in \hat{G}^{(q)}_k$ as

$$[\Xi^{(p)}, \Xi^{(q)}] \in \hat{G}^{(p+q)}_k .$$

This commutation relation shows that the linear subspace $\hat{G}^{(0)}_k$ is a Lie algebra, whereas $\hat{G}^{(p)}_k (p = 1, 2, 3, \ldots)$ is not a Lie algebra. The direct sum $\hat{G}_k \equiv \bigoplus_{p=0}^{\infty} \hat{G}^{(p)}_k$ is obviously a Lie algebra. Thus, the linear subspace $\hat{G}^{(0)}_k$ forms a subalgebra of $\hat{G}_k$. We consider the Lie groups $\hat{G}_k$ and $\hat{G}^{(0)}_k$ associated with $\hat{G}_k$ and $\hat{G}^{(0)}_k$, respectively. Here, both $\hat{G}_k$ and $\hat{G}^{(0)}_k$ are Lie subgroups of the affine Lie group $\hat{G}_k$. Since $\hat{G}^{(0)}_k$ is a subgroup of $\hat{G}_k$, we can consider the coset manifold $\hat{G}_k / \hat{G}^{(0)}_k$. We introduce the scalar field $\Phi^{(p)}[x]$ on loop space so as to parameterize the coset representative of $\hat{G}_k / \hat{G}^{(0)}_k$.

$$\mathcal{V}[\Phi] = \exp \left( -i \sum_{p=1}^{\infty} \Phi^{(p)}[x] \right) .$$

Here $\Phi^{(p)}[x]$ is an element of $\hat{G}_k^{(p)}$:

$$\Phi^{(p)}[x] = \Phi^{(p)}_Y[x] + \Phi^{(p)}_U[x] .$$

The (finite) transformation rule of $\mathcal{V}[\Phi] \rightarrow \mathcal{V}[\Phi']$ for $\mathcal{X} \in \hat{G}_k$ is given by

$$\mathcal{V}[\Phi'] \rightarrow \mathcal{V}[\Phi] = \mathcal{X} \mathcal{V}[\Phi] \mathcal{V}^{-1}[\Phi, \mathcal{X}] ,$$

where $\mathcal{X} = \exp(-i \sum_{p=0}^{\infty} \Xi^{(p)})$ and $\mathcal{V} = \exp(-i \Xi^{(0)})$. (Here, $\mathcal{V}$ depends on $\mathcal{X}$ and $\Phi$.)
Next, we define the vector field $\hat{A}_{\mu\sigma}$ by using $A_{\mu\sigma}$ and $\mathcal{V}[\Phi]$:

$$\hat{A}_{\mu\sigma} \equiv \mathcal{V}^{-1}A_{\mu\sigma}\mathcal{V} - i\mathcal{V}^{-1}\partial_{\mu\sigma}\mathcal{V}. \quad (4.18)$$

Substituting (4.5) and (4.15) into (4.18), we can express $\hat{A}_{\mu\sigma}$ as a linear combination: $\hat{A}_{\mu\sigma} = \sum_{p=0}^{\infty} \hat{A}_{\mu\sigma}^{(p)}$. The concrete forms of $\hat{A}_{\mu\sigma}^{(0)}$ and $\hat{A}_{\mu\sigma}^{(1)}$ are given by

$$\hat{A}_{\mu\sigma}^{(0)} = A_{\mu\sigma}^{(0)}, \quad (4.19)$$

$$\hat{A}_{\mu\sigma}^{(1)} = A_{\mu\sigma}^{(1)} - (\partial_{\mu\sigma} \Phi^{(1)} + i[A_{\mu\sigma}^{(0)}, \Phi^{(1)}]), \quad (4.20)$$

respectively. Under the transformation (4.17) and the finite gauge transformation $A_{\mu\sigma} \rightarrow \tilde{A}_{\mu\sigma} = \mathcal{X}A_{\mu\sigma}\mathcal{X}^{-1} - i\mathcal{X}\partial_{\mu\sigma}\mathcal{X}^{-1}$, we obtain the finite gauge transformation rule of $\hat{A}_{\mu\sigma}$:

$$\hat{A}_{\mu\sigma} \rightarrow \tilde{\hat{A}}_{\mu\sigma} = \mathcal{Y}\hat{A}_{\mu\sigma}\mathcal{Y}^{-1} - i\mathcal{Y}\partial_{\mu\sigma}\mathcal{Y}^{-1}. \quad (4.21)$$

Consequently, the (naive) field strength $\hat{F}_{\mu\rho,\nu\sigma} \equiv \partial_{\mu\rho}\hat{A}_{\nu\sigma} - \partial_{\nu\sigma}\hat{A}_{\mu\rho} + i[A_{\mu\rho}, \hat{A}_{\nu\sigma}]$ obeys the transformation rule

$$\hat{F}_{\mu\rho,\nu\sigma} \rightarrow \tilde{\hat{F}}_{\mu\rho,\nu\sigma} = \mathcal{Y}\hat{F}_{\mu\rho,\nu\sigma}\mathcal{Y}^{-1}. \quad (4.22)$$

Replacing $\Xi^{(0)}$ with the infinitesimal function $\Lambda^{(0)}$ in $\mathcal{Y}$, we obtain the infinitesimal transformation rule of $\hat{F}_{\mu\rho,\nu\sigma}$ from (4.22):

$$\delta\hat{F}_{\mu\rho,\nu\sigma} = i[\hat{F}_{\mu\rho,\nu\sigma}, \Lambda^{(0)}]. \quad (4.23)$$

As in (4.8), we express $\hat{F}_{\mu\rho,\nu\sigma}$ as a linear combination, $\hat{F}_{\mu\rho,\nu\sigma} = \sum_{p=0}^{\infty} \hat{F}_{\mu\rho,\nu\sigma}^{(p)}$, with

$$\hat{F}_{\mu\rho,\nu\sigma}^{(p)} = \hat{F}_{\mu\rho,\nu\sigma}^{(Y(p))} + \hat{F}_{\mu\rho,\nu\sigma}^{(U(p))}, \quad (4.24)$$

and

$$\hat{F}_{\mu\rho,\nu\sigma}^{(Y(p))} = \partial_{\mu\rho}\hat{A}_{\nu\sigma}^{(Y(p))} - \partial_{\nu\sigma}\hat{A}_{\mu\rho}^{(Y(p))} + i\sum_{k=0}^{p} [\hat{A}_{\mu\rho}^{(Y(k))}, \hat{A}_{\nu\sigma}^{(Y(p-k))}]^{Y}, \quad (4.25)$$

$$\hat{F}_{\mu\rho,\nu\sigma}^{(U(p))} = \partial_{\mu\rho}\hat{A}_{\nu\sigma}^{(U(p))} - \partial_{\nu\sigma}\hat{A}_{\mu\rho}^{(U(p))} + i\sum_{k=0}^{p} [\hat{A}_{\mu\rho}^{(U(k))}, \hat{A}_{\nu\sigma}^{(U(p-k))}]^{U}. \quad (4.26)$$

Here, $\hat{F}_{\mu\rho,\nu\sigma}^{(0)}$ is identical with $F_{\mu\rho,\nu\sigma}^{(0)}$. From (4.23), we see that the infinitesimal transformation rules of $\hat{F}_{\mu\rho,\nu\sigma}^{(p)}$ ($p = 0, 1, 2, \ldots$) are given by $\delta\hat{F}_{\mu\rho,\nu\sigma}^{(p)} = i[\hat{F}_{\mu\rho,\nu\sigma}^{(p)}, \Lambda^{(0)}]$. Note that every $\hat{F}_{\mu\rho,\nu\sigma}^{(p)}$ ($p = 0, 1, 2, \ldots$) obeys the same transformation rule as $F_{\mu\rho,\nu\sigma}^{(p)}$. From $\hat{F}_{\mu\rho,\nu\sigma}^{(p)}$, we can construct the modified field strengths of $A_{\mu\sigma}^{(p)}$ ($p = 1, 2, 3, \ldots$) in the manner discussed in Sec. 2.
We define the field strength of $A^{(p)}_{\mu\sigma}$ in the same way as (2.14):

$$\tilde{H}^{(p)}_{\mu\rho,\nu\sigma} = \tilde{F}^{(p)}_{\mu\rho,\nu\sigma} + \tilde{H}^{(p)}_{\mu\rho,\nu\sigma}, \quad (4.27)$$

with

$$\tilde{H}^{(p)}_{\mu\rho,\nu\sigma} \equiv \tilde{F}^{(p)}_{\mu\rho,\nu\sigma} + k \int \frac{d\omega}{2\pi} x^\lambda(\omega) \text{Tr} \left[ A^{(0)}_{\lambda\omega} \tilde{F}^{(p)}_{\mu\rho,\nu\sigma} \right], \quad (4.28)$$

where $\tilde{H}^{(p)}_{\mu\rho,\nu\sigma}$ becomes gauge-invariant owing to the condition (2.7).

