Skewness as a probe of non-Gaussian initial conditions

We compute the skewness of the matter distribution arising from non-linear evolution and from non-Gaussian initial perturbations. We apply our result to a very generic class of models with non-Gaussian initial conditions and we estimate analytically the ratio between the skewness due to non-linear clustering and the part due to the intrinsic non-Gaussianity of the models. We finally extend our estimates to higher moments.

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The source of the initial density fluctuations which have led to the formation of structure, observed in the Universe today is unknown. Determining its nature will certainly be of utmost importance for the fruitful relation between high energy physics and cosmology.

In models which presently attract most attention, initial density fluctuations are generated during an inflationary phase. In the simplest inflationary models, the initial fluctuations obey Gaussian statistics. If this picture is correct, the deviations from Gaussianity we observe today were induced by nonlinear gravitational instability [1,2]. However, it is also conceivable that the present deviations from Gaussianity have two components: gravitationally induced and intrinsic, coming from the initial conditions rather than nonlinear dynamics [3–6]. Here, we investigate to what extent an intrinsic component can be washed out by nonlinear dynamics and on which scales it could be either detected or constrained from above in future galaxy surveys.

We start by deriving a general expression for the so-called skewness parameter, $S_3$, including the effect of an initial non-Gaussianity, non-linear evolution and smoothing. We then estimate the normalized $N$–point cumulant, $S_N$, for a wide class of models and compare it with the result obtained in Gaussian models due to mild non-linearities.

If the galaxies trace the spatial mass distribution, galaxy surveys [18] can be used to estimate the cumulants of the mass density contrast field, given by

$$M_N(R) = \langle (\delta_R)^N(x, \eta_0) \rangle_c$$

of the smoothed density field $\delta_R(x, \eta) \equiv \int d^3x' W_R(|x - x'|) \delta(x', \eta)$, where $\delta(x, \eta)$ is the density field, $\eta$ and $\eta_0$ the conformal time and its value today, and $W_R$ is a window function (e.g. Gaussian or top–hat) of width $R$. The brackets in (1) denote an ensemble average and the subscript $c$ indicates that we deal with the connected part of the $N$–point function. For a Gaussian field, all cumulants of order $N > 2$ vanish: $M_N = 0$. $M_2$ is the variance while $M_3$ is a measure of the asymmetry of the distribution, known as skewness. We will also use the more common normalized cumulant,

$$S_N(R) = M_N(R)/(M_2(R))^{(N-1)/2}.$$  

This ratio is constant (independent of $R$) in the weakly non-linear regime [7]. To calculate the general expression for $M_3(R)$ in the weakly nonlinear regime, we follow the method developed in [2]. Expanding $\delta(x, \eta)$ in a perturbative series, $\delta_1 + \delta_2 + O(3)$ and solving the system of coupled Euler, Poisson and continuity equations at second order leads, in Fourier space, to $\Delta_1(\eta, k) = D(\eta, k)$ and

$$\Delta_2(\eta, k) = (2\pi)^{-3/2} \int d^3q J(q, k - q)D(\eta, q)D(\eta, k - q)$$

where we consider only the fastest growing modes and we use the convention

$$\Delta_N(\eta, k) = (2\pi)^{-3/2} \int \delta_N(\eta, x) e^{-ik \cdot x} d^3x.$$  

At late times where a possible source term or seed has decayed, the time and space dependence of the function $D$ can be factorized, $D(\eta, k) = D_+(\eta)z(k)$, where $D_+$ is the standard linear growing mode [1]. Perturbation theory gives [2]

$$J(k, q) = \frac{2}{3} \left( 1 + \kappa \right) + (q/k)P_1(\mu) + \frac{2}{3} \left( \frac{2}{3} - \kappa \right)P_2(\mu),$$

where $P_\ell$ is the Legendre polynomial of order $\ell$, $\mu \equiv k \cdot q/kq$. The quantity $\kappa$ is a weak function of $\Omega$; for $\Omega > 0.01$, $\kappa \approx (3/14)\Omega^{-0.03}$ [8]. The smoothing applies order by order. In Fourier space, we have

$$\Delta_3(\eta, k) = D(\eta, k) W_k \Delta_3, W_k being the Fourier transform of the window function. To fifth order, the skewness is

$$M_3 = \langle \delta_{R,1}^3 \rangle + \frac{9}{4} \langle \delta_{R,1}^2 \delta_{R,2} \rangle + O(5).$$

