FINITE VOLUME FLOWS AND MORSE THEORY

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by

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§0. Introduction. In this paper we present a new approach to Morse theory based on the de Rham - Federer theory of currents. The full classical theory is derived in a transparent way. The methods carry over uniformly to the equivariant and the holomorphic settings. Moreover, the methods are substantially stronger than the classical ones and have interesting applications to geometry. They lead, for example, to formulas relating characteristic forms and singularities of bundle maps.

The ideas came from addressing the following.

Question. Given a smooth flow \( \varphi_t : X \to X \) on a manifold \( X \), when does the limit

\[
P(\alpha) \equiv \lim_{t \to -\infty} \varphi_t^* \alpha
\]

exist for a given smooth differential form \( \alpha \)?

We do not demand that this limit be smooth. As an example consider the gradient flow of a linear function restricted to the unit sphere \( S^n \subset \mathbb{R}^{n+1} \). Note that for any \( n \)-form \( \alpha \), \( P(\alpha) = c[p] \) where \( c = \int_X \alpha \) and \( [p] \) is the point measure of the bottom critical point \( p \).

Our key to the problem is the concept of a finite volume flow - a flow for which the graph of the relation \( x \prec y \), defined by the forward motion of the flow, has finite volume in \( X \times X \). Such flows are abundant. They include generic gradient flows, flows for which the graph is analytic (e.g., algebraic flows), and any flow with fixed points on \( S^1 \).

We show that for any flow of finite volume, the limit \( P \) exists for all \( \alpha \) and defines a continuous linear operator

\[
P : \mathcal{E}^*(X) \to \mathcal{D}^*(X)
\]

from smooth forms to generalized forms, i.e., currents. Furthermore, this operator is chain homotopic to the inclusion \( I : \mathcal{E}^*(X) \hookrightarrow \mathcal{D}^*(X) \), that is, there exists a continuous operator \( T : \mathcal{E}^*(X) \to \mathcal{D}^*(X) \) of degree -1 such that

\[
d \circ T + T \circ d = I - P.
\]

(0.1)

By de Rham [deR], \( I \) induces an isomorphism in cohomology. Hence so does \( P \).

Now let \( f : X \to \mathbb{R} \) be a Morse function with critical set \( C(f) \), and suppose there is a riemannian metric on \( X \) for which the gradient flow \( \varphi_t \) of \( f \) has the following properties:

1. \( \varphi_t \) is of finite volume.
2. The stable and unstable manifolds, \( S_p, U_p \) for \( p \in C(f) \), are of finite volume in \( X \).
3. \( p < q \Rightarrow \lambda_p < \lambda_q \) for all \( p, q \in C(f) \), where \( \lambda_p \) denotes the index of \( p \) and where \( p < q \) means there is a piecewise flow line connecting \( p \) in forward time to \( q \).

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Metrics yielding such gradients always exist. In fact Morse-Smale gradient systems have these properties [HL1]. Under the hypotheses (1)-(3) we prove that the operator $P$ has the following simple form

$$P(\alpha) = \sum_{p \in C^r(f)} \left( \int_{U_p} \alpha \right) [S_p]$$

where $\int_{U_p} \alpha = 0$ if $\deg \alpha \neq \lambda_p$. Thus $P$ gives a retraction

$$P : \mathcal{E}^*(X) \longrightarrow S_f \overset{\text{def}}{=} \text{span}_\mathbb{R} \left\{ [S_p] \right\}_{p \in C^r(f)}$$

onto the finite dimensional subspace of currents spanned by the stable manifolds of the flow.

By (0.1) we see that $S_f$ is $d$-invariant and $H^*(S_f) \cong H^*_{\text{dRham}}(X)$. This immediately yields the strong Morse inequalities.

Applying Stokes' Theorem (cf. [L]) one sees that the restriction of $d$ to $S_f$ has the form

$$d|_{S_f} = \sum_{q \in C^r(f)} n_{pq} [S_q]$$

where the constants $n_{pq}$ are integers which are non-zero only when $\lambda_q = \lambda_p - 1$ and are computed, in the Morse-Smale case, by counting flow lines from $p$ to $q$ (cf. §4). This follows directly from Stokes' Theorem. One concludes that $S_f^\mathbb{Z} \equiv \text{span}_\mathbb{Z} \{ [S_p] \}_{p \in C^r(f)}$ is a finite-rank subcomplex of the integral currents $I_*(X)$ whose inclusion into $I_*(X)$ induces an isomorphism $H \left( S_f^\mathbb{Z} \right) \cong H_*(X; \mathbb{Z})$.

Our method of proving (0.1) employs the kernel calculus of [HP] to convert current equations on $X \times X$ to operator equations.

The method applies to quite general flows and has been used in [HL2] to derive a local version of a formula of MacPherson [Mac1,2] which relates singularities of a bundle map $A : E \to F$ to characteristic forms of $E$ and $F$.

The approach also works for holomorphic $C^*$-actions with fixed-points on Kähler manifolds. One finds a complex analogue of the current $T$. General results of Sommese imply that all the stable and unstable manifolds of the flow are analytic subvarieties. One retrieves classical results of Bialynicki-Birula [BB] and Carrell-Lieberman-Sommese [CL],[CS]. The approach also fits into MacPherson's Grassmann graph construction and the construction of transgression classes in the refined Riemann-Roch Theorem [GS].

The method has many other extensions. It applies to the multiplication and comultiplication operators in cohomology whose kernel is the triple diagonal in $X \times X \times X$ (§11). These ideas can be extended as in [BC]. The method also fits into constructions of invariants of knots and 3-manifold from certain "Feynman graphs" (cf. [K], [BT]).

The cell decomposition of a manifold by the stable manifolds of a gradient flow is due to Thom [T], [S]. That these stable manifolds embed into the de Rham complex of currents was first observed by Landenbach [La].

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**Notation.** For an $n$-manifold $X$, $\mathcal{E}^k(X)$ will denote the space of smooth $k$-forms on $X$, and $\mathcal{D}^k(X)$ the space of $k$-forms with distribution coefficients (the currents of dimension
n - k on X). Under the obvious inclusion $\mathcal{E}^k(X) \subset \mathcal{D}^k(X)$, exterior differentiation $d$ has an extension to all of $\mathcal{D}^k(X)$, and $d = (-1)^{k+1}\partial$ where $\partial$ denotes the current boundary. (See [deR] for full definitions.) We assume here that $X$ is oriented, but this assumption is easily dropped and all results go through.