Finally, we obtain the action for $A^{(p)}_{\mu\sigma}$ as

$$S^{(p)}_R = \frac{1}{V_R} \int [dx] \left( \mathcal{L}^{(p)} + \mathcal{L}^{(p)}_U \right) \exp \left( -\frac{L}{\ell^2} \right), \quad (4.29)$$

with the Lagrangians

$$\mathcal{L}^{(p)} = -\frac{1}{4} N^{(p)} Y^{\kappa\rho,\lambda\sigma} G^{\mu\chi,\nu\omega} \text{Tr} \left[ \tilde{F}^{(p)}_{\mu\rho,\nu\sigma} \tilde{F}^{(p)}_{\chi\omega,\lambda} \right], \quad (4.30)$$

$$\mathcal{L}^{(p)}_U = -\frac{1}{4} N^{(p)} U^{\kappa\rho,\lambda\sigma} G^{\mu\chi,\nu\omega} \tilde{H}^{(p)}_{\mu\rho,\nu\sigma} \tilde{H}^{(p)}_{\chi\omega,\lambda}, \quad (4.31)$$

where the same damping factor is introduced as in (2.17). Obviously, the Lagrangians $\mathcal{L}^{(p)}$ and $\mathcal{L}^{(p)}_U$ are gauge invariant. Also, the action $S^{(p=0)}_R$ is identical with the action given in Sec. 3. In this way, the suitable action for $A^{(p)}_{\mu\sigma}$ can be systematically derived using the formalism of the non-linear realization.

5 The Chapline-Manton coupling for higher rank tensor fields

Referring to the previous section, we now derive the local field theory with couplings among non-abelian tensor fields of second rank, local Yang-Mills fields and abelian tensor fields of third rank. Let us consider the next simplest solutions (2.7) and (2.8) consisting of local functions on $M^D$,

$$\Lambda^{(1)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} g_1 Q^\mu(\sigma) \lambda^a_\mu(x(\sigma)) T_a(\sigma), \quad (5.1)$$

$$\Lambda^{(1)}_U[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} q_1 \sqrt{x^2(\sigma)Q^\nu(\sigma)Q^\nu(\sigma)} \xi_{\mu\nu}(x(\sigma)) + \int_0^{2\pi} \frac{d\sigma}{2\pi} e_1 Q^{\mu(\sigma)} \kappa_\mu(x(\sigma)), \quad (5.2)$$

where $\lambda^a_\mu(x)$ and $\kappa_\mu(x)$ are infinitesimal vector functions, and $\xi_{\mu\nu}(x)$ is an infinitesimal tensor function of second rank. The constants $g_1$ and $e_1$ are of dimension $[\text{length}]^{\frac{D-1}{2}}$, and $q_1$ is of
dimension $[\text{length}]^{\frac{D-3}{2}}$. The infinitesimal functions (5.1) and (5.2) correspond to (4.1) and (4.2) with $p = 1$, respectively. In this case, however, we can rewrite (5.2) in a more simple form as

$$\Lambda^{U(1)}[\pi] = \int_{0}^{2\pi} d\sigma q_{1} \sqrt{-x'^{2}(\sigma)} Q^\mu(\sigma)Q^\nu(\sigma) \lambda_{\mu\nu}(x(\sigma))$$

(5.3)

with a redefinition of the infinitesimal tensor function: $\lambda_{\mu\nu}(x) \equiv \xi_{\mu\nu}(x) - (e_{1}/2q_{1}) \partial_{(\mu} \kappa_{\nu)}(x)$. In deriving (5.3), we carried out a partial integration over $\sigma$ for the second term on the right-hand side of (5.2). Corresponding to (5.1) and (5.3), we take the Yang-Mills field $A_{\mu\sigma}^{Y(1)}$ and the $U(1)$ gauge field $\tilde{A}_{\mu\sigma}^{U(1)}$ consisting of local fields on $M^D$ as

$$A_{\mu\sigma}^{Y(1)}[\pi] = g_{1} Q^\nu(\sigma) B_{\mu\nu}(x(\sigma)) T_{\sigma}(\sigma) - \frac{1}{2} g_{1} Q^\nu(\sigma) Q^\sigma(\sigma) C_{\nu\sigma}(x(\sigma)) T_{\sigma}(\sigma)$$

$$\tilde{A}_{\mu\sigma}^{U(1)}[\pi] = q_{1} \sqrt{-x'^{2}(\sigma)} Q^\mu(\sigma) Q^\nu(\sigma) U_{\mu\nu}(x(\sigma))$$

(5.4)

where $\Pi^\mu_{\nu}(\sigma) \equiv \delta^\mu_{\nu} + Q^\mu(\sigma)Q^\nu(\sigma)$. The differential $D_{\sigma}$ is defined as $D_{\sigma} P_{\sigma}(\sigma) \equiv dP_{\sigma}(\sigma)/d\sigma + g_{0}x'^{\mu}(\sigma) A_{\mu\nu}(x(\sigma)) f_{ab} c C_{\nu\sigma}(\sigma)$ with $P_{\sigma}(\sigma)$ an arbitrary function of $\sigma$, where the $A_{\mu\nu}(x)$ are the local Yang-Mills fields. Here, $U_{\mu\nu}(x)$ is a tensor field of third rank with the symmetric property $U_{\lambda\mu\nu} = U_{\lambda\nu\mu}$, $V_{\mu\nu\lambda}(x)$ is a totally symmetric tensor field of third rank, $B_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$ are antisymmetric tensor fields of second rank, $C_{\mu\nu}(x)$ and $C_{\mu\nu}(x)$ are symmetric tensor fields of second rank, and $\phi_{\mu\nu}(x)$ are vector fields. The constants $\tilde{g}_{1}$ and $\tilde{q}_{1}$ are of dimension $[\text{length}]^{\frac{D-3}{2}}$ and $[\text{length}]^{\frac{D-5}{2}}$, respectively.

From (4.7), the gauge-transformation rules of $A_{\mu\sigma}^{Y(1)}$ and $\tilde{A}_{\mu\sigma}^{U(1)}$ are given by

$$\delta A_{\mu\sigma}^{Y(1)} = \partial_{\mu\sigma} \Lambda^{Y(1)} + i[A_{\mu\sigma}^{Y(1)}, \Lambda^{Y(0)}]^{Y} + i[A_{\mu\sigma}^{Y(0)}, \Lambda^{Y(1)}]^{Y},$$

(5.6)

$$\delta \tilde{A}_{\mu\sigma}^{U(1)} = \partial_{\mu\sigma} \Lambda^{U(1)} + i[A_{\mu\sigma}^{U(1)}, \Lambda^{U(0)}]^{U} + i[A_{\mu\sigma}^{U(0)}, \Lambda^{U(1)}]^{U}.$$

(5.7)

Substituting (3.1), (3.3), (5.1) and (5.4) into (5.6), we obtain the infinitesimal transformation rules of the local fields $B_{\mu\nu}(x)$, $C_{\mu\nu}(x)$ and $\phi_{\mu\nu}(x)$ as

$$\delta B_{\mu\nu}(x) = D_{[\mu\lambda}] a_{\nu]}(x) - g_{0} B_{\mu\nu}(x) \lambda^{e}(x) f_{bc} a,$$