We introduce the two–three– and four–point power spectra as (12) \equiv \mathcal{P}_2(k_1, k_2),

$$\langle 123 \rangle \equiv \mathcal{P}_3(k_1, k_2, k_3),$$

$$\langle 1234 \rangle \equiv \mathcal{P}_4(k_1, k_2, k_3, k_4).$$

(The Dirac $\delta$ is a simple consequence of statistical homogeneity which we assume throughout.) Here $\langle 12 \ldots N \rangle \equiv \langle D(\eta, k_1)D(\eta, k_2) \ldots D(\eta, k_N) \rangle$. The functions $\mathcal{P}_2$ and
inhomogeneous energy density and pressure of the seeds. of the cosmic fluid.
d with vanishing potential, \( \delta \) is given by the components of the energy momentum tensor which are quadratic in the field. Numerical calculations, however, have shown that this non-linearity is purely due to the statistical homogeneity.

For a Gaussian field, \( P_4 = P_3 = 0 \) and the only non-vanishing contribution comes from the second term in the above expression. For a top hat window, this term gives \( M_2 = (34/7 - \gamma) M_2^2 \) with \( \gamma = -d \log M_2(R)/d \log R \). Note also that \( \gamma(R) \) is the logarithmic slope of the two-point correlation function of the density fluctuations - the Fourier transform of \( P_2(k) \). It is usually assumed that \( \gamma > 0 \) (condition of hierarchical clustering, see e.g. [1]).

The class of models we want to analyze are those where the cosmic matter radiation fluid evolves according to [9,10]

\[
\dot{\delta} + \left( 1 - 6w + 3c_s^2 \right) \ddot{\delta} + k^2 c_s^2 H^2 \delta = S(k, \eta), \tag{6}
\]

with \( S \equiv (1 + w)4\pi G(f_p + 3f_F) \), \( f_p \) and \( f_F \) being the inhomogeneous energy density and pressure of the seeds. When the seed is a scalar field \( \phi \) with vanishing potential, \( f_p + 3f_F = \dot{\phi}^2 \), \( G \) is Newton’s constant, \( a \) denotes the cosmic scale factor, \( \dot{a} \) refers to the derivative with respect to conformal time, \( H \equiv \dot{a}/a \) and \( c_s^2 \equiv P/\rho \) are respectively the enthalpy and the adiabatic sound speed of the cosmic fluid.

Equation (6) can be solved by a Green’s function, \( G \),

\[
D(k, \eta) = \int_{\eta_1}^{\eta} G(k, \eta, \eta') S(k, \eta') d\eta', \tag{7}
\]

where \( \eta_1 \) is some early initial time deep in the radiation era. For the linear part of the reduced \( N \)-point function we then obtain

\[
(D(k_1, \eta_1) \cdots D(k_N, \eta_N))_c = \int_{\eta_1}^{\eta} d\eta_1 \cdots d\eta_N \rho(k_1, \eta_1) \cdots \rho(k_N, \eta_N) \langle S(k_1, \eta_1) \cdots S(k_N, \eta_N) \rangle_c. \tag{8}
\]

We define the connected \( N \)-point function of the source by

\[
\langle S(1) \cdots S(N) \rangle_c \equiv F_N(k_1, \cdots; k_N; \eta_1, \cdots, \eta_N) \delta \left( \sum_i k_i \right),
\]

where \( \langle \rangle \equiv \langle k_i, \eta_i \rangle \). Again, the \( \delta \) function of the sum of all momenta is a consequence of the statistical homogeneity.

We now assume that the reduced \( N \)-point function of the source can be replaced by its ‘perfectly coherent approximation’ given by

\[
F_N(k_1, \cdots; k_{N-1}; \eta_1, \cdots, \eta_{N}) \simeq \text{sign}(F_N) \times \frac{1}{\sqrt{|F_N(k_1, \cdots; k_{N-1}; \eta_1, \cdots, \eta_{N})|}} \tag{9}
\]

(here and below, \( k_N \) is always given by \( k_N = -k_1 - \cdots - k_{N-1} \)). This approximation is exact if the evolution equation for \( S \) is linear and the randomness is entirely due to initial conditions. Then the source term is of the form \( S(k, \eta) = R(k) s(\eta) \), where only \( R \) is a random variable and \( s \) is a deterministic solution to the linear evolution equation of \( S \) which can be taken out of the average \( \langle \rangle \). This is the key property which renders the \( N \)-point function decoherent. Then \( F_N \) can be written as

\[
F_N(k_1, \cdots; k_{N-1}; \eta_1, \cdots, \eta_{N}) \simeq s(1) \cdots s(N) \langle R(k_1) \cdots R(k_{N}) \rangle_c \tag{10}
\]

which is clearly of the form (9).