§1. Finite Volume Flows. Let $X$ be a compact oriented manifold of dimension $n$, and let $\varphi_t : X \to X$ be the flow generated by a smooth vector field $V$ on $X$. For each $t > 0$ consider the compact submanifold with boundary

$$T_t \equiv \{(s, \varphi_s(x), x) : 0 \leq s \leq t \text{ and } x \in X\} \subset \mathbb{R} \times X \times X,$$

oriented so that

$$\partial T_t = \{0\} \times \Delta - \{t\} \times P_t,$$

where $\Delta \subset X \times X$ is the diagonal and $P_t = [\text{graph}(\varphi_t)] = \{(\varphi_t(x), x) : x \in X\}$ is the graph of the diffeomorphism $\varphi_t$. Let $\text{pr} : \mathbb{R} \times X \times X \to X \times X$ denote projection and consider the push-forward

$$T_t \equiv (\text{pr})_*(T_t).$$

of $T_t$ as a de Rham current. Since $\partial$ commutes with $(\text{pr})_*$, we have that

$$\partial T_t = [\Delta] \cdot P_t.$$

Note that the current $T_t$ can be equivalently defined by

$$T_t = \Phi_*(\{0,t\} \times X),$$

where $\Phi : \mathbb{R} \times X \to X \times X$ is the smooth mapping defined by $\Phi(s, x) = (\varphi_s(x), x)$. This mapping is an immersion exactly on the subset $\mathbb{R} \times (X - Z(V))$ where $Z(V) = \{x \in X : V(x) = 0\}$. Thus if we fix a riemannian metric $g$ on $X$, then $\Phi^*(g \times g)$ is a non-negative symmetric tensor which is $> 0$ exactly on the subset $\mathbb{R} \times (X - Z(V))$.

Definition 1.1. A flow $\varphi_t$ on $X$ is called a finite volume flow if $\mathbb{R}^+ \times (X - Z(V))$ has finite volume with respect to the metric induced by the immersion $\Phi$. (This concept is independent of the choice of riemannian metric on $X$.)

Proposition 1.2. Let $\varphi_t$ be a finite volume flow on a compact manifold $X$, and let $T_t$ be the family of currents defined above with

(A) $\partial T_t = [\Delta] - P_t$

Then both the limits

(B) $P \equiv \lim_{t \to \infty} P_t = \lim_{t \to \infty} [\text{graph} \varphi_t]$ and $T \equiv \lim_{t \to \infty} T_t$

exist as currents, and taking the boundary of $T$ gives the equation of currents

(C) $\partial T = [\Delta] - P$ on $X \times X$. 

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Proof. Since \( \varphi_t \) is a finite-volume flow, the current \( T \equiv \Phi_t((0, \infty) \times X) \) is the limit in the mass norm of the currents \( T_t = \Phi_t((0, t) \times X) \) as \( t \to \infty \). The continuity of the boundary operator and equation (A) imply the existence of \( \lim_{t \to \infty} P_t \) and also establish equation (C).

We shall see in the next section that Proposition 1.2 has an important reinterpretation in terms of operators on the differential forms on \( X \).

**Remark 1.3.** If we define a relation on \( X \times X \) by setting \( x \preceq y \) if \( y = \varphi_t(x) \) for some \( 0 \leq t < \infty \), then \( T \) is just the (reversed) graph of this relation. This relation is always transitive and reflexive, and it is antisymmetric if and only if \( \varphi_t \) has no periodic orbits (i.e., \( \preceq \) is a partial ordering precisely when \( \varphi_t \) has no periodic orbits).

**Remark 1.4.** The immersion \( \Phi : \mathbb{R} \times (X - Z(V)) \to X \times X \) is an embedding outside the subset \( \mathbb{R} \times P_{cr}(V) \) where

\[
P_{cr}(V) = \{ x \in X : \varphi_t(x) = x \text{ for some } t > 0 \}
\]

are the non-trivial periodic points of the flow. Thus, if \( P_{cr}(V) \) has measure zero, then \( T_t \) is given by integration over the embedded finite-volume submanifold \( \Phi(R_t) \), where \( R_t = (0, t) \times \tilde{X} \) and \( \tilde{X} = X - Z(V) \cup P_{cr}(V) \). If furthermore the flow has finite volume, then \( T \) is given by integration over the embedded, finite-volume submanifold \( \Phi(R_\infty) \).

There is evidence that any flow with periodic points cannot have finite volume. However, a gradient flow never has periodic points, and such flows are generally of finite volume.

Note that any flow with fixed points on \( S^1 \) has finite volume.

**Remark 1.5.** A standard method for showing that a given flow is finite volume can be outlined as follows. Pick a coordinate change \( t \mapsto \rho \) which sends \( +\infty \) to 0 and \( [t_0, \infty] \) to \( [0, \rho_0] \). Then show that

\[
\tilde{T} = \{ (\rho, \varphi_t(\rho)(x), x) : 0 < \rho < \rho_0 \} \text{ has finite volume in } \mathbb{R} \times X \times X.
\]

Pushing forward to \( X \times X \) then yields the current \( T \) with finite mass. Perhaps the most natural such coordinate change is \( r = 1/t \). Another natural choice (if the flow is considered multiplicatively) is \( s = e^{-t} \). Of course finite volume in the \( r \) coordinate insures finite volume in the \( s \) coordinate since \( r \mapsto s = e^{-1/r} \) is a \( C^\infty \)-map.

Many interesting flows can be seen to be finite volume as follows.

**Lemma 1.6.** If \( X \) is analytic and \( \tilde{T} \subset \mathbb{R} \times X \times X \) is contained in a real analytic subvariety of dimension \( n + 1 \), then \( \varphi_t \) is a finite volume flow.

**Proof.** The manifold points of a real analytic subvariety have (locally) finite volume.

**Example 1.7.** Consider \( S^n \) with two coordinate charts \( \mathbb{C}^n \) and the coordinate change \( x = y/|y|^2 \). In the \( y \)-coordinates consider the translational flow \( \varphi_t(y) = y + tu \) where \( u \in \mathbb{C}^n \) is a fixed unit vector. In the \( x \)-coordinates this becomes

\[
\varphi_t(x) = \frac{x + t|u|^2 u}{|u + tx|^2}.
\]
To see that \( \varphi_t \) is finite volume flow let \( r = 1/t \) and note that \( \hat{T} \) is defined by algebraic equations so that Lemma 1.6 is applicable. The vector field \( V \) has only one zero and is therefore not a gradient vector field. It is however a limit of gradient vector fields.

\[ \text{ §2. The Operator Equations. } \]

The current equations (A), (B), (C) of Proposition 1.2 can be translated into operator equations by a kernel calculus which was introduced in [HP] and which we shall now briefly review. For clarity of exposition we assume our manifolds to be orientable. However, the discussion here extends straightforwardly to the non-orientable case and to forms and currents with values in a flat bundle.

Let \( X \) and \( Y \) be compact oriented manifolds, and let \( \pi_Y \) and \( \pi_X \) denote projection of \( Y \times X \) onto \( Y \) and \( X \) respectively. Then each current (or kernel) \( K \in \mathcal{D}'(Y \times X) \), determines an operator \( K: \mathcal{E}^*(Y) \rightarrow \mathcal{D}'^*(X) \) by the formula

\[ K(\alpha) = (\pi_X)_*(K \wedge \pi_Y^*\alpha), \]

i.e., for a smooth form \( \beta \) of appropriate degree on \( X \)

\[ K(\alpha)(\beta) = K(\pi_Y^*\alpha \wedge \pi_X^*\beta). \]

**Example 2.1.** Suppose \( \varphi: X \rightarrow Y \) is a smooth map, and let

\[ P_\varphi = \{ (\varphi(x), x) : x \in X \} \subset Y \times X, \]

be the graph of \( \varphi \) with orientation induced from \( X \). Then \( P_\varphi \) is just the pull-back operator \( P_\varphi(\alpha) \equiv \varphi^*(\alpha) \) on differential forms \( \alpha \). In particular if \( X = Y \) and \( K = [\Delta] \subset X \times X \) is the diagonal (the graph of the identity), then \( K = I: \mathcal{E}^*(X) \rightarrow \mathcal{D}'(X) \) is the standard inclusion of the smooth forms into the currents on \( X \).

