Extreme black hole entropy obtained in an operational approach

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Abstract

The entropy of anti-de Sitter Reissner-Nordström black hole is found to be stored in the material which gathers to form it and equals to $A/4$ regardless of material states. Extending the study to two kinds of extreme black holes, we find different entropy results for the first kind of extreme black hole due to different material states. However for the second kind of extreme black hole the results of entropy are uniform independently of the material states. Relations between these results and the stability of two kinds of extreme black holes have been addressed.

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Traditionally it is widely believed that black holes have a gravitational entropy given by \( S_{BH} = A/4 \), where \( A \) is its area and units are such that \( c = G = h = k = 1 \). However until now the full understanding of the origin of this entropy is still lacking, though some possible explanations have been raised [1-3]. Recently, Pretorius, Vollick and Israel have made significant progress on this problem [4]. By examining the reversible contraction of a thin spherical shell down to the Reissner-Nordstrøm (RN) black hole event horizon, they suggested that \( S_{BH} \) is the equilibrium thermodynamic entropy that would be stored in the material which gathers to form the black hole, if one imagines all of this material compressed into a thin layer near its gravitational radius. It is of interest to extend their study to other black hole models and investigate whether their operational definition for black hole entropy is valid for other black holes. This is the first motivation of the present paper. In view of recent interest in anti-de Sitter geometries, we will extend the study of ref. [4] to an asymptotically anti-de Sitter version of RN black hole [5].

The second motivation of this paper is to extend the operational approach to extreme black hole (EBH) entropy. Recently there have been heated discussions on EBH entropies and different results obtained by using different treatments [6-10]. Starting with the original RN EBH, Hawking et al claimed that a RN EBH has zero entropy, infinite proper distance \( l \) between the horizon and any fixed point [6,7]. However, in the grand canonical ensemble, Zaslavskii argued that EBH can be obtained as the limit of nonextreme counterpart by first adopting the boundary condition \( r_+ = r_B \), where \( r_+ \) is the event horizon and \( r_B \) is boundary of the cavity, and then the extreme condition. The final extreme hole is in the topological sector of nonextreme configuration and its entropy still obeys the Bekenstein-Hawking formula [8-10]. Recently by using these two treatments, the geometry and intrinsic thermodynamics have been investigated in detail for a wide class of EBHs including 4D and two-dimensional (2D) cases [11-13]. It was found that these different treatments lead to two different topological objects represented by different Euler characteristics and show drastically different intrinsic thermodynamical properties both classically and quantum-mechanically. Based upon these results it was suggested that there maybe two kinds of EBHs in nature: the first kind suggested by Hawking et al with the extreme topology and zero entropy, which can only be formed by pair creation in the early universe; on the other hand, the second kind, suggested by Zaslavskii, has the topology of the nonextreme sector and the entropy is still described by the Bekenstein-Hawking formula, which can be
developed from its nonextreme counterpart through second order phase transition [11-13].

This speculation has been further confirmed recently in a Hamiltonian framework [14] and using the grand canonical ensemble [15] as well as canonical ensemble [16] formulation for RN anti-de Sitter black hole by finding that the Bekenstein-Hawking entropy and zero entropy emerge for extreme cases respectively. It is worth extending the operational approach [4] to investigate the EBH entropy, especially for these two kinds of EBHs, and compare the operational definitions of EBH entropy to the results available. In [4] some attempts have been made to study the EBH entropy and some ambiguous results have been obtained, however their study is only limited in the first kind of RN EBH. By extending the study to two kinds of RN EBH as well as anti-de Sitter RN EBH, we will show that ambiguous results for EBH entropy do appear for the first kind of EBH, disappearing for the second kind of EBH. Some physical understanding on this problem will be given.