An important example are models with no sources but with non-Gaussian initial conditions for \( D \). Such models, like e.g. the recent \( \chi^2 \) Peebles model [11], are always perfectly coherent and therefore included in our analysis: In this case \( D(k, \eta) = R(k) d(k, \eta) \), where \( R \) is a non-Gaussian random variable given by the initial condition and \( d \) is a deterministic homogeneous solution of Eq. (6). Clearly, if we choose \( S(k, \eta) = R(k) \delta(\eta - \eta_n) \) and \( G(k, \eta, \eta') = d(k, \eta) \), \( D \) is of the form (7). Therefore, models where the non-Gaussianity is purely due to initial conditions are always perfectly coherent. As the equation of motion for \( D \) is second order, the homogeneous solution has in principle two modes, \( D = R_1(k) d_1(k, \eta) + R_2(k) d_2(k, \eta) \), but since we shall evaluate the \( N \)-point functions deeply in the matter era, the decaying mode will have disappeared and may thus be neglected in our analysis.

Models where the source term is due to a scalar field which evolves linearly in time are not perfectly coherent, since \( S \) is given by the components of the energy momentum tensor which are quadratic in the fields. Numerical calculations, however, have shown that this non-linearity is not severe and perfect coherence is a relatively good approximation [12,13]. One example of this kind are axion seeds in pre-big bang cosmology [14–16] for which decoherence has been tested and is found to be on the level of less than 5% for the CMB power spectrum. In Fig. 1 the functions \( D_2(k, \eta) \) and \( D_3(k, k, \eta) \) as obtained...
by a full numerical calculation are compared to their coherent approximation (9) for the large-N limit of global $O(N)$ symmetric scalar fields. This is another example where the scalar field evolution is linear and the only non-linearity in the source term is due to the energy momentum tensor being quadratic in the field [17,13,12].

For topological defects, especially for cosmic strings, the perfectly coherent approximation misses several important features (like the ‘smearing out’ of secondary acoustic peaks). However, we believe that our generic scaling result holds also in this case, as is indicated by numerical simulations of global texture: even though global texture show considerable decoherence [12], the same scaling law for higher moments which we derive here has been discovered numerically [5].

Under the perfectly coherent approximation Eq. (8) can be factorized as the product of the $N$ solutions, $D_{Nj}(\mathbf{k}_1,\ldots,\mathbf{k}_{N-1},\eta)$ of the equations (6) with source term $[F_N(\mathbf{k}_1,\ldots,\mathbf{k}_{N-1};\eta,\ldots,\eta)]^{1/N}$, where $\mathbf{k}_j$ is the wave number $\mathbf{k}$ appearing in the term $c_j^2k^2$ on the left hand side of (6) and the other wave numbers have to be considered like parameters of the source term,

$$
(D(\mathbf{k}_1,\eta)\cdots D(\mathbf{k}_N,\eta))_{c} \simeq \prod_{j=1}^{N} \delta(\sum \mathbf{k}_i)
$$

$$
\equiv \mathcal{P}_N(\mathbf{k}_1,\ldots,\mathbf{k}_{N-1},\eta)\delta(\sum \mathbf{k}_i). \quad (11)
$$

To continue, we assume that $F_N$ is a simple power law in the $k_i$ on super-Hubble scales and that it decays after Hubble crossing. This behavior is certainly correct for all examples discussed in the literature so far. We can then make the following ansatz

$$
F_N \simeq \begin{cases} 
\prod_{n=1}^{N} \frac{k_i^n}{k_0^n} (f(\eta)\eta)^N\eta^{-3} & \text{if } k_i\eta \leq 1, \forall i \in \{1,\ldots,N\} \\
0 & \text{otherwise.}
\end{cases} \quad (12)
$$

Here $f$ is a dimensionless function and $k_0$ is an arbitrary scale. For scale invariant seeds (e.g. topological defects) $f$ is just a constant and $\alpha = 0$. For axion seeds generated during a pre-big bang phase, $\alpha$ depends on the spectral index of the axion field, which in turn is determined by the evolution law of the extra dimension [15]. For the Peebles model $\alpha$ is given by the power spectrum of the scalar field $\phi$ and $f$ is a delta-function. Since $F_N$ is symmetric in the variables $\mathbf{k}_i$ we can order them such that $k_1 \geq k_2 \geq \cdots \geq k_N$.