**Lemma 2.2.** Suppose an operator \( T: \mathcal{E}^*(Y) \rightarrow \mathcal{D}'*(X) \) has kernel \( T \in \mathcal{D}'*(Y \times X) \). Suppose that \( T \) lowers degree by one, or equivalently, that \( \deg(T) = \dim Y - 1 \). Then

\[ (2.1) \quad \text{The operator } d \circ T + T \circ d \text{ has kernel } \partial T. \]

**Proof.** The boundary operator \( \partial \) is the dual of exterior differentiation. That is \( \partial T \) is defined by \( (\partial T)(\pi_Y^*\alpha \wedge \pi_X^*\beta) = T(d(\pi_Y^*\alpha \wedge \pi_X^*\beta)) \). Also, by definition, \( T(d\alpha)(\beta) \equiv T(\pi_Y^*(d\alpha) \wedge \pi_X^*\beta) \), and

\[ d(T(\alpha))(\beta) \equiv (-1)^{\deg \alpha} T(\alpha)(d\beta) = (-1)^{\deg \alpha} T(\pi_Y^*(\alpha) \wedge \pi_X^*d\beta), \]

since \( T(\alpha) \) has degree equal to \( \deg \alpha - 1 \).

The results that we need are summarized in the following table.

<table>
<thead>
<tr>
<th>Operators</th>
<th>Kernels</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>[\Delta]</td>
</tr>
<tr>
<td>( P_t \equiv \varphi_t^* )</td>
<td>( P_t \equiv [\text{graph } \varphi_t] )</td>
</tr>
<tr>
<td>T</td>
<td>( T )</td>
</tr>
<tr>
<td>( d \circ T + T \circ d )</td>
<td>( \partial T )</td>
</tr>
</tbody>
</table>

From this table Proposition 1.2 can be reformulated as follows.
Theorem 2.3. Let \( \varphi_t \) be a finite volume flow on a compact manifold \( X \), and let \( P_t : \mathcal{E}^*(X) \rightarrow \mathcal{E}^*(X) \) be the operator given by pull-back

\[
P_t(\alpha) = \varphi_t^*(\alpha).
\]

Let \( T_t : \mathcal{E}^k(X) \rightarrow \mathcal{E}^{k-1}(X) \) be the family of operators associated to the currents \( T_t \) defined in \( \S 1 \). Then for all \( t \) one has that

(A) \[
d \circ T_t + T_t \circ d = I - P_t.
\]

Furthermore, the limits

(B) \[
T = \lim_{t \to \infty} T_t \quad \text{and} \quad P = \lim_{t \to \infty} P_t
\]

exist as operators from forms to currents, and they satisfy the equation

(C) \[
d \circ T + T \circ d = I - P.
\]

§3. Morse-Stokes Gradients; Axioms. Let \( f \in C^\infty(X) \) be a Morse function on a compact \( n \)-dimensional manifold \( X \), and let \( \text{Cr}(f) \) denote the (finite) set of critical points of \( f \). Recall that \( f \) is a \textit{Morse function} if its Hessian at each critical point is non-degenerate. The standard Morse Lemma asserts that in a neighborhood of each \( p \in \text{Cr}(f) \) of index \( \lambda \), there exist \textit{canonical local coordinates} \( (u_1, \ldots, u_\lambda, v_1, \ldots, v_{n-\lambda}) \) for \( |u| < r, |v| < r \) with \( (u(p), v(p)) = (0,0) \) such that

(3.1) \[
f(u,v) = f(p) - |u|^2 + |v|^2.
\]

Fix a riemannian metric on \( X \) and let \( \varphi_t \) denote the flow associated to \( \nabla f \). We assume our metric has the form \( |du|^2 + |dv|^2 \) in some canonical coordinate system \( (u,v) \) about each \( p \in \text{Cr}(f) \). Therefore, in these coordinates the gradient flow is given by

(3.2) \[
\varphi_t(u,v) = (e^{-t}u, e^{t}v)
\]

Metrics with this property will be called \textit{canonically flat near} \( \text{Cr}(f) \).

Now to each \( p \in \text{Cr}(f) \) are associated the \textit{stable} and \textit{unstable manifolds} of the flow, defined respectively by

\[
S_p = \{ x \in X : \lim_{t \to -\infty} \varphi_t(x) = p \} \quad \text{and} \quad U_p = \{ x \in X : \lim_{t \to -\infty} \varphi_t(x) = p \}.
\]

For coordinates \( (u,v) \) at \( p \), chosen as above, we consider the disks

\[
S_p(\epsilon) = \{ (u,0) : |u| < \epsilon \} \quad \text{and} \quad U_p(\epsilon) = \{ (0,v) : |v| < \epsilon \}
\]

and observe that

(3.3) \[
S_p = \bigcup_{-\infty < \epsilon < 0} \varphi_{\epsilon}(S_p(\epsilon)) \quad \text{and} \quad U_p = \bigcup_{0 < \epsilon < +\infty} \varphi_{\epsilon}(U_p(\epsilon)).
\]
Hence, $S_p$ and $U_p$ are contractible submanifolds (but not closed subsets) of $X$ with

\begin{equation}
\dim S_p = \lambda_p \quad \text{and} \quad \dim U_p = n - \lambda_p
\end{equation}

where $\lambda_p$ is the index of the critical point $p$. For each $p$ we choose an orientation on $U_p$. This gives an orientation on $S_p$ via the splitting $T_pX = T_pU_p \oplus T_p S_p$.

The flow $\varphi_t$ induces a partial ordering on $X$ by setting $x \prec y$ if there is a continuous path consisting of a finite number of forward-time orbits, which begins with $x$ and ends with $y$. This is the closure of the partial ordering of Remark 1.3.

**Definition 3.1.** The gradient flow of a smooth function $f$ on a Riemannian manifold $X$ is called **Morse-Stokes** if

(i) $f$ is a Morse function.
(ii) The flow is a finite-volume flow.
(iii) Each of the stable and unstable manifolds $S_p$ and $U_p$ for $p \in \text{Cr}(f)$ has finite volume.
(iv) If $p \prec q$ and $p \neq q$ then $\lambda_p < \lambda_q$, for all $p, q \in \text{Cr}(f)$.