(5.8)
\[ \delta C_{\mu \nu}^a(x) = D_{(\mu \lambda)}^a(x) - g_0 C_{\mu \nu}^b(x) \lambda^c(x) f_{bc}^a, \quad (5.9) \]

\[ \delta \phi_\mu^a(x) = m_{q1} \phi_\mu^a(x) - g_0 \phi_\mu^b(x) \lambda^c(x) f_{bc}^a, \quad (5.10) \]

where \( m_{q1} \equiv g_1/g_1 \) is a constant of dimension \([\text{length}]^{-1}\) and \( D_\mu \) denotes the covariant derivative given by (3.5). We see that the local fields \( B_{\mu \nu}^a(x) \), \( C_{\mu \nu}^a(x) \) and \( \phi_\mu^a(x) \) obey non-abelian gauge transformation rules [14]. Similarly, substituting (3.1), (3.2), (3.3), (5.1), (5.3), (5.4) and (5.5) into (5.7) yields the infinitesimal transformation rules of the local fields \( U_{\mu \nu \lambda}(x) \), \( V_{\mu \nu \lambda}(x) \), \( B_{\mu \nu}(x) \), \( C_{\mu \nu}(x) \) and \( \phi_{\mu \nu}(x) \):

\[ \delta B_{\mu \nu}(x) = \frac{\tilde{k}_1}{2} \left( m_{q1} A_{\mu \nu}^a(x) \lambda_{\nu, a}(x) - \phi_{\mu}^a(x) \partial_{\nu} \lambda_a(x) \right), \quad (5.11) \]

\[ \delta C_{\mu \nu}(x) = 2m_{q1} \lambda_{\mu \nu}(x) + \frac{\tilde{k}_1}{2} \left( m_{q1} A_{\mu \nu}^a(x) \lambda_{\nu, a}(x) - \phi_{\mu}^a(x) \partial_{\nu} \lambda_a(x) \right), \quad (5.12) \]

\[ \delta U_{\mu \nu \lambda}(x) = \partial_{\mu} \lambda_{\nu \lambda}(x) - \partial_{(\nu} \lambda_{\lambda)\mu}(x) \]
\[ - \frac{\tilde{k}_1}{2} \left( A_{\mu}^a(x) \partial_{(\nu} \lambda_{\lambda)\mu}(x) + B_{\mu (\nu}^a(x) \partial_{\lambda)} \lambda_a(x) - \frac{1}{m_{g1}} \phi_{\mu}^a(x) D_{(\nu} \partial_{\lambda)} \lambda_a(x) \right), \quad (5.13) \]

\[ \delta V_{\mu \nu \lambda}(x) = -\frac{1}{6} \partial_{(\mu} \lambda_{\nu \lambda)}(x) \]
\[ + \frac{\tilde{k}_1}{12} \left( C_{(\mu \nu}^a \partial_{\lambda)} \lambda_a(x) + \frac{2}{m_{g1}} \phi_{(\mu}^a(x) D_{(\nu} \partial_{\lambda)} \lambda_a(x) \right), \quad (5.14) \]

\[ \delta \phi_{\mu \nu}(x) = m_{q1} \lambda_{\mu \nu}(x) - \frac{\tilde{k}_1}{2} m_{g1} \phi_{(\mu}^a(x) \partial_{\nu)} \lambda_a(x). \quad (5.15) \]

Here \( m_{q1} \equiv q_1/\tilde{q}_1 \) and \( \tilde{k}_1 \equiv k g_0 g_1/g_1 \) are constants of dimension \([\text{length}]^{-1}\) and \([\text{length}]\), respectively. The local fields \( U_{\mu \nu \lambda}(x) \), \( V_{\mu \nu \lambda}(x) \), \( B_{\mu \nu}(x) \), \( C_{\mu \nu}(x) \) and \( \phi_{\mu \nu}(x) \) obey abelian gauge transformation rules. As we expected, the non-abelian local fields \( A_{\mu}^a, B_{\mu \nu}^a, C_{\mu \nu}^a \) and \( \phi_{\mu}^a \) appear in the infinitesimal transformation rules of these abelian local fields.

As in the case of (3.8), we can find the gauge invariant tensor fields under the transformation rules (5.8)–(5.15) and (3.5). We obtain

\[ \tilde{B}_{\mu \nu}(x) \equiv B_{\mu \nu}(x) - \frac{\tilde{k}_1}{2} m_{g1} A_{(\mu}^a(x) \partial_{\nu)} \lambda_a(x), \quad (5.16) \]

\[ \tilde{C}_{\mu \nu}(x) \equiv C_{\mu \nu}(x) - 2 \phi_{\mu \nu}(x) - \frac{\tilde{k}_1}{2} m_{g1} A_{(\mu}^a(x) \partial_{\nu)} \lambda_a(x), \quad (5.17) \]

\[ 6) X_{(\mu \nu \lambda Z)} \equiv X_{\mu \nu \lambda } + X_{\nu \lambda } Z_{\nu} + X_{\lambda } Y_{\mu } Z_{\nu} + X_{\lambda } Y_{\nu } Z_{\mu } + X_{\nu } Y_{\lambda } Z_{\nu } + X_{\nu } Y_{\mu } Z_{\lambda } + X_{\lambda } Y_{\mu } Z_{\nu } \]
\[
\widetilde{V}_{\mu\nu\lambda}(x) \equiv V_{\mu\nu\lambda}(x) - S_{\mu\nu\lambda}(x) - \frac{\tilde{k}_1}{12} C_{(\mu\nu}^a(x)A_{\lambda)\alpha}(x),
\]
(5.18)

where \( S_{\mu\nu\lambda}(x) \) is the irreducible component of \( U_{\mu\nu\lambda}(x) \) given by \( S_{\mu\nu\lambda} \equiv (U_{\mu\nu\lambda} + U_{\nu\lambda\mu} + U_{\lambda\mu\nu})/3 \).

Eliminating \( B_{\mu\nu}, C_{\mu\nu} \) and \( V_{\mu\nu\lambda} \) from (5.5) by using (5.16), (5.17) and (5.18), we can rewrite (5.5) in terms of the abelian local fields \( \tilde{B}_{\mu\nu}, \tilde{C}_{\mu\nu}, \tilde{V}_{\mu\nu\lambda}, U_{\mu\nu\lambda}, S_{\mu\nu\lambda} \) and \( \phi_{\mu\nu} \) and the non-abelian local fields \( C_{\mu\nu}^a, \phi_{\mu}^a \) and \( A_{\mu}^a \). As in Sec. 3, we next remove the gauge invariant tensor fields \( \tilde{B}_{\mu\nu}, \tilde{C}_{\mu\nu} \) and \( \tilde{V}_{\mu\nu\lambda} \) from (5.5). Then (5.5) can be rewritten as

\[
\mathcal{A}^{U(1)}_{\mu\sigma}[x] = \mathcal{A}^{U(1)}_{\mu\sigma}[x] - \tilde{k}_1 q_1 \frac{1}{m_g} Q^\nu(\sigma)A_{\mu}^a(x(\sigma))\phi_{\nu\alpha}(x(\sigma))
\]
\[
+ \frac{\tilde{k}_1}{2} q_1 \sqrt{-x'^2(\sigma)}Q_{\mu}(\sigma)Q_{\nu}(\sigma)Q^a(\sigma)Q^c(\sigma)C_{\nu\kappa}(x(\sigma))A_{\lambda\alpha}(x(\sigma)),
\]
(5.19)

with

\[
\mathcal{A}^{U(1)}_{\mu\sigma}[x] = q_1 \sqrt{-x'^2(\sigma)}\{\delta_{\mu}^{[\nu}Q_{\kappa]}(\sigma)Q^{\zeta}(\sigma) - \Pi_{\mu}^{\zeta}(\sigma)Q_{\nu}(\sigma)Q^c(\sigma)\}A_{\nu\kappa}(x(\sigma))
\]
\[- q_1 \sqrt{-x'^2(\sigma)}\{Q_{\mu}(\sigma)Q_{\nu}(\sigma)Q^c(\sigma) + 2\Pi_{\mu}^{\nu}(\sigma)Q^c(\sigma)\}Q^{\zeta}(\sigma)\phi_{\nu\kappa}(x(\sigma)),
\]
(5.20)