Let us discuss the temporal behavior of the variables $D_{Nj}$. As long as $k_1\eta < 1$, the term $c_j^2k^2D$ can be neglected in Eq. (6) and the Green’s function is a power law. At $k_1\eta \sim 1$ the source term decays and as long as a perturbation remains super horizon, it just grows like $\eta^2$, so that for $k_j\eta < 1 < k_1\eta$, $D_{Nj} \approx g(1/k_1)k_1^{-2+3/N} (\eta k_1)^2 \prod_{n=1}^{N} (k_n/k_0)^\alpha$

where

$$g(\eta) = \frac{4\pi G}{\eta^2-3/N} \int_{\eta_n}^{\eta} G(\eta,\eta')f(\eta')\eta'^{(2-3/N)} d\eta',$$

and we have to take the part of the integral above which converges when $\eta_n \to 0$.

Once the perturbation enters the horizon it either starts oscillating with roughly constant amplitude or continues to grow $\propto \eta^2$, depending on whether $k_1$ enters during the radiation or matter dominated era. At late time, $\eta > \eta_{eq}$ and $k_1 > 1$, we therefore obtain

$$D_{Nj} \approx g(1/k_1)k_1^{-2+3/N} (k_i/k_0)^2 \prod_{n=1}^{N} (k_n/k_0)^\alpha \left\{ \begin{array}{ll}
\left( \frac{\eta_n}{\eta_{eq}} \right)^2 & \text{if } k_j\eta_{eq} > 1 \\
\left( \frac{\eta_{eq}}{\eta_j} \right)^2 & \text{if } k_j\eta_{eq} < 1
\end{array} \right\}$$
we can express
\[ P_1 \] for all \( j \) where
\[ \eta_{eq} \]
are the input of the skewness \((5)\).

For the class of models considered and under the assumption of perfect coherence, we have determined the connected \( N \)-point power spectra in the linear regime which are the input of the skewness \((5)\).

\( M_1 \) has two contributions: A linear one due to the initial non-Gaussianity (contained in \( P_3 \)) and one due to non-linear clustering which induces skewness even in an originally Gaussian distribution of perturbations; it contains a Gaussian part \((P_2^G)\) and a non-Gaussian term \((P_4)\). We decompose the skewness as
\[ M_3 = M_3^{(L)} + M_3^{(NL)} \]

We want to estimate the ratio of these two contributions. Under our approximation \((14)\), the first term of \((5)\) reduces to
\[ M_3^{(L)} = \int \frac{d^3k\,d^3k'}{(2\pi)^6} W_k W_q W_{|k+k'|} \sqrt{P_2(k)P_2(q)P_2(|k+q|)} \]
\[ k_{\text{max}}^{-3/2} \left[ \frac{g(1/k_{\text{max}}^3 g(|k+q|)^{3/2}}{g(1/k)g(1/q)g(1/|k+q|)k_{\text{max}}^{3/2}} \right], \quad (15) \]

where \( k_{\text{max}} = \max\{k, q, |k+q|\} \). \( M_3^{(NL)} \) is given by the second and third terms in \((5)\).

To estimate analytically the ratio \( M_3^{(L)}/M_3^{(NL)} = S_3^{(L)}/S_3^{(NL)} \), we make the following approximations:

- We assume that \( P_2 \) is a simple power law within the range of scales of interest, namely all the modes which enter the horizon during the radiation era, this is \( 0.1h^{-1}\text{Mpc} \lesssim 2\pi/k \lesssim 20h^{-1}\text{Mpc} \), name. \( P_2(k) = k^{-3}(k/k_{\text{eq}})^\gamma \).
- We also assume that \( g(\eta) \propto \eta^\gamma \).
- We replace the window function by a simple cut-off at \( k = 1/R \).
- For symmetry reasons we may integrate over the triangle \( q \leq k \leq R \) and then multiply the result by 2.

- Since in our integration region, \( q \leq k \), we replace \(|k+q| \) by \( k \).