**Remark 3.2.** If the gradient flow of $f$ is Morse-Smale, then it is Morse-Stokes. Furthermore, for any Morse function $f$ on a compact manifold $X$ there exist Riemannian metrics on $X$ for which the gradient flow of $f$ is Morse-Stokes. These metrics are constructed to be canonically flat near $\text{Cr}(f)$ and are dense in all metrics with this property. See [HL1].

**Theorem 3.3.** Let $f \in C^\infty(X)$ be a Morse function on a compact Riemannian manifold whose gradient flow is Morse-Stokes. Then there is an equation of integral currents

\begin{equation}
\partial T = |\Delta| \cdot P
\end{equation}

on $X \times X$, where $T$ is an embedded submanifold of finite volume (essentially the graph of the relation $\prec$), $\Delta \subset X \times X$ is the diagonal, and

\begin{equation}
P = \sum_{p \in \text{Cr}(f)} [U_p] \times [S_p]
\end{equation}

where $U_p$ and $S_p$ are oriented so that $U_p \times S_p$ agrees in orientation with $X$ at $p$.

**Proof.** For each critical point $p \in \text{Cr}(f)$ we define

\[\tilde{U}_p \overset{\text{def}}{=} \bigcup_{p \prec q} U_q = \{x \in X : p \prec x\} \].

**Lemma 3.4.** Let $f \in C^\infty(X)$ be any Morse function whose gradient flow is of finite volume, and let $\text{spt} P \subset X \times X$ denote the support of the current $P$ defined in Theorem 1.2. Then

\[\text{spt} P \subset \bigcup_{p \in \text{Cr}(f)} \tilde{U}_p \times S_p\].

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Proof. Since \( P = \lim_{t \to -\infty} P_t \) and \( P_t = \{(\varphi_t(x), x) : x \in X\} \), it is clear that \((y, x) \in \text{spt} P\) only if there exist sequences \(x_i \to x\) in \(X\) and \(s_i \to \infty\) in \(\mathbb{R}\) such that \(y_i \equiv \varphi_{s_i}(x_i) \to y\). Let \(L(x_i, y_i)\) denote the oriented flow line from \(x_i\) to \(y_i\). Since the lengths of these lines are bounded, compactness implies that a subsequence converges to a piecewise flow line \(L(x, y)\) from \(x\) to \(y\). By the continuity of the boundary operator on currents, \(\partial L(x, y) = [y] - [x]\). Finally, since \(s_i \to \infty\) there must be at least one critical point on \(L(x, y)\), and we define \(p = \lim_{s \to \infty} \varphi_s(x)\).

Consider now the compact subset

\[
\Sigma \equiv \bigcup_{\substack{p < q \in \mathbb{Q} \cap \mathbb{R} \neq q}} U_q \times S_p \subset X \times X.
\]

Note that from (3.4) and Axiom (iv) in Definition 3.1 that

\[
(3.7) \quad \dim(\Sigma) \leq n - 1.
\]

Set \(\Sigma' = \Sigma \cup \{(p, p) : p \in \text{Cr}(f)\}\).

Lemma 3.5. Let \(f \in C^\infty(X)\) be a Morse function and consider the embedded submanifold

\[
T = \{(y, x) : x \notin \text{Cr}(f), \text{ and } y = \varphi_t(x) \text{ for some } 0 < t < \infty\}.
\]

of \(X \times X \setminus \Sigma'\). Then the closure \(\overline{T}\) of \(T\) in \(X \times X \setminus \Sigma'\) is a proper \(C^\infty\)-submanifold with boundary

\[
\partial \overline{T} = \Delta - \sum_{p \in \text{Cr}(f)} U_p \times S_p.
\]

Proof. We first show that it will suffice to prove the assertion in a neighborhood of \((p, p) \in X \times X\) for \(p \in \text{Cr}(f)\). Consider \((\tilde{y}, \tilde{x}) \in \overline{T} - T\). If \((\tilde{y}, \tilde{x}) \notin \Sigma'\), then the proof of Lemma 3.4 shows that either \((\tilde{y}, \tilde{x}) \in \Delta\) or \((\tilde{y}, \tilde{x}) \in U_p \times S_p\) for some \(p \in \text{Cr}(f)\). Near points \((\tilde{x}, \tilde{x}) \in \Delta, \tilde{x} \notin \text{Cr}(f)\), one easily checks that \(\tilde{T}\) is a submanifold with boundary \(\Delta\). If \((\tilde{y}, \tilde{x}) \in U_p \times S_p\), then for sufficiently large \(s > 0\), the diffeomorphism \(\psi_s(y, x) \equiv (\varphi_{-s}(y), \varphi_s(x))\) will map \((y, x)\) into any given neighborhood of \((p, p)\). Note that \(\psi_s\) leaves the subset \(U_p \times S_p\) invariant, and that \(\psi_{s}^{-1}\) maps \(T\) into \(T\). Hence, if \(\partial \overline{T} = \Delta - U_p \times S_p\), in a neighborhood of \((p, p)\), then \(\partial \overline{T} = -U_p \times S_p\) near \((\tilde{y}, \tilde{x})\).

Now in a neighborhood \(\mathcal{O}\) of \((p, p)\) we may choose coordinates as in (3.2) so that \(T\) consists of points \((y, x) = (u, v, u, v)\) with \(\tilde{u} = e^{-t}u\) and \(\tilde{v} = e^{t}v\) for some \(0 < t < \infty\). Consequently, in \(\mathcal{O}\) the set \(\overline{T}\) is given by the equations

\[
\tilde{u} = su \quad \text{and} \quad \tilde{v} = sv \quad \text{for some} \quad 0 \leq s \leq 1.
\]

This obviously defines a submanifold in \(\mathcal{O} - \{(p, p)\}\) with boundary consisting of \(\Delta\) and the set \(\{u = 0, v = 0\} \cong (U_p \times S_p) \cap \mathcal{O}\).
Lemma 3.5 has the following immediate consequence

\[(3.8) \quad \text{spt} \left\{ P - \sum_{p \in \text{Cr}(f)} [U_p] \times [S_p] \right\} \subset \Sigma' \]

We now apply the following elementary but key result of Federer.

**Proposition 3.6.** ([F, 4.1.15]) Let \([W] \) be a current in \(\mathbb{R}^n\) defined by integration over a \(k\)-dimensional oriented submanifold \(W\) of locally finite volume. Suppose \(\text{supp} (d[W]) \subset \mathbb{R}^l\), a linear subspace of dimension \(\ell < k - 1\). Then \(d[W] = 0\).

Combining this with (3.7) and (3.8) proves (3.6) and completes the proof of Theorem 3.3. \(\blacksquare\)

§4. Morse Theory. Combining Theorems 2.3 and 3.3 gives the following.