where \( A_{\mu\nu\lambda}(x) \) is a tensor field of third rank defined by \( A_{\mu\nu\lambda} \equiv (U_{\mu\nu\lambda} - 3S_{\mu\nu\lambda})/2 \) and has the symmetry property \( A_{\lambda\mu\nu} = A_{\lambda\nu\mu} \). The infinitesimal transformation rule of \( A_{\mu\nu\lambda}(x) \) is given by

\[
\delta A_{\mu\nu\lambda}(x) = \partial_{\mu}\lambda_{\nu\lambda}(x)
\]
\[
+ \frac{\tilde{k}_1}{4} \left( \frac{1}{m_g} A_{(\nu}^a(x)\partial_{\kappa)}\lambda_{\mu,\alpha}(x) + \partial_{\mu}\lambda_{(\nu}^a(x)A_{\kappa)\alpha}(x) - B_{(\mu}^{(\nu}(x)\partial_{\kappa)}\lambda_{\alpha}(x)
\]
\[- \frac{1}{m_g} A_{(\mu}^{(\nu}(x)D_{\kappa)}\partial_{\lambda)}\lambda_{\alpha}(x) - \frac{1}{m_g} D_{\mu}\partial_{(\nu}\lambda_{\alpha}(x)\phi_{\kappa),\alpha}(x) \right). \]
(5.21)

The \( U(1) \) gauge field \( \mathcal{A}^{U(1)}_{\mu\sigma} \) is reduced to \( \mathcal{A}^{U(1)}_{\mu\sigma} \) by setting \( k = 0 \). As we have seen in Sec. 3, we can also check that \( \mathcal{A}^{U(1)}_{\mu\sigma} \) does not satisfy the condition \( x'^\mu(\sigma)\mathcal{A}^{U(1)}_{\mu\sigma} = 0 \) if \( k \neq 0 \), while \( \mathcal{A}^{U(1)}_{\mu\sigma} \), which is written in terms of the abelian local tensor fields, satisfies the condition \( x'^\mu(\sigma)\mathcal{A}^{U(1)}_{\mu\sigma} = 0 \). [17]

Next, let us consider the scalar field \( \Phi^{(1)} = \Phi^{Y(1)} + \Phi^{U(1)} \), where \( \Phi^{(1)} \) corresponds to (4.16) with \( p = 1 \). From (4.17), we see that \( \Phi^{Y(1)} \) and \( \Phi^{U(1)} \) obey the infinitesimal transformation rules

\[
\delta \Phi^{Y(1)} = \Lambda^{Y(1)} + i[\Phi^{Y(1)}, \Lambda^{Y(0)}]Y,
\]
(5.22)

\[
\delta \Phi^{U(1)} = \Lambda^{U(1)} + i[\Phi^{U(1)}, \Lambda^{Y(0)}]U.
\]
(5.23)
We express the scalar fields Φ(1) and Φ(0) in terms of the local fields on M. Since the infinitesimal scalar functions Λ(1), Λ(0) and Λ(0) are reparametrization invariant, Φ(1) and Φ(0) also have to be reparametrization invariant. Taking account of this, we express Φ(1) and Φ(0) in terms of the local fields φ_μ^a and φ_μν, respectively, as

\[ \Phi^{(1)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} \tilde{g}_1 Q^\mu(\sigma) \phi_\mu^a(x(\sigma)) T_a(\sigma) , \]

\[ \Phi^{(0)}[x] = \int \frac{d\sigma}{2\pi} \tilde{g}_1 \sqrt{-x^2(\sigma)} Q^\mu(\sigma) Q^\nu(\sigma) \phi_{\mu\nu}(x(\sigma)) , \]

where \( \tilde{g}_1 \) and \( \tilde{q}_1 \) are the constants appearing in (5.4) and (5.5). The transformation rules (5.22) and (5.23) lead to infinitesimal transformation rules of φ_μν and φ_μ^a. However, we find that the transformation rules of φ_μν and φ_μ are compatible with (5.10) and (5.15).

From (4.20), the vector fields \( \tilde{A}^{(1)}_{\mu\sigma} \) and \( \tilde{A}^{(0)}_{\mu\sigma} \) are given by

\[ \tilde{A}^{(1)}_{\mu\sigma} \equiv A^{(1)}_{\mu\sigma} - \partial_{\mu\sigma} \Phi^{(1)} + i [ A^{(0)}_{\mu\sigma} , \Phi^{(1)} ]^{(1)} , \]

\[ \tilde{A}^{(0)}_{\mu\sigma} \equiv A^{(0)}_{\mu\sigma} - \partial_{\mu\sigma} \Phi^{(0)} + i [ A^{(0)}_{\mu\sigma} , \Phi^{(0)} ]^{(0)} . \]

Substituting (5.4), (5.24), and (3.3) into (5.26), we obtain

\[ \tilde{A}^{(1)}_{\mu\sigma} = g_1 Q^\nu(\sigma) \tilde{B}_\mu^a(x(\sigma)) T_a(\sigma) \]

\[ - \frac{1}{2} g_1 Q_\mu(\sigma) Q^\nu(\sigma) Q_\lambda(\sigma) \tilde{C}_{\nu\lambda}^a(x(\sigma)) T_a(\sigma) , \]

with

\[ \tilde{B}_\mu^a(x) \equiv B_\mu^a(x) - \frac{1}{m_1} D_{\mu\nu} \phi_\nu^a(x) , \]

\[ \tilde{C}_\mu^a(x) \equiv C_\mu^a(x) - \frac{1}{m_1} D_{\mu\nu} \phi_\nu^a(x) . \]

Under the transformation rules (5.8), (5.9), (5.10) and (3.5), \( \tilde{B}_\mu^a \) and \( \tilde{C}_\mu^a \) transform homogeneously:

\[ \delta \tilde{B}_\mu^a(x) = -g_0 B_\mu^b(x) x^c(x) f_{bc}^a , \]

\[ \delta \tilde{C}_\mu^a(x) = -g_0 C_\mu^b(x) x^c(x) f_{bc}^a . \]

The transformation rules (5.31) and (5.32) are compatible with the gauge transformation rules \( \delta \tilde{A}^{(1)}_{\mu\sigma} = i [ \tilde{A}^{(0)}_{\mu\sigma} , \Lambda^{(0)} ]^{(1)} \). Also, substitution of (5.19), (5.24), (5.25) and (3.3) into (5.27) yields

\[ \tilde{A}^{(1)}_{\mu\sigma}[x] = q_1 \sqrt{-x^2(\sigma)} Q^\nu(\sigma) Q^\lambda(\sigma) B_{\mu\lambda}(x(\sigma)) \]

\[ - \frac{1}{2} q_1 \sqrt{-x^2(\sigma)} Q_\mu(\sigma) Q^\nu(\sigma) Q^\lambda(\sigma) Q^\psi(\sigma) C_{\nu\lambda\psi}(x(\sigma)) . \]
with
\[ B_{\mu\nu\lambda}(x) \equiv \tilde{A}_{\mu\nu\lambda}(x) - \tilde{A}_{(\nu\lambda)\mu}(x) + \frac{k_1}{2m_1} A_{\mu}^{a}(x) \partial_{(\nu\phi_{\lambda})a}(x) , \] (5.34)
\[ C_{\mu\nu\lambda}(x) \equiv \frac{1}{3} \tilde{A}_{(\mu\nu\lambda)}(x) - \frac{k_1}{6} C_{(\mu\nu}^{a}(x) A_{\lambda)a}(x) , \] (5.35)
and \( \tilde{A}_{\mu\nu\lambda}(x) \equiv A_{\mu\nu\lambda}(x) - (1/m_1) \partial_{\mu}\phi_{\nu\lambda}(x) \). Here, \( \tilde{A}_{\mu\nu\lambda} \) is a third-rank tensor field that is gauge invariant under \( k = 0 \). [17] By setting \( k = 0 \), the abelian tensor fields \( B_{\mu\nu\lambda} \) and \( C_{\mu\nu\lambda} \) are reduced to the (irreducible) components of \( \tilde{A}_{\mu\nu\lambda} \). Under the transformation rules of (5.9), (5.10), (5.15), (5.21) and (3.5), we see that \( B_{\mu\nu\lambda} \) and \( C_{\mu\nu\lambda} \) obey simple transformation rules:
\[ \delta B_{\mu\nu\kappa}(x) = -\frac{k_1}{2} \tilde{B}_{\mu}^{a}(x) \partial_{\kappa} \lambda_a(x) , \] (5.36)
\[ \delta C_{\mu\nu\kappa}(x) = -\frac{k_1}{6} \tilde{C}_{(\mu\nu}^{a}(x) \partial_{\kappa} \lambda_a(x) . \] (5.37)
These transformation rules are also compatible with the gauge transformation rule \( \delta \tilde{A}_{\mu\rho}^{Y(1)} = i [ \tilde{A}_{\mu\rho}^{Y(1)}, \Lambda^{Y(0)} ]^U \).