With these approximations the angular dependence of the integrand disappears and the integrals over \( k \) and \( q \) in \((5)\) can be trivially performed leading to
\[ M_3^{(L)}(R) \approx \frac{4(k_\ast R)^{-3\gamma/2}}{(2\pi)^3(3+\gamma/2+r)} \]
for \( \gamma > 0 \) and \( 3+\gamma/2+r > 0 \)
\[ M_3^{(NL)}(R) \approx \frac{(k_\ast R)^{-2\gamma}}{(2\pi)^{4\gamma/2}} \text{ for } \gamma > 0, \]
where we have just considered the Gaussian contribution, \( P_2^G \) to \( M_3^{(NL)} \).

Since \( k_\ast \) is just the scale beyond which the density contrast \( (D(x)^2)_{R=1/k} \sim P_2(k)k^3 \) is larger than unity and non-linearities become important, we define the non-linearity scale \( R_{\text{lin}} = 1/k_\ast \). The ratio between the skewness due to the non-Gaussianity in the linear perturbation and the one due to dynamical non-linearities is then
\[ \frac{S_3^{(L)}}{S_3^{(NL)}} \approx \frac{4\gamma}{3(3+\gamma/2-r)} \left( \frac{R}{R_{\text{lin}}} \right)^{\gamma/2}. \quad (17) \]

This is our main result. It is readily checked that the non-Gaussian contribution, \( P_4 \), to \( M_3^{(NL)} \) behaves just like the contribution \( M_3^{(NL)} \) and thus only modifies the pre-factor in \((17)\), which should not be taken too seriously in view of the relatively crude approximations which we have employed to obtain our result.

This computation of the skewness is easily generalized to higher moments. As our computation shows, linear non-Gaussianities scale like
\[ M_N^{(L)}(R) \propto (R/R_{\text{lin}})^{-N\gamma/2}. \quad (18) \]

The dominant non-linear contribution to the connected \( N \)-point function which is also present in Gaussian theories contains \( N - 2 \) second order terms \( D_2 \) \[7\] and therefore scales like
\[ M_N^{(NL,\text{Gauss})}(R) \propto (R/R_{\text{lin}})^{-(N-1)\gamma}. \quad (19) \]

The lowest order non-linearity for a generic non-Gaussian model, however just comes from the non-Gaussian term with \( N + 1 \) factors of \( D \). The non-Gaussian non-linear corrections therefore generically scale like
\[ M_N^{(NL,\text{noGauss})}(R) \propto (R/R_{\text{lin}})^{-(N+1)\gamma/2}. \quad (20) \]

Only for \( N = 3 \) the two terms \((19)\) and \((20)\) scale in the same way. For all higher \( N \)'s the non-Gaussian contribution dominates in the mildly non-linear regime, \( R \gtrsim R_{\text{lin}} \). From Eq. \((20)\) we infer that in on large scales the ratios for all reduced \( N \)-point functions very generically scale like
\[ \frac{S_N^{(L)}(R)}{S_N^{(N)}(R)} \propto \left( \frac{R}{R_{\text{lin}}} \right)^{\gamma/2}. \]  

(21)

This expression agrees with other analytic predictions [3] as well as numerical simulations in a global texture model [5]. The agreement with the texture simulations which are decoherent suggests that the validity of our result extends beyond the conditions under which Eq. (21) was derived. More important than decoherence is that the source term decays at late times and therefore the density perturbations just evolve according to the homogeneous solution. This implies that at late times the \(N\)-point functions behave like the homogeneous growing mode to the \(N\)th power, while the reduced \(N\)-point function induced by non-linear clustering from Gaussian perturbations scales like the growing mode to the \(2(N-1)\)th power. Since topological defect sources decay on sub-horizon scales, we conclude that the derived scaling behavior is also valid for this argument (this will be expanded in our follow up publication [9]).

Our result implies that on small scales (\(R \lesssim R_{\text{lin}}\)), the dominant contribution to the cumulants comes from non-linear Newtonian gravitational clustering, and the Gaussian term actually dominates. Intrinsic deviations from linear Newtonian gravitational clustering, and the Gaussian term contribute little to the cumulants. We plan to follow these calculations with more detailed predictions for coherent seed models and to confront our analytic results with numerical simulations as well as observational data from galaxy surveys [9]. Let us also repeat that the derived scaling laws seem to be more general than their derivation as they have been obtained numerically for global texture which are decoherent seeds. We actually believe that the origin of the scaling laws is not coherence but mainly the decay of the sources at late time and we therefore conjecture that they hold also for topological defects.