**Theorem 4.1.** Let \(f \in C^\infty(X)\) be a Morse function on a compact riemannian manifold \(X\) whose gradient flow \(\varphi_t\) is Morse-Stokes. Then for every differential form \(\alpha \in \mathcal{E}^k(X)\), \(0 \leq k \leq n\), one has

\[P(\alpha) = \lim_{t \to \infty} \varphi^*_t \alpha = \sum_{p \in \text{Cr}(f)} r_p(\alpha)[S_p] \]

where the "residue" \(r_p(\alpha)\) of \(\alpha\) at \(p\) is defined by \(r_p(\alpha) = \int_{U_p} \alpha\) if \(k = n - \lambda\) and 0 otherwise. Furthermore, there is an operator \(T\) of degree -1 on \(\mathcal{E}^\ast(X)\) with values in flat currents, such that

\[(4.1) \quad d \circ T + T \circ d = I - P\]

Note that \(P : \mathcal{E}^\ast(X) \to \mathcal{D}^\ast(X)\) maps onto the finite-dimensional subspace

\[(4.2) \quad S_f^\ast \overset{\text{def}}{=} \text{span}\{[S_p]\}_{p \in \text{Cr}(f)}\]

and that from (4.1) we have

\[P \circ d = d \circ P.\]

Hence, Theorem 4.1 has the following immediate corollary.

**Theorem 4.2.** The subspace \(S_f^\ast\) is \(d\)-invariant and is therefore a subcomplex of \(\mathcal{D}^\ast(X)\). The linear map

\[P : \mathcal{E}^\ast(X) \to S_f^\ast\]

is a map of cochain complexes which induces an isomorphism

\[P : H^\ast_{d\text{er}}(X) \overset{\cong}{\longrightarrow} H^\ast(S_f^\ast).\]
The vector space $S_f^\mathbb{Z}$ has a distinguished lattice
\[ S_f^\mathbb{Z} \; \text{def} \; \text{span}_\mathbb{Z} \{ [S_p] \}_{p \in Cr(f)} \subset S_f \]
generated by the stable manifolds. Note that $S_f^\mathbb{Z}$ is a subgroup of the integral currents $\mathcal{I}(X)$ on $X$.

**Theorem 4.3.** The lattice $S_f^\mathbb{Z}$ is preserved by $d$, i.e., $(S_f^\mathbb{Z}, d)$ is a subcomplex of $(S_f, d)$. Furthermore, the inclusion of complexes $(S_f^\mathbb{Z}, d) \subset (\mathcal{I}(X), d)$ induces an isomorphism
\[ H(S_f^\mathbb{Z}) \cong H_*(X; \mathbb{Z}) \]

**Proof.** Theorem 4.2 implies that for any $p \in Cr(f)$ we have
\[ d[S_p] = \sum_{\lambda_q = \lambda_p - 1} n_{p,q} [S_q] \]
for real numbers $n_{p,q}$. In particular, $d[S_p]$ has finite mass, and so by results of Federer [F] it is an integral current. This implies that $n_{p,q} \in \mathbb{Z}$ for all $p, q$, and the first assertion is proved. Now the domain of the operator $P$ extends to include any $C^1$ chain $c$ which is transversal to the submanifolds $U_p$, $p \in Cr(f)$, while the domain of $T$ extends to any $C^1$ chain $c$ for which $X \times c$ is transversal to $T$. Standard transversality arguments show that such chain groups (over $\mathbb{Z}$) compute $H_*(X; \mathbb{Z})$. The result then follows from (4.1).

**Corollary 4.4.** Let $G$ be a finitely generated abelian group. Then there are natural isomorphisms
\[ H(S_f^\mathbb{Z} \otimes \mathbb{Z} G) \cong H_*(X; G). \]

The first assertion of Theorem 4.3 has a completely elementary proof whenever the flow is Morse-Smale, i.e., when $S_p$ is transversal to $U_q$ for all $p, q \in Cr(f)$. Suppose the flow is Morse-Smale and that $p, q \in Cr(f)$ are critical points with $\lambda_q = \lambda_p - 1$. Then $U_q \cap S_p$ is the union of a finite set of flow lines from $q$ to $p$ which we denote $\Gamma_{p,q}$. To each $\gamma \in \Gamma_{p,q}$ we assign an index $n_\gamma$ as follows. Let $B_\varepsilon \subset S_p$ be a small ball centered at $p$ in a canonical coordinate system (cf. (3.1) ), and let $y$ be the point where $\gamma$ meets $\partial B_\varepsilon$. The orientation of $S_p$ induces an orientation on $T_y(\partial B_\varepsilon)$, which is identified by flowing backward along $\gamma$ with $T_q(S_q)$. If this identification preserves orientations we set $n_\gamma = 1$, and if not, $n_\gamma = -1$. We then define
\[ N_{p,q} \; \text{def} \; \sum_{\gamma \in \Gamma_{p,q}} n_\gamma. \]
As in [La] Stokes' Theorem now directly gives us the following (cf. [W]).

10
Proposition 4.5. When the gradient flow of $f$ is Morse-Smale, the coefficients in (4.3) are given by

$$n_{p,q} = (-1)^{\lambda_p} N_{p,q}$$

Proof. Given a form $\alpha$ of degree $\lambda_p - 1$, we have

$$(-1)^{\lambda_p} d[S_p](\alpha) = \int_{S_p} d\alpha = \lim_{r \to \infty} \int_{dS_p(r)} \alpha$$

where $S_p(r) = \varphi_{-r}(S_p(\epsilon))$ as in §3. It suffices to consider forms $\alpha$ with support near $q$ where $\lambda_q = \lambda_p - 1$. Near such $q$, the set $S_p(r)$, for large $r$, consists of a finite number of manifolds with boundary, transversal to $U_q$. There is one for each $\gamma \in \Gamma_{p,q}$. As $r \to \infty$ along one such $\gamma$, $dS_p(r)$ converges to $\pm S_q$ where the sign is determined by the agreement (or not) of the orientation of $dS_p(r)$ with the chosen orientation of $S_q$. \(\blacksquare\)

Remark 4.6. The integers $N_{p,q}$ have a simple definition in terms of currents. Set $S_p(r) = \varphi_{-r}(S_p(\epsilon))$ and $U_q(r) = \varphi_{r}(U_q(\epsilon))$ (cf. (3.3)). Then for all $r$ sufficiently large

$$(4.5) \quad N_{p,q} = \int_X [U_q(r)] \wedge d[S_p(r)]$$

where the integral denotes evaluation on the fundamental class.

§5. Poincaré Duality. In this context there is a simple proof of Poincaré duality. Given two oriented submanifolds $A$ and $B$ of complementary dimensions in $X$ which meet transversally in a finite number of points, let $A \cdot B = \int_A [A] \wedge [B]$ denote the algebraic number of intersections points. Then for any $k$ we have

$$(5.1) \quad U_q \bullet S_p = \delta_{pq} \quad \text{for all } p, q \in Cr_k(f)$$

where $Cr_k(f) \equiv \{ p \in Cr(f) : \lambda_p = k \}$. This gives a formal identification

$$(5.2) \quad U_f^2 \overset{\text{def}}{=} Z \cdot \{ [U_p] \}_{p \in Cr(f)} \cong \text{Hom} (S_f^2, Z).$$

Therefore, taking the adjoint of $d$ gives a differential $\delta$ on $U_f^2$ with the property that $H_n - *(U_f^2, \delta) \cong H^*(X; Z)$. On the other hand the arguments of §§1–4 (with $f$ replaced by $-f$) show that $U_f^2$ is $d$-invariant with $H_*(U_f^2, d) \cong H_*(X; Z)$. However, these two differentials on $U_f^2$ agree up to sign as we see in the next lemma.