We next express the field strength \( \tilde{H}_{\mu\nu\rho\sigma}^{(1)} \) in terms of the local fields, where \( \tilde{H}_{\mu\nu\rho\sigma}^{(1)} \) corresponds to (4.27) with \( p = 1 \). The concrete forms of \( \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)} \) and \( \tilde{H}_{\mu\nu\rho\sigma}^{U(1)} \) are given by
\[ \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)} \equiv \partial_{\mu\nu}\tilde{A}_{\rho\sigma}^{(1)} - \partial_{\rho\sigma}\tilde{A}_{\mu\nu}^{(1)} + i [ \tilde{A}_{\mu\rho}^{(1)}, \tilde{A}_{\nu\sigma}^{(1)} ]^Y + i [ \tilde{A}_{\mu\rho}^{(1)}, \tilde{A}_{\nu\lambda}^{(0)} ]^U , \] (5.38)
\[ \tilde{H}_{\mu\nu\rho\sigma}^{U(1)} \equiv \partial_{\mu\nu}\tilde{A}_{\rho\sigma}^{U(1)} - \partial_{\rho\sigma}\tilde{A}_{\mu\nu}^{U(1)} + i [ \tilde{A}_{\mu\rho}^{U(1)}, \tilde{A}_{\nu\sigma}^{(1)} ]^U + i [ \tilde{A}_{\mu\rho}^{U(1)}, \tilde{A}_{\nu\lambda}^{(0)} ]^U \]
\[ + k \int \frac{d\omega}{2\pi} x^\lambda(\omega) \text{Tr} \left[ \lambda_{\lambda\omega} \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)} \right] . \] (5.39)
As was mentioned in Sec. 4, \( \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)} \) obeys the homogeneous gauge transformation rule \( \delta \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)} = i [ \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)}, \Lambda^{Y(0)} ]^Y \), while \( \tilde{H}_{\mu\nu\rho\sigma}^{U(1)} \) is gauge invariant: \( \delta \tilde{H}_{\mu\nu\rho\sigma}^{U(1)} = 0 \). Substituting (5.28) and (3.3) into (5.38), we obtain \( \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)} \) in terms of the local fields:
\[ \tilde{F}_{\mu\nu\rho\sigma}^{Y(1)}[x] = \frac{1}{2} g_1 \delta(\rho - \sigma) \left[ Q^\lambda(\rho) \tilde{H}_{\lambda\mu\nu}^{a}(x(\rho)) T_a(\rho) + \tilde{B}_{\mu\nu}^{a}(x(\rho)) D_{\rho} \left( \frac{T_a(\rho)}{\sqrt{-x^2(\rho)}} \right) \right. \]
\[ - \frac{1}{2} \tilde{H}_{\mu}^{a}(x(\rho)) D_{\rho} \left( Q_{\nu}(\rho) Q^\lambda(\rho) \frac{T_a(\rho)}{\sqrt{-x^2(\rho)}} \right) \]
\[ - \frac{1}{2} Q^\lambda(\rho) Q_{\mu}(\rho) \tilde{J}_{\nu}^{a}(\rho) T_a(\rho) \]
\[ - \frac{1}{2} g_1 \delta'(\rho - \sigma) \left[ \{ Q^\lambda(\rho) Q_{\mu}(\rho) \tilde{J}_{\nu}^{a}(\rho) \}^a(x(\rho)) \right] . \]
in terms of the local fields: (5.28), (5.33), (5.40) and (3.3) into (5.39). After a little tedious calculation, we obtain

\[ \tilde{H}_{\lambda\mu\nu}^a(x) \equiv D_\lambda \tilde{B}_{\mu\nu}^a(x) + D_\mu \tilde{B}_{\nu\lambda}^a(x) + D_\nu \tilde{B}_{\lambda\mu}^a(x) , \]  

(5.41)

\[ \tilde{I}_{\mu\nu}^a(x) \equiv \tilde{B}_{\mu\nu}^a(x) - \tilde{C}_{\mu\nu}^a(x) , \]  

(5.42)

\[ \tilde{J}_{\lambda\mu\nu}^a(x) \equiv D_\mu \tilde{B}_{\lambda\nu}^a(x) + D_\nu \tilde{C}_{\lambda\mu}^a(x) + D_\lambda \tilde{C}_{\mu\nu}^a(x) . \]  

(5.43)

It is obvious that \( \tilde{H}_{\lambda\mu\nu}^a(x) \), \( \tilde{I}_{\mu\nu}^a(x) \) and \( \tilde{J}_{\lambda\mu\nu}^a(x) \) transform homogeneously. We next substitute (5.28), (5.33), (5.40) and (3.3) into (5.39). After a little tedious calculation, we obtain \( \tilde{F}_{\mu\rho,\nu\sigma}^{(1)} \) in terms of the local fields:

\[ \tilde{F}_{\mu\rho,\nu\sigma}^{(1)} = -\frac{1}{4} q_1 \delta(\rho - \sigma) \times \left[ -x^2(\rho) \left\{ -2Q^\kappa(\rho)Q^\lambda(\rho)\delta_{\mu\nu}^\kappa + 2Q^\kappa(\rho)Q^\lambda(\rho)\delta_{\nu\mu}^\kappa \right. \right. 
\]
\[ + Q^\kappa(\rho)Q^\lambda(\rho)Q^\xi(\rho)Q_{[\mu}(\rho) \partial_{\nu]} \tilde{B}_{\kappa\lambda}(x(\rho)) \] 
\[ + \sqrt{-x^2(\rho)} \left\{ -Q^\kappa(\rho)Q_{[\mu}(\rho)\delta_{\nu]}^\kappa \right. \right. 
\]
\[ + \frac{3}{2} Q^\kappa(\rho)Q_{[\mu}(\rho)\delta_{\nu]}^\kappa \} \partial_\omega \tilde{C}_{\kappa\lambda\xi}(x(\rho)) \] 
\[ + 2\tilde{k}_1 \sqrt{-x^2(\rho)}Q^\kappa(\rho)Q^\lambda(\rho)\tilde{C}_{\kappa[\mu}(x(\rho))F_{\nu]\lambda,a}(x(\rho)) 
\] 
\[ + \tilde{k}_1 \sqrt{-x^2(\rho)}Q^\kappa(\rho)Q^\lambda(\rho)Q^\xi(\rho)Q_{[\mu}(\rho)\tilde{C}_{\kappa\lambda}^a(x(\rho))F_{\nu]\xi,a}(x(\rho)) 
\] 
\[ + \frac{d}{d\rho}\left\{ 2Q^\lambda(\rho)\delta_{\mu\nu}^\kappa + Q^\kappa(\rho)Q^\lambda(\rho)Q_{[\mu}(\rho) \tilde{B}_{\nu]\kappa\lambda}(x(\rho)) \right. \] 
\[ + 3 \frac{d}{d\rho}\left\{ Q^\kappa(\rho)Q^\lambda(\rho)Q_{[\mu}(\rho)\delta_{\nu]}^\kappa \right. \right. 
\]
\[ \left. \left. \tilde{C}_{\kappa\lambda\xi}(x(\rho)) \right] \right. \] 
\[ + \frac{1}{2} q_1 \delta(\rho - \sigma) \times \left[ - \left\{ 2Q^\lambda(\rho)\delta_{\mu\nu}^\kappa + Q^\kappa(\rho)Q^\lambda(\rho)Q_{[\mu}(\rho) \right. \right. 
\] 
\[ + \left\{ (3Q^\rho(\rho)Q^\rho(\rho) + \eta_{\mu\nu})Q^\kappa(\rho)Q^\lambda(\rho)Q^\xi(\rho) 
\]
\[ + \frac{3}{2} Q^\kappa(\rho)Q^\lambda(\rho)Q_{[\mu}(\rho)\delta_{\nu]}^\kappa \left\} \tilde{C}_{\kappa\lambda\xi}(x(\rho)) \right] \right] \right] \]
\[-(\text{all of the above terms with } \mu \leftrightarrow \nu \text{ and } \rho \leftrightarrow \sigma )\] 