The RN black hole solution of Einstein equations in free space with a negative cosmological constant $\Lambda = -\frac{3}{l^2}$ is given by [5]

$$\text{ds}^2 = -h dt^2 + h^{-1} dr^2 + r^2 d\Omega^2$$ \hspace{1cm} (1)

where

$$h = 1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{Q^2}{r^2} + \frac{r^2}{l^2}$$ \hspace{1cm} (2)

The asymptotic form of this spacetime is anti-de Sitter. There is an outer horizon located at $r = r_+$. The mass of the black hole is given by

$$m = \frac{1}{2} \left( r_+ + \frac{r_+^3}{l^2} + \frac{Q^2}{r_+} \right)$$ \hspace{1cm} (3)

In the extreme case $r_+, Q$ satisfy the relation

$$1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2} = 0.$$ \hspace{1cm} (4)

Now we consider compressing a spherical shell reversibly from an infinite radius down to the black hole event horizon. We will concentrate our attention on the NEBH case at the beginning with the left-hand-side of Eq.(4) bigger than zero. To maintain reversibility at each stage the shell must be in equilibrium with the acceleration radiation that would be measured by an observer on the shell. To ensure this equilibrium, an “adiabatic” diaphragm must be interposed between the faces. As done in [4], we picture the shell as a pair of concentric spherical plates, with inner and outer masses $M_1$ and $M_2$, separated by a massless
and thermally inert interstitial layer of negligible thickness. These two plates separate three concentric spherical regions: an inner region where \( h(r) = h_1(r) \), a very thin intermediate flat region with \( h(r) = 1 \) and an outer region where \( h(r) = h_2(r) \). The local temperature \( T_i \) of the plates is given by the expression
\[
T_i = \frac{h_i^4}{4\pi \sqrt{h_i}} \quad (i = 1, 2)
\]

Introducing the Gaussian normal coordinates near every point on the plate \( \Sigma \), the coordinates on \( \Sigma \) are \( (\tau, \theta, \phi) \), where \( \tau \) is the proper time for an observer on \( \Sigma \). Defining \( \vec{N} \) as the unit spacelike vector orthogonal to \( \Sigma \) and \( \vec{U} \) the velocity of a mass element of this surface, the orthogonal condition becomes \( \vec{N} \cdot \vec{U} = 0 \). The velocity is \( \vec{U} = \dot{t} \partial_t + \dot{r} \partial_r \) where the overdot denotes differentiation with respect to \( \tau \). We obtain \( \vec{N} = (|g_{tt}|)^{-1} \dot{r} \partial_r + |g_{tt}| \dot{t} \partial_t \). The normalization conditions are \( \vec{N} \cdot \vec{N} = 1, \vec{U} \cdot \vec{U} = -1 \). The extrinsic curvatures relative to the Gaussian normal coordinates are simply \( K_{\tau \tau} = N_{\tau \tau} = U^\nu U\nu N_{\mu \nu} \) and \( K_{xy} = N_{x y}(x, y = \theta, \phi) \).

Evaluating the jump \( \gamma^i_j \) in the extrinsic curvatures between different regions and employing Israel’s equation \[17\]
\[
\gamma^i_j - \delta^i_j \text{Tr} \gamma_{ij} = -8\pi s^i_j,
\]
we can obtain masses \( M_i \) and surface pressures \( P_i (i = 1, 2) \) for the inner and outer plates,
\[
M_i = R \xi_i (1 - \sqrt{h_i(R)})
\]
\[
16\pi P_i = \left( \frac{\xi_i h_i'(R)}{\sqrt{h_i(R)}} - \frac{2M_i}{R^2} \right)
\]
where \( R \) is the common radius of the two plates and \( \xi_i = (-1)^i \).