Lemma 5.1. One has

$$(5.3) \quad (dU_q) \bullet S_p = (-1)^{n-k} U_q \bullet (dS_p)$$

for all $p \in Cr_k(f)$ and $q \in Cr_{k-1}(f)$, and for any $k$.\[11\]
Proof. One can see directly from the definition that the integers \( N_{p,q} \) are invariant (up to a global sign) under time-reversal in the flow. However, for a simple current-theoretic proof consider the 1-dimensional current \([U_q(r)] \wedge [S_p(r)]\) consisting of a finite sum of oriented line-segments in the flow lines of \( \Gamma_{p,q} \) (cf. Remark 4.6). Note that

\[
d[U_q(r)] \wedge [S_p(r)] = (d[U_q(r)]) \wedge [S_p(r)] + (-1)^{n-k+1} [U_q(r)] \wedge (d[S_p(r)])
\]

and apply (4.5). \( \blacksquare \)

Corollary 5.2. (Poincaré Duality)

\[
H^{n-k}(X; \mathbb{Z}) \cong H_k(X; \mathbb{Z}) \quad \text{for all } k.
\]

Note 5.3. The Poincaré duality isomorphism can be realized in our operator picture as follows. Consider the "total graph" of the flow: in \( X \times X \)

\[
T_{\text{tot}} = T^* + T = \{(y, x) : y = \varphi_t(x) \text{ for some } t \in \mathbb{R}\}
\]

with corresponding operator \( T_{\text{tot}} \). Then one has the operator equation

\[
d \circ T_{\text{tot}} + T_{\text{tot}} \circ d = \mathcal{P} - \bar{\mathcal{P}}
\]

where

\[
\mathcal{P} = \sum_{p \in Cr(f)} [U_p] \times [S_p] \quad \text{and} \quad \bar{\mathcal{P}} = \sum_{p \in Cr(f)} [S_p] \times [U_p].
\]

This chain homotopy induces an isomorphism \( H_*(\mathcal{U}^{2}_f) \cong H_*(\mathcal{S}^{2}_f) \), which after identifying \( \mathcal{U}^{2}_f \) with the cochain complex via (5.1) and (5.3), gives the duality isomorphism 5.2. When \( X \) is not oriented, a parallel analysis yields Poincaré duality with mod 2 coefficients.

§6. Generalizations. As seen in §2, for any flow \( \varphi_t \) of finite volume the operator \( \mathcal{P}(\alpha) = \lim_{t \to -\infty} \varphi_t^* (\alpha) \) exists and is chain homotopic to the identity. This situation occurs often. Suppose for example that \( f : X \to \mathbb{R} \) is a smooth function whose critical set is a finite disjoint union

\[
Cr(f) = \bigsqcup_{j=1}^{\nu} F_j
\]

of compact submanifolds \( F_j \) in \( X \) and that Hess(\( f \)) is non-degenerate on the normal spaces to \( Cr(f) \). Then for each \( j \), there are stable and unstable manifolds

\[
S_j = \{ x \in X : \lim_{t \to -\infty} \varphi_t(x) \in F_j \} \quad \text{and} \quad U_j = \{ x \in X : \lim_{t \to -\infty} \varphi_t(x) \in F_j \}
\]

with projections

\[
S_j \xrightarrow{\tau_j} F_j \xleftarrow{\sigma_j} U_j.
\]

(6.1)
where 
\[ \tau_j(x) = \lim_{t \to -\infty} \varphi_t(x) \quad \text{and} \quad \sigma_j(x) = \lim_{t \to -\infty} \varphi_t(x). \]

For each \( j \), let \( n_j = \dim(F_j) \) and set \( \lambda_j = \dim(S_j) - n_j \). Then \( \dim(U_j) = n - \lambda_j \). For \( p \in F_j \) we define \( \lambda_p \equiv \lambda_j \) and \( n_p \equiv n_j \).

**Definition 6.1.** The gradient flow \( \varphi_t \) of a smooth function \( f \in C^\infty(X) \) on a Riemannian manifold \( X \) is called a **generalized Morse-Stokes flow** if:

(i) The critical set of \( f \) consists of a finite number of submanifolds \( F_1, \ldots, F_\nu \) on the normals of which \( \text{Hess}(f) \) is non-degenerate.

(ii) The manifolds \( T \), and \( T^* \), and the stable and unstable manifolds \( S_j, U_j \) for \( 1 \leq j \leq \nu \) are submanifolds of finite volume. Furthermore, for each \( j \), the fibres of the projections \( \tau_j \) and \( \sigma_j \) are of uniformly bounded volume.

(iii) \( p < q \implies \lambda_p + n_p < \lambda_q \quad \forall p, q \in \text{Crit}(f). \)

**Theorem 6.2.** Suppose \( \varphi_t \) is a gradient flow satisfying the generalized Morse-Stokes conditions 6.1 on a compact oriented manifold \( X \). Then there is an equation of currents

\[ \partial T = [\Delta] - P \]

on \( X \times X \), where \( T \), \( \Delta \), and \( P \) are as in Theorem 1.2, and

\[ P = \sum_{j=1}^\nu [U_j \times_{F_j} S_j] \]

where \( U_j \times_{F_j} S_j \equiv \{(y, x) \in U_j \times S_j \subset X \times X : \sigma_j(y) = \tau_j(x)\} \) denotes the fibre product of the projections (6.1)

**Proof.** The argument follows closely the proof of Theorem 3.3. Details are omitted. ■

In [L] Janko Latschev has found a Smale-type condition which yields this result in cases where 6.1 (iii) fails. In particular, his condition implies only that: \( p < q \implies \lambda_p < \lambda_q \).

The result above can be translated into the following operator form.

**Theorem 6.3.** Let \( \varphi_t \) be a gradient flow satisfying the generalized Morse-Stokes Conditions on a manifold \( X \) as above. Then for all smooth forms \( \alpha \) on \( X \), the limit

\[ P(\alpha) = \lim_{t \to -\infty} \varphi_t^*(\alpha) \]

exists and defines a continuous linear operator \( P : \mathcal{E}^\ast(X) \to \mathcal{D}^\ast(X) \) with values in flat currents on \( X \). This operator fits into a chain homotopy

\[ d \circ T + T \circ d = I - P. \]

Furthermore, \( P \) is given by the formula

\[ P(\alpha) = \sum_{j=1}^\nu \text{Res}_j(\alpha) [S_j] \]

13
where

\begin{equation}
\text{Res}_j(\alpha) = (\sigma_j)_* \{ (\sigma_j)_* \{ \alpha |_{U_j} \} \}
\end{equation}

**Proof.** This is a direct consequence of Theorem 6.2 except for the formulae (6.4)-(6.5). To see this consider the pull-back square

\begin{equation}
\begin{array}{ccc}
U_j \times_{F_j} S_j & \xrightarrow{t_j} & U_j \\
\downarrow s_j & & \downarrow \sigma_j \\
S_j & \xrightarrow{\tau_j} & F_j
\end{array}
\end{equation}

where $t_j$ and $s_j$ are the obvious projections. One sees from the definitions that

\[ P(\alpha) = \sum_{j=1}^{\nu} (s_j)_* \{ (t_j)_* \{ \alpha |_{U_j} \} \}. \]

The commutativity of the diagram (6.6) allows us to rewrite these terms as in (6.5).