\begin{equation}
\tilde{B}_{\mu\nu\lambda}(x) \equiv B_{\mu\nu\lambda}(x) + \frac{1}{2} \kappa_1 \tilde{B}_{\mu(\nu}{}^a(x)A_{\lambda)\lambda}(x), \quad (5.45)
\end{equation}

\begin{equation}
\tilde{C}_{\mu\nu\lambda}(x) \equiv C_{\mu\nu\lambda}(x) + \frac{1}{6} \kappa_1 \tilde{C}_{\mu(\nu}{}^a(x)A_{\lambda)\lambda}(x). \quad (5.46)
\end{equation}

Here $F_{\mu\nu}{}^a$ is the field strength of $A_{\mu}{}^a$. Obviously, $\tilde{B}_{\mu\nu\lambda}$ and $\tilde{C}_{\mu\nu\lambda}$ are invariant under the transformation rules (5.31), (5.32), (5.36), (5.37) and (3.5). Besides $\tilde{B}_{\mu\nu\lambda}$ and $\tilde{C}_{\mu\nu\lambda}$ and their derivatives, the two distinctive terms $\tilde{B}_{\mu\nu}{}^a F_{\kappa\lambda}{}^a$ and $\tilde{C}_{\mu\nu}{}^a F_{\kappa\lambda}{}^a$ occur in (5.44). These terms are also invariant under the transformation rules of (5.31), (5.32) and (3.5). Consequently, $\tilde{H}_{\mu\rho,\nu\sigma}^{U(1)}$ remains invariant under the transformation of the local fields. This invariance is compatible with the fact that $\delta \tilde{H}_{\mu\rho,\nu\sigma}^{U(1)} = 0$. Note that the terms $\tilde{B}_{\mu\nu}{}^a F_{\kappa\lambda}{}^a$ and $\tilde{C}_{\mu\nu}{}^a F_{\kappa\lambda}{}^a$ take the form of products of the field strengths $F_{\mu\nu}{}^a$ and the non-abelian tensor fields $\tilde{B}_{\mu\nu}{}^a$ or $\tilde{C}_{\mu\nu}{}^a$. These terms take the same form as the (non-abelian) BF-term, except for a totally antisymmetric property. [7] Accordingly, we refer to these terms as “BF-like terms” hereafter. We can regard the BF-like terms in (5.44) as a kind of generalization of the Chern-Simons terms $\Omega_{\mu\nu\lambda}$ in (3.13) for the non-abelian tensor fields $\tilde{B}_{\mu\nu}{}^a$ and $\tilde{C}_{\mu\nu}{}^a$.

Finally, we consider the action $S_R^{(1)}$ corresponding to (4.29) with $p = 1$. We divide $S_R^{(1)}$ into $S_Y^{(1)}$ and $S_U^{(1)}$ as

\begin{equation}
S_R^{(1)} = \frac{1}{V_R} \int \left[ dx \right] L_Y^{(1)} \exp \left( - \frac{L}{L_2} \right), \quad (5.47)
\end{equation}

\begin{equation}
S_R^{(1)} = \frac{1}{V_R} \int \left[ dx \right] L_U^{(1)} \exp \left( - \frac{L}{L_2} \right), \quad (5.48)
\end{equation}

with the Lagrangians

\begin{equation}
L_Y^{(1)} = -\frac{1}{4} N_Y G^{\kappa\rho,\lambda\sigma} G^{\mu\chi,\nu\omega} \text{Tr} \left[ \tilde{F}_Y^{(1)} \tilde{F}_Y^{(1)} \right], \quad (5.49)
\end{equation}

\begin{equation}
L_U^{(1)} = -\frac{1}{4} N_U G^{\kappa\rho,\lambda\sigma} G^{\mu\chi,\nu\omega} \tilde{H}_U^{U(1)} \tilde{H}_U^{U(1)} \tilde{H}_U^{U(1)} \tilde{H}_U^{U(1)}. \quad (5.50)
\end{equation}

From the definition of (5.47), it is obvious that $S_R^{Y(1)}$ does not include the central charge $k$. This means that the action $S_R^{Y(1)}$ is not affected by the central extension of the gauge group. In other words, the action $S_R^{Y(1)}$ is coincident with the action whose gauge group is the loop group. The action $S_R^{Y(1)}$ written in terms of the local fields is derived in Ref. 14). It describes a massive

\footnote{\comment{\small
$X_{\mu} Y_{\nu} Z_{\lambda} \equiv X_{\mu} Y_{\nu} Z_{\lambda} - X_{\mu} Y_{\nu} Z_{\mu}$, \quad $X_{(\mu \nu} Z_{\lambda)} \equiv X_{\mu} Y_{\nu} Z_{\lambda} + X_{\mu} Y_{\nu} Z_{\mu}$.
}}
tensor field theory as non-abelian St"uckelberg formalism for the tensor fields $\tilde{B}_{\mu\nu}^a$ and $\tilde{C}_{\mu\nu}^a$. The local interactions among $\tilde{B}_{\mu\nu}^a$, $\tilde{C}_{\mu\nu}^a$ and the local Yang-Mills fields $A_{\mu}^a$ are determined by the non-linear realization method developed for the loop gauge group. The central charge $k$ does not appear in these interactions.

Let us now derive the action $S_{U(1)}^R$ written in terms of the local fields. As was done in Sec. 3, substituting (5.44) into (5.50), we obtain $L^{U(1)}$ in terms of the local fields. Next, inserting the Lagrangian into (5.48), and carrying out the integrations over $x^{\mu n}$ ($n \neq 0$) for (5.48), we obtain the action $S_{U(1)}^R$ written in terms of the local fields. The concrete form of $S_{U(1)}^R$ is given by