The shell serves merely as the working substance and the nature of the material in the shell is irrelevant, provided that the first law of thermodynamics,
\[
dS = \beta dM + \beta P dA - adN,
\]
is satisfied (we drop the index \( i \) for the moment). Above, \( \beta = 1/T, \alpha = \mu/T \) and \( \mu \) is the chemical potential. \( N \) here is introduced as the number of particles in the shell to make the representation of differential \( dS \) complete \[4\]. Using the Gibbs-Duhem relation
\[
S = \beta(M + PA) - \alpha N
\]
we have
\[
n d\alpha = \beta dP + (\sigma + P) d\beta
\]
where \( n = N/A \) and \( \sigma \) is the surface mass density which satisfies \( \sigma = \frac{M}{4\pi R^2} \). Substituting Eqs(5,7) into (10), we get

\[
nd\alpha = \sigma^2 d\left(\frac{2\pi \sqrt{h_i}}{\sigma h_i'}\right)
\]

(11)

The functions \( n \) and \( \alpha \) can be chosen arbitrarily if they satisfy (11). For simplicity, we can choose plate materials fulfilling the state equation

\[
n_i^* = \alpha_i^2, \quad \text{and}, \quad \alpha_i^* = \frac{2\pi \sqrt{h_i}}{\sigma_i h_i'}.
\]

(12)

Considering the chemical potential \( \mu_i^* = T_i \alpha_i^* \), we have

\[
\mu_i^* n_i^* = \sigma_i/2
\]

(13)

Substituting the above into Eq(9), we arrive at the entropy density \( s_i = S_i/A \) of the plates as

\[
s_i^* = \beta_i P_i + \beta_i \sigma_i - \alpha_i^* n_i^* = \beta_i P_i + \frac{\beta_i \sigma_i}{2}.
\]

(14)

Using Eq(7) for the surface pressure and the local temperature of the plate Eq(5), we find

\[
s_2^* = 1/4
\]

(15)

For the outer plate, when it reaches the black hole horizon, its entropy is one quarter of its area in Planck units.

This result is also valid choosing general functions \( n, \alpha \) which satisfy (11). The most general way of proceeding along these lines is

\[
\alpha_i = g_i(\alpha_i^*), n_i = n_i^*/g_i'(\alpha_i^*)
\]

(16)

and

\[
\frac{\mu_i n_i}{\sigma_i} = \frac{g_i}{2\alpha_i^* g_i'}
\]

(17)

The most general expression for the entropy density of the plates is

\[
s_i = \frac{1}{4} \left[ \xi_i + \frac{4\pi \sqrt{h_i}}{h_i^*}(2\sigma_i - 8\mu_i n_i) \right]
\]

(18)

When the outer plate approaches the horizon, \( h_2 \to 0 \) as \( R \to r_+ \). Therefore \( s_2 = 1/4 \) again. This means that in the anti-de Sitter RN NEBH, regardless of equations of states, the entropy of a shell made of materials approaches \( A/4 \) as the shell approaches its event horizon. This result is in agreement with [4] for RN NEBH.
Now it is of interest to extend the above discussion to EBH cases. As stated in [11-13], two kinds of EBHs emerge due to two different treatments. The first kind of EBH obtained by Hawking et al is the original EBH with zero entropy, zero Euler characteristic and arbitrary imaginary time period $\beta$ because of its peculiar topology and no conical singularity for its spacetime [6,7]. While the second kind of EBH proposed by Zaslavskii has entropy equaling $A/4$ and the same topology as that of NEBH [8-10]. We hope that using operational approach can give a deeper understanding of these two kinds of EBHs' entropies.

In [4], attempts have been given to find the RN EBH entropy by using operational approach. However in their study, the authors only consider the first kind of RN EBH case with $\beta$ arbitrary and arrive at $s_{EBH}^*(1) = 0$ and $s_{EBH}(1) = \frac{1}{2}(1 - \frac{\mu n}{\sigma})$ for different states of materials in the extremally charged spherical shell collapsing onto the hole. They concluded that entropy of RN EBH may depend on their prior history. This result is not valid for the second kind of RN EBH as we will show in the following. This kind of EBH, obtained by first taking the boundary limit and then the extreme limit in the grand canonical ensemble, has the same topology as that of NEBH, and still has a conical singularity in the spacetime. Therefore, $\beta$ cannot be arbitrary, being given by $1/T$, where $T$ is the nonzero local temperature [8]. We keep this fact in mind and let a nonextreme shell collapse to black hole horizon first and make it become extreme afterwards as we have done for obtaining the second kind of EBH. We thus find, for the simplest choice of $n, \alpha$ as (22) in [4],

$$s_{EBH}^*(2) = \beta_2 P_2$$
$$= \frac{4\pi V_2}{f_2} \left( \frac{\xi f'}{V_2} - \frac{2M_2}{R^2} \right) = 1/4,$$

where we first took the boundary limit $R \rightarrow r_+$, which leads to $V_2 = \frac{\sqrt{f_2(R)}}{0}$.