**Corollary 6.4.** Suppose that $\lambda_p + n_p + 1 < \lambda_q$ for all critical points $p < q$. Then the homology of $X$ is spanned by the images of the groups $H_{\lambda_j + \ell}(S_j)$ for $j = 1, \ldots, \nu$ and $\ell \geq 0$.

**Proof.** Under this hypothesis $\partial(U_j \times F_j, S_j) = 0$ for all $j$, and so (6.2) yields a decomposition of $P$ into operators that commute with $d$.

Each map $\tau_j : S_j \to F_j$ can be given the structure of a vector bundle of rank $\lambda_j$. The closure $\overline{S_j} \subset X$ is a compactification of this bundle with a complicated structure at infinity (cf. [CJS]). There is nevertheless a homomorphism $\Theta_j : H_*(F_j) \to H_{\lambda_j + \ell}(\overline{S_j})$ which after pushing forward to the one-point compactification of $S_j$, is the Thom isomorphism. This leads to the following (cf. [AB]).

**Corollary 6.5.** Suppose that $\lambda_p + n_p + 1 < \lambda_q$ for all critical points $p < q$ and that $X$ and all $F_j$ and $S_j$ are oriented. Then there is an isomorphism

\[ H_*(X) \cong \bigoplus_{j} H_*(F_j) - \lambda_j \]

One can drop the orientation assumptions by taking homology with appropriately twisted coefficients. Extensions of 6.3 and 6.5 to integral homology groups are found in [L]. Latschev also derives a spectral sequence, with geometrically computable differentials, associated to any Bott-Smale function satisfying a certain transversality hypothesis. Assuming that everything is oriented, the $E^1$-term is given by $E^1_{p,q} = \bigoplus_{\lambda_p = p} H_q(F_j; Z)$ and $E^k_{p,q} \Rightarrow H_*(X; Z)$.
Remark 6.6. In standard Morse Theory one considers a proper exhaustion function $f : X \to \mathbb{R}$ and studies the change in topology as one passes from $\{x : f(x) \leq a\}$ to $\{x : f(x) < b\}$. Our operator approach is easily adapted to this case (see [HL1]).

§7. The local MacPherson formula. These ideas can be applied to the study of curvature and singularities. Suppose

$$\alpha : E \to F$$

is a map between smooth vector bundles with connection over a manifold $X$. Let $G = G_k(E \oplus F) \to X$ denote the Grassmann bundle of $k$-planes in $E \oplus F$ where $k = \text{rank}(E)$.

There is a flow $\varphi_t$ on $G$ induced by the flow $\psi_t : E \oplus F \to E \oplus F$ where $\psi_t(e, f) = (te, f)$. This is a very simple generalized Morse-Stokes flow on $G$. On the “affine chart” $\text{Hom}(E, F)$, one has that $\varphi_t(A) = \frac{1}{t} A$.

Note that here it is more natural to consider the flow multiplicatively ($\varphi_{ts} = \varphi_t \circ \varphi_s$) than additively. Consider $\mathbb{R}^+ = \{[1 : t] \in \mathbb{P}^1(\mathbb{R}) : 0 < t < \infty\} \subset \mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. Note that this inclusion is compatible with the inclusion of the complex multiplicative group $\mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C})$.

Definition 7.1. The section $\alpha$ is said to be geometrically atomic if the graph

$$T(\alpha) \overset{\text{def}}{=} \{(t, x, \frac{1}{t} \alpha(x)) : 0 < t \leq 1 \text{ and } x \in X\}$$

has finite volume in $\mathbb{P}^1(\mathbb{R}) \times X \times G$.

This hypothesis is sufficient to guarantee the existence of $\lim_{t \to 0} \alpha_t^* \Phi$ where $\alpha_t \equiv \frac{1}{t} \alpha$ and where $\Phi$ is any differential form on $G$. Choosing $\Phi = \Phi_0(\Omega_U)$ where $\Phi_0$ is an $\text{Ad}$-invariant polynomial on $\mathfrak{g}_k(\mathbb{R})$ and $\Omega_U$ is the curvature of the tautological $k$-plane bundle over $G$, one can establish a local version of a basic formula of MacPherson [Mac1,2]. Details appear in [HL2].

§8. Equivariant Morse Theory. Our approach carries over to equivariant cohomology by using Cartan’s equivariant de Rham theory (cf. [BV]). Suppose $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$ acting on a compact $n$-manifold $X$. By an equivariant differential form on $X$ we mean a $G$-equivariant polynomial map $\alpha : \mathfrak{g} \to \mathcal{E}^*(X)$. The set of equivariant forms is denoted by

$$\mathcal{E}^*_G(X) = \{S^p(\mathfrak{g}^*) \otimes \mathcal{E}^*(X)\}^G$$

and is graded by declaring elements of $S^p(\mathfrak{g}^*) \otimes \mathcal{E}^q(X)$ to have total degree $2p + q$. The equivariant differential $d_G : \mathcal{E}^*_G(X) \to \mathcal{E}^{*+1}_G(X)$ is defined by setting

$$(d_G \alpha)(V) = d\alpha(V) - i_V^* \alpha(V)$$

for $V \in \mathfrak{g}$ where $i_V^*$ denotes contraction with the vector field $\tilde{V}$ on $X$ corresponding to $V$.

The complex $\mathcal{E}^*_G(X)$ of equivariant currents is similarly defined by replacing smooth forms $\mathcal{E}^*(X)$ by forms $\mathcal{C}^*(X)$ with distribution coefficients.
Consider a $G$-invariant function $f \in C^\infty(X)$ and suppose $X$ is given a $G$-invariant Riemannian metric. Suppose the ($G$-invariant) gradient flow has finite volume and let

\[(8.1) \quad \partial T = \Delta - P\]

be the current equation derived in §3. Let $G$ act on $X \times X$ by $g \cdot (x, y) = (gx, gy)$.

**Lemma 8.1.** The current $T$ satisfies:

(i) $g_* T = T$ for all $g \in G$, and

(ii) $i_V T = 0$ for all $V \in \mathfrak{g}$.

So also does the current $P$.

**Proof.** The current $T$ corresponds to integration over the finite volume submanifold \[\{(x, y) \in X \times X - \Delta : y = \varphi_t(x) \text{ for some } 0 < t < \infty\}.\] Since $g \varphi_t(x) = \varphi_t(gx)$, assertion (i) is clear. The invariance of $T$ implies the invariance of $P$ by (8.1). For assertion (ii) note that for any $n$-form $\beta$ on $X \times X$ \[\left( i_V T \right)(\beta) = \int_T i_V \beta = 0\]

since $V$ is tangent to $T$. Similarly, $i_V \Delta = 0$. Since $d \circ i_V + i_V \circ d = L_V$ (Lie derivative) and $L_V T = 0$, we conclude that $i_V P = 0$.