$$S_{U(1)}^R = \int d^D x \left\{ -\frac{1}{4} \left\{ R_{\mu\gamma\xi\eta}^{\nu\zeta\kappa\lambda} \partial^{\mu} \tilde{B}^{\gamma\xi\eta}(x) \partial^{\nu} \tilde{B}^{\zeta\kappa\lambda}(x) 
+ a_1 R_{\xi\eta}^{\kappa\lambda} \left( \partial_{\kappa} \tilde{B}_{\mu\nu\lambda}(x) + \partial_{\lambda} \tilde{B}_{\mu\nu}(x) \right) 
\times \left( \partial^{\xi} \tilde{B}^{\mu\eta\nu}(x) + \partial^{\mu} \tilde{B}^{\nu\xi\eta}(x) \right) 
+ 4a_2 R_{\xi\eta}^{\mu\kappa\lambda} \partial^{\gamma} \tilde{B}^{\mu\xi\eta}(x) 
\times \left( 4\partial_{\kappa} \tilde{B}_{\nu\mu\lambda}(x) + 4\partial_{\lambda} \tilde{B}_{\nu\mu}(x) + \partial_{\mu} \tilde{B}_{\nu\kappa\lambda}(x) \right) 
+ a_3 R_{\mu\gamma\xi\eta}^{\nu\kappa\lambda} \partial^{\mu} \tilde{C}^{\gamma\xi\eta}(x) \partial^{\nu} \tilde{C}^{\zeta\kappa\lambda}(x) 
+ a_2 R_{\xi\eta}^{\kappa\lambda\zeta} \left( 2\partial_{\mu} \tilde{C}_{\kappa\lambda\zeta}(x) - 3\partial_{\kappa} \tilde{C}_{\mu\lambda\zeta}(x) \right) 
\times \left( 2\partial^{\mu} \tilde{C}^{\xi\eta\gamma}(x) - 3\partial^{\xi} \tilde{C}^{\nu\eta\gamma}(x) \right) 
- 2a_2 R_{\xi\eta}^{\mu\kappa\lambda} \left( 2\partial^{\nu} \tilde{C}^{\xi\eta\gamma}(x) - 3\partial^{\xi} \tilde{C}^{\nu\eta\gamma}(x) \right) 
\times \left( 2\partial_{\kappa} \tilde{B}_{\nu\mu\lambda}(x) + 2\partial_{\lambda} \tilde{B}_{\nu\mu}(x) + \partial_{\mu} \tilde{B}_{\nu\kappa\lambda}(x) \right) 
- 2a_3 R_{\mu\gamma\xi\eta}^{\nu\kappa\lambda} \partial^{\mu} \tilde{B}_{\zeta\kappa\lambda}(x) \partial^{\nu} \tilde{C}^{\gamma\xi\eta}(x) 
+ 2k_1 a_1 R_{\xi\eta}^{\kappa\lambda} \tilde{B}_{\mu a}(x) F^{\nu\eta a}(x) \left( \partial_{\kappa} \tilde{B}_{\mu\nu\lambda}(x) + \partial_{\lambda} \tilde{B}_{\mu\nu}(x) \right) 
- 4k_1 a_2 R_{\xi\eta}^{\mu\kappa\lambda} \tilde{C}_{\xi\eta a}(x) F^{\nu\gamma\eta a}(x) 
\times \left( 2\partial_{\kappa} \tilde{B}_{\nu\mu\lambda}(x) + 2\partial_{\lambda} \tilde{B}_{\nu\mu}(x) + \partial_{\mu} \tilde{B}_{\nu\kappa\lambda}(x) \right) 
+ 2k_1 a_2 R_{\xi\eta}^{\kappa\lambda\zeta} \tilde{C}_{\xi\eta a}(x) F^{\nu\gamma\eta a}(x) 
\times \left( 2\partial_{\mu} \tilde{C}_{\kappa\lambda\zeta}(x) - 3\partial_{\kappa} \tilde{C}_{\mu\lambda\zeta}(x) \right) 
+ k_1^2 a_1 R_{\xi\eta}^{\kappa\lambda} \tilde{B}_{\mu a}(x) F_{\nu\lambda a}(x) \tilde{B}_{\mu b}(x) F^{\nu\eta\gamma b}(x) 
+ 4k_1^2 a_2 R_{\xi\eta}^{\kappa\lambda\zeta} \tilde{C}_{\mu a}(x) F_{\zeta\lambda a}(x) \tilde{C}_{\mu b}(x) F^{\nu\gamma\eta b}(x) \right\} 
+ \frac{1}{2} m q_1 \left\{ \tilde{B}_{\mu
u\kappa}(x) \tilde{B}^{\mu
u\kappa}(x) + b_1 \tilde{B}_{\mu
u\kappa}(x) \tilde{B}^{\nu\mu\kappa}(x) 
+ b_2 \tilde{B}_{\mu\kappa}^{\kappa}(x) \tilde{B}^{\mu\lambda\lambda}(x) + b_3 \tilde{B}_{\kappa}^{\kappa\mu}(x) \tilde{B}^{\mu\lambda\lambda}(x) 
+ b_4 \tilde{B}_{\kappa}^{\kappa\mu}(x) \tilde{B}^{\mu\lambda\lambda}(x) + b_5 \tilde{C}_{\mu
u\kappa}(x) \tilde{C}^{\mu
u\kappa}(x) 
+ b_6 \tilde{C}_{\kappa}^{\kappa\mu}(x) \tilde{C}^{\mu\lambda\lambda}(x) + b_7 \tilde{B}_{\mu
u\kappa}(x) \tilde{C}^{\mu
u\kappa}(x) \right\} \right\}$$
Here, with normalization conditions

\[
+ b_8 \tilde{B}_{\mu \kappa}(x) \tilde{C}^{\mu \kappa}(x) + b_9 \tilde{B}_{\kappa \mu}(x) \tilde{C}^{\mu \lambda}(x) \right) ,
\]

(5.51)

with

\[
R_{\mu \nu}^{\kappa \lambda} = \delta_{(\mu}^{\kappa} \delta_{\nu)}^{\lambda} + \eta_{\mu \nu} \eta^{\kappa \lambda} ,
\]

(5.52a)

\[
R_{\mu \nu \xi}^{\kappa \lambda \gamma} = \delta_{(\mu}^{\kappa} \delta_{\nu}^{\lambda} \delta_{\xi)}^{\gamma} + \frac{1}{4} \eta_{(\mu \nu} \eta^{(\kappa \lambda \gamma)} \delta_{\xi)}^{\gamma} ,
\]

(5.52b)

\[
R_{\mu \nu \xi \eta}^{\kappa \lambda \gamma \zeta} = \delta_{(\mu}^{\kappa} \delta_{\nu}^{\lambda} \delta_{\xi}^{\gamma} \delta_{\eta)}^{\zeta} + \frac{1}{8} \eta_{(\mu \nu \eta} \eta^{(\kappa \lambda \gamma \zeta)} \delta_{\xi)}^{\zeta} .
\]

(5.52c)

Here \(a_i (i = 1, 2, 3)\) and \(b_i (i = 1, 2, \ldots, 9)\) are constants given by

\[
a_1 = (D + 4)(D + 6) , \quad a_2 = \frac{D + 6}{16} , \quad a_3 = \frac{1}{16} ,
\]

\[
b_1 = \frac{1}{2} J \left\{ 4K(2D - 1)(D + 3)\delta(0) - 16(D - 1)(D^2 + 4D + 1)\delta''(0) \right\} ,
\]

\[
b_2 = -\frac{1}{2} J \left\{ K(3D^2 + 22D - 41)\delta(0) + 4(D - 1)(3D + 11)\delta''(0) \right\} ,
\]

\[
b_3 = J \left\{ 4K(D + 1)^2\delta(0) + 16(D - 1)(D + 3)\delta''(0) \right\} ,
\]

\[
b_4 = -J \left\{ 12K(D + 1)\delta(0) + 16(D - 1)\delta''(0) \right\} ,
\]

\[
b_5 = -\frac{1}{2} J \left\{ 27K(D + 1)^2\delta(0) + 12(D - 1)(D + 21)\delta''(0) \right\} ,
\]

\[
b_6 = \frac{1}{4} J \left\{ 9K(D^2 - 18D - 7)\delta(0) - 36(D - 1)(D - 13)\delta''(0) \right\} ,
\]

\[
b_7 = 6J \left\{ 3K(D + 1)^2\delta(0) - 2(D - 1)(3D + 7)\delta''(0) \right\} ,
\]

\[
b_8 = -3J \left\{ K(D^2 + 10D - 3)\delta(0) + 2(D - 1)^2\delta''(0) \right\} ,
\]

\[
b_9 = 3J \left\{ 2K(D + 1)(D - 5)\delta(0) - 4(D - 1)(3D + 7)\delta''(0) \right\} ,
\]

with

\[
\nonumber K \equiv -\frac{1}{V_R} \int_0^{2\pi} dx \int_0^{2\pi} \frac{d\sigma}{2\pi} Q'_\mu(\sigma)Q''_\mu(\sigma) \exp \left( -\frac{L}{L^2} \right) (\geq 0) ,
\]

\[
J^{-1} \equiv 2K(4D^3 + 19D^2 - 20D - 15)\delta(0) - 8(D - 1)(2D^2 + 9D + 5)\delta''(0) .
\]

In deriving the action (5.51), we have set the free parameters \(k_u, q_1,\) and \(\tilde{q}_1\) so as to satisfy the normalization conditions

\[
\frac{k_u q_1^2 l^2}{4(D + 2)(D + 4)(D + 6)} \delta(0)^2 = -1 ,
\]