However when $\alpha, n$ fulfill the general formulas (29,30) of [4], we still have

$$s_{EBH}(2) = \beta_2 P_2 + \beta_2 \sigma_2(1 - \frac{\mu_2 n_2}{\sigma_2})$$

$$= \frac{4\pi V_2}{f_2} \left( \frac{\xi f'}{V_2} - \frac{2M_2}{R^2} \right) + \frac{4\pi V_2}{f_2} \sigma_2(1 - \frac{\mu_2 n_2}{\sigma_2}) = 1/4$$

in the boundary limit, taking $R \rightarrow r_+$ and $V_2 \rightarrow 0$.

These results indicate that unlike the results obtained for the first kind of RN EBH, the operational approach leads to universal results for the entropy of the second kind of RN EBH. These results also hold in Anti-de Sitter RN EBH.
For the first kind of Anti-de Sitter RN EBH, because of their peculiar topology, it has been shown that its imaginary time period $\beta$ is arbitrary [5,18]. Taking account of this fact, from Eq.(14) and (7), we find for the extreme shell satisfying the equation of state (12)

$$s_{EBH}^*(1) = \frac{\xi_2}{16\pi} \frac{h'_i}{\sqrt{h_i}} = \beta \frac{\xi_2}{16\pi} 2(\sqrt{h_i})' = 0 \quad (21)$$

We took $R \to r_+(h_i \to 0)$. It is the same as that in the RN first kind of EBH case. However it is worthy to point out that the entropy here decreases as the extreme shell approaches the black hole horizon, unlike the first kind of RN EBH where $s_{EBH}^*(1) = 0$ at all stages. This is because in the first kind of RN EBH, the surface pressure for the extremely charged shell material is always zero, while this does not hold for the first kind of anti-de Sitter RN EBH.

For general state satisfying (16,17),

$$s_{EBH}(1) = \frac{\beta}{4}(2\sigma_i - 8\mu_i n_i) \quad (22)$$

when the extreme shell collapse to the black hole horizon. These results again support the argument that the entropy of the first kind of EBHs may depend on their prior history [4].

Now we turn to study entropy for the second kind of anti-de Sitter RN EBH. It has been shown that it has the same topology as that of the NEBH, therefore it has a conical singularity with $\beta = 1/T$ [15,18]. $T$ here is the local temperature $T = T_H/|h(r_B)|^{1/2}$ and $T_H$ is the Hawking temperature. For the second kind of EBH $T$ is nonzero though $T_H = 0$ [18]. And in the grand canonical ensemble actually only the local temperature $T$ has physical meaning, whereas $T_H$ can always be rescaled without changing observable quantities [19]. Therefore, this kind of EBH can be achieved with no contradiction with the third law of thermodynamics. Let the nonextreme shell collapse to the black hole and make it extreme afterwards, which corresponds to the treatment of Zaslavskii by first adopting the boundary limit and then the extreme limit. We have

$$s_{EBH}^*(2) = \left\{ \frac{4\pi\sqrt{h_2}}{h'_2} \frac{\xi_2 h'_2}{16\pi \sqrt{h_2}} \right\}_{R \to r_+} |_{extr} = 1/4 \quad (23)$$

for the equation of state of the shell material given by (12).

For the shell material satisfying the general equations of (16,17)

$$s_{EBH}(2) = \left\{ \frac{4\pi\sqrt{h_2}}{h'_2} \frac{\xi_2 h'_2}{16\pi \sqrt{h_2}} + \frac{\sigma_2}{4} (2 - \frac{8\mu_2 n_2}{\sigma_2}) \right\}_{R \to r_+} |_{extr} = 1/4, \quad (24)$$
by taking $R \to r_+$ first ($h_2 \to 0$ first). Therefore the entropies of this second kind of EBH are independent of their prior history as that of the second kind of RN EBH case.