**Corollary 8.2.** Consider $T \equiv 1 \otimes T \in 1 \otimes \mathcal{E}_G^{n-1}(X \times X)^G$ as an equivariant current of total degree $(n-1)$ on $X \times X$. Consider $\Delta$ and $P$ similarly as equivariant currents of degree $n$. Then

\[(8.2) \quad \partial_G T = \Delta - P \quad \text{on } X \times X.\]

The correspondence between operators and kernels discussed in §2 carries over to the equivariant context. Currents in $\mathcal{E}^n_G - \ell(X \times X)$ yield $G$-equivariant operators $\mathcal{E}^n(X) \rightarrow \mathcal{E}^{n+\ell}(X)$, and hence operators $\mathcal{E}^n_G(X) \rightarrow \mathcal{E}^{n+\ell}_G(X)$. Equations of type (8.2) translate into operator equations

\[(8.3) \quad d_G \circ T + T \circ d_G = I - P.\]

Applying the arguments of §4 proves the following.

**Proposition 8.3.** Let $\varphi_t$ be an invariant flow on a compact $G$-manifold $X$. If $\varphi_t$ has finite volume, then the limit

\[(8.4) \quad P(\alpha) = \lim_{t \to \infty} \varphi_t^* \alpha\]

exists for all $\alpha \in \mathcal{E}^n_G(X)$ and defines a continuous linear operator $P : \mathcal{E}^n_G(X) \rightarrow \mathcal{D}^n_G(X)$ of degree 0, which is equivariantly chain homotopic to the identity on $\mathcal{E}^n_G(X)$.

The fact that $S^* g^* G \cong H^*(BG)$ yields the following result.
**Theorem 8.4.** Let $f \in C^\infty(X)$ be an invariant Morse function on a compact riemannian $G$-manifold whose gradient flow $\varphi_t$ is Morse-Stokes. Then the continuous linear operator (8.4) defines a map of equivariant complexes

$$
P : \mathcal{E}^*_G(X) \longrightarrow S^*(\mathfrak{g}^*)^G \otimes S_f$$

where $S_f = \text{span}\{[S]_\mu\}_{\mu \in \mathcal{C}(f)}$ as in (4.2) and where the differential on $S^*(\mathfrak{g}^*)^G \otimes S_f$ is $1 \otimes \partial$. This map induces an isomorphism

$$H^*_G(X) \isom H^*(BG) \otimes H^*(X)$$

Examples of this phenomenon arise in moment map constructions. For a simple example consider $G = (S^1)^{n+1}/\Delta$ acting on $\mathbb{P}^n_G$ via the standard action on homogeneous coordinates $[z_0, ..., z_n]$, and set $f([z]) = \sum k|z_k|^2/\|z\|^2$. One sees immediately the well-known fact that $H^*_G(\mathbb{P}^n_G)$ is a free $H^*(BG)$-module with one generator in each dimension $2k$ for $k = 0, ..., n$. This extends to all generalized flag manifolds and to products.

J. Latschev has pointed out that there exists an invariant Morse function for which no choice of invariant metric gives a Morse-Stokes flow.

However, the method applies to quite general functions and yields results as in §§6-7. Suppose for example that $f$ is an invariant function whose critical set consists of a finite number of non-degenerate critical orbits $O_i = G/H_i$, $i = 1, ..., N$ (the generic case). There is a spectral sequence with (assuming for simplicity that everything is oriented)

$$E^{p,*}_1 = \bigoplus_{\lambda_i = p} H^*_G(O_i) = \bigoplus_{\lambda_i = p} H^*(BH_i)$$

and computable differentials such that $E^k_{p,*} \Rightarrow H^*_G(X)$ (cf. [L]).

**§9. Holomorphic Flows and the Carrell-Lieberman-Sommese Theorem.** This method also applies to the holomorphic case. For example, given a $C^*$-action $\varphi_t$ on a compact Kähler manifold $X$, there is a complex graph

$$\mathcal{T} \overset{\text{def}}{=} \{(t, \varphi_t(x), x) \in C^* \times X \times X : t \in C^* \text{ and } x \in X\} \subset \mathbb{P}^1(C) \times X \times X$$

analogous to the graphs considered above. The following is a result of Sommese [So].

**Theorem 9.1.** If $\varphi_t$ has fixed-points, then $\mathcal{T}$ has finite volume and its closure $\overline{\mathcal{T}}$ in $\mathbb{P}^1(C) \times X \times X$ is an analytic subvariety.

The relation of $C^*$-actions to Morse-Theory is classical. The action $\varphi_t$ decomposes into an "angular" $S^1$-action and a radial flow. Averaging a Kähler metric over $S^1$ and applying an argument of Frankel [Fr], we find a function $f : X \rightarrow \mathbb{R}$ of Bott-Morse type whose gradient generates the radial action. Theorem 9.1 implies that the associated current $\mathcal{T}$ on $X \times X$, which is a real slice of $\mathcal{T}$, is real analytic and hence of finite volume. One can then apply the methods of this paper, in particular those of §6.
Thus when $\varphi_t$ has fixed-points, $T$ gives a rational equivalence between the diagonal $\Delta$ in $X \times X$ and an analytic cycle $P$ whose components consist of fibre products of stable and unstable manifolds over components of the fixed-point set of the action.

When the fixed-points are all isolated, $P$ becomes a sum of analytic K"unneth components

$$P = \sum \overline{S}_p \times \overline{U}_p,$$

and we recover the well-known fact that the cohomology of $X$ is freely generated by the stable subvarieties $(\overline{S}_p)_{p \in \text{prim}(\varphi)}$. It follows that $X$ is algebraic and that all cohomology theories on $X$ (e.g., algebraic cycles modulo rational equivalence, algebraic cycles modulo algebraic equivalence, singular cohomology) are naturally isomorphic. (See [BB], [ES], [Fr].)

When the fixed-point set has positive dimension, one can recover results of Carrell-Lieberman-Sommese for $C^*$-actions ([CL], [CS]), which assert among other things that if $\dim(X \times C^*) = k$, then $H^{p,q}(X) = 0$ for $|p - q| > k$.

§10. Local coefficient systems. Our method applies immediately to forms with coefficients in a flat bundle $E \to X$. In this case the kernels of $\delta$ are currents on $X \times X$ with coefficients in $\text{Hom}(\pi_1^* E, \pi_2^* E)$. Given a Morse-Stokes flow $\varphi_t$ on $X$ we consider the kernel $T_E = h \circ T$ where $T$ is defined as in §3 and $h : E_{\varphi_t(x)} \to E_x$ is parallel translation along the flow line. One obtains the equation $\partial T_E = \Delta_E - P_E$ where $\Delta_E = \text{Id} \