(5.53a)

\[
\frac{k_u \tilde{q}_1^2}{D(D - 1)(D + 2)(D + 4)} \times \left\{ 2K(4D^3 + 19D^2 - 20D - 15)\delta(0) - 8(D - 1)(2D^2 + 9D + 5)\delta''(0) \right\} = -1 .
\]

(5.53b)
The action (5.51) describes the massive tensor fields theory for $\tilde{B}_{\mu\nu\lambda}$ and $\tilde{C}_{\mu\nu\lambda}$ without spoiling the gauge invariance. This property is also possessed by Stückelberg formalism. Reflecting the non-abelian gauge theory, however, the action (5.51) includes non-trivial couplings via the $BF$-like terms $\tilde{B}_{\mu\nu}^a F_{\lambda\kappa,a}$ and $\tilde{C}_{\mu\nu}^a F_{\lambda\kappa,a}$. It is obvious that these couplings are due to the central extension of the gauge group. Indeed, all the interaction terms occurring in (5.51) include the central charge $k$. By setting $k = 0$, we find that the gauge invariant tensor fields $\tilde{B}_{\mu\nu\lambda}$ and $\tilde{C}_{\mu\nu\lambda}$ reduce to the components of $\tilde{A}_{\mu\nu\lambda}$, and all the interaction terms occurring in (5.51) vanish. Hence, (5.51) becomes the action for the massive tensor field $\tilde{A}_{\mu\nu\lambda}$ without interactions. The gauge invariance still holds, because $\tilde{A}_{\mu\nu\lambda}$ is invariant under the transformation rules of (5.15) and (5.21) with $k = 0$. Therefore, (5.51) results in the action of the Stückelberg formalism for the abelian tensor field of third rank $\tilde{A}_{\mu\nu\lambda}$. [17] Consequently, we can regard (5.51) as the action of the “generalized” Stückelberg formalism for the tensor fields of third rank $\tilde{B}_{\mu\nu\lambda}$ and $\tilde{C}_{\mu\nu\lambda}$ in a broad sense.

We next comment on the interactions in (5.51). Although the types of interactions in (5.51) are somewhat complicated, we can find some features of the interactions. First, the abelian tensor fields $\tilde{B}_{\mu\nu\lambda}$ and $\tilde{C}_{\mu\nu\lambda}$ couple with the non-abelian tensor fields $\tilde{B}_{\mu\nu}^a$ and $\tilde{C}_{\mu\nu}^a$ and the local Yang-Mills fields $A_{\mu}^a$ via the $BF$-like terms. (Here, $\tilde{C}_{\mu\nu\lambda}$ does not couple with $\tilde{B}_{\mu\nu}^a F_{\lambda\kappa,a}$.) Second, the second power of the $BF$-like terms give couplings among the non-abelian fields $\tilde{B}_{\mu\nu}^a$, $\tilde{C}_{\mu\nu}^a$ and $A_{\mu}^a$ that are obviously different from the minimal interactions resulting from the covariant derivative. [14] We would like to emphasize that these features are analogous to those of the couplings in the action (3.19). Instead of the Chern-Simons term, the $BF$-like terms contribute to the non-trivial couplings in the action (5.51). We may regard the couplings between the abelian tensor fields and the $BF$-like terms as a kind of generalization of the Chapline-Manton coupling.

6 Conclusion and discussion

In this paper, we have considered the EYMT in loop space whose gauge group is the affine Lie group. In the EYMT, central extension of the gauge group leads to a coupling between the Yang-Mills fields $A_{\mu\nu}^Y$ and the $U(1)$ gauge field $A_{\mu\sigma}^U$. The coupling is different from the minimal coupling and the coupling via the Pauli terms existing in the Standard Model. [18] The coupling yields non-trivial couplings between non-abelian local fields included in the Yang-Mills fields $A_{\mu\sigma}^Y$ and an abelian local field included in the $U(1)$ gauge field, $A_{\mu\sigma}^U$. The Chapline-Manton coupling, which was originally introduced in order to combine a supergravity and a super Yang-Mills system, can be systematically derived within the framework of the Yang-Mills theory. In the supergravity theory, the Chapline-Manton coupling is derived using the local supersymmetry. [2] It is interesting to study the relation of the central extension of the gauge
group and local supersymmetry.

By using the formalism of the non-linear realization developed for the affine Lie gauge group, furthermore, we can derive the “generalized” Chapline-Manton coupling for higher-rank tensor fields. This coupling is given by the couplings among the local Yang-Mills fields $A_{\mu}^a$, the non-abelian tensor fields of second rank $\hat{B}_{\mu\nu}^a$ and $\hat{C}_{\mu\nu}^a$, and the abelian tensor fields of third rank $\check{B}_{\mu\nu\lambda}$ and $\check{C}_{\mu\nu\lambda}$ via the BF-like terms. In the (bosonic) string theory, an abelian antisymmetric tensor field of rank appearing as massless excited states, while an abelian tensor field of third rank having the same symmetric property as $\check{A}_{\mu\nu\lambda}$ ($\check{A}_{\mu\nu\lambda} = \check{A}_{\mu\lambda\nu}$) appears as massive excited states. [19] The Chapline-Manton coupling is realized in the type I supergravity theory. This theory is obtained as the low energy effective theory of the type I (or heterotic) superstring theory. The generalized Chapline-Manton coupling including the abelian tensor field of third rank might be realized in a massive mode in string theory.

If an abelian tensor field couples with the BF term, then it must have the totally antisymmetric property, because BF terms have this property. However, both the abelian tensor fields of third rank $\check{B}_{\mu\nu\lambda}$ and $\check{C}_{\mu\nu\lambda}$ have some specific symmetric properties. For this reason, these abelian tensor fields cannot couple with the BF term. Such a difficulty as this might be settled by considering the EYMT in closed p-manifold space $\Omega^pM^D$, which is the configuration space for closed p-branes. [21] [20] A $U(1)$ gauge field $A_{\mu\sigma}^{U(0)}[x]$ on $\Omega^pM^D$ consisting of an abelian local tensor fields is given by

$$A_{\mu\sigma}^{U(0)}[x] = \frac{g_0}{p!} \Sigma^{\nu_1 \nu_2 \cdots \nu_p}(\check{\sigma}) \, B_{\mu\nu_1 \nu_2 \cdots \nu_p}(x(\check{\sigma})),$$  

where $B_{\mu\nu_1 \nu_2 \cdots \nu_p}(x)$ is an (abelian) totally antisymmetric tensor field of rank $(p + 1)$ on $M^D$ and $\Sigma^{\nu_1 \nu_2 \cdots \nu_p}(\check{\sigma}) \equiv x^{[\nu_1}_1(\check{\sigma}) x^{\nu_2}_2(\check{\sigma}) \cdots x^{\nu_p}_{p]}(\check{\sigma})$. [Here, $\check{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ represents the parameters describing a closed p-brane and $x^{\mu}_n(\check{\sigma}) \equiv \partial x^\mu_n(\check{\sigma})/\partial \sigma_n \ (1 \leq n \leq p)$] We can indeed obtain the local field theory of $B_{\mu\nu_1 \nu_2 \cdots \nu_p}(x)$ from the $U(1)$ gauge theory in closed p-manifold space. [20] In order to carry out a similar extension to the Yang-Mills theory, we have to find a suitable gauge group other than the affine Lie gauge group. It is conceivable that the suitable gauge group for the Yang-Mills theory in closed p-manifold space is the Mickelson-Faddeev group (and its generalization to higher dimensions). [11] [22] [23] The commutation relations of the generators of the (generalized) Mickelson-Faddeev group are given by

$$[T_a(\hat{\rho}), T_b(\check{\sigma})] = i f_{abc} T_c(\check{\sigma}) \delta^p(\hat{\rho} - \check{\sigma})$$

$$+ k \epsilon^{j_1 j_2 \cdots j_p} \partial_{j_1} T_{(ab) j_2 \cdots j_{p-1}}(\check{\sigma}) \partial_{j_p} \delta^p(\hat{\rho} - \check{\sigma}).$$

Setting $p = 1$, we find that (6.2) coincides with (2.1). The commutation relations (6.2) are a natural extension of (2.1). We hope to discuss this subject in the future.
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