At first sight, it seems hard to believe why the operational approach leads to drastically different results for two kinds of EBHs. Especially the changable results for entropy of the first kind of EBH concerning different equations of state of collapsing materials. We know that entropy is a function of state. We have shown in our previous papers [11-13,18] that although both of these two kinds of EBHs satisfy the extreme condition, their topological properties differ drastically, so we cannot treat them as the same state. This understanding may help us to understand the different entropies for two kinds of EBHs. But how can we explain different operational definitions of the entropy for the first kind of EBH?

As an example, let us first go over the issue of stability discussed for RN black hole [20]. The heat capacity at constant electrostatic potential difference and cavity radius can be computed from $C_{\phi,r_B} = -\beta \frac{\partial S}{\partial \beta}_{\phi,r_B}$, which leads to $C = 4\pi r_B^2 x^3 (1 - x)/(3x^2 - 2x - q^2)$, where $x = r_+/r_B$, $q = e/r_B$ Eq(5.17) of [20]. For the first kind of EBH obtained by Hawking et al’s treatment (adopting extreme condition at the very beginning) $C = 4\pi r_B^2 x^3 (1 - x)/2x(x-1) < 0$. The negative sign here determine that this kind of EBH is locally unstable. In addition, it follows from Eq(5.9-5.11) of [20], that the second derivatives of the action $I$ with respect to entropy $S$ and the mean charge value $< Q >$ diverge by starting with the original EBH [for example, $\partial^2 I/\partial < Q >^2 = 4\pi(1 - x)(1 - q^2/x^2)^{-1}(1 - q^2/x)^{-1}$, $q = x$ for the original EBH]. This means that fluctuations of the charge and entropy are infinite for the first kind of EBH. Because of this instability of the first kind of EBH, different states of materials collapses onto it will of course lead to different entropy results, while for the second kind of EBH, the entropy is uniform because of its stability, as shown in [8]. The extension of the stability results to two kinds of anti-de Sitter RN EBHs is obvious.

In summary, we have extended the operational definition of RN black hole entropy [4] to an interesting anti-de Sitter RN NEBH model and found that the entropy for this NEBH is described by the Bekenstein-Hawking formula as well. Extending the operational approach to two kinds of EBHs we suggested [11-13], we arrived at different results for their entropy. For the first kind of EBH, the entropy values depends on different material equations of state which collapse onto the hole, which is in agreement with the results of [4]. However, for the second kind of EBH, uniform entropy emerges, regardless of the material equation of state. These different results can be attributed to the issues of stability for these two kinds...
of EBHs. Using the argument given in [20], it is easy to find that the first kind of EBH is unstable. Thus its entropy changes in case of different states of the material collapsing onto it. However the second kind of EBH is stable [8], and it has the same values of entropy, regardless of material states. Therefore, although two kinds of EBHs can be created, due to stability only the second kind of EBH can last for long in nature.

It is worth pointing out that all discussions for obtaining EBH in this paper are just from a theoretical viewpoint and refers to the mathematical treatment only. Physical realization for creating EBHs leads to the problem about how to satisfy the third law of thermodynamics. Although it is clear that quantum processes like evaporation, which typically involve the absorption of negative energy, can violate Nernst’s form of the third law, in all classical studies it holds [4,21-23]. The fact that the entropy of the first kind of EBH tends, as \( T \to 0 \), to an absolute zero, ensures that the strong version of the third law holds in this kind of EBH. For the second kind of EBH obtained by first letting the nonextreme shell collapse to the black hole horizon and next making it become extreme does not upset the third law as well, because the final extreme state can be obtained at nonzero temperature. Therefore no challenge to the third law arise. Besides mathematical treatments, physical processes for obtaining EBH still need further studies. This is still an open question and we shall discuss it elsewhere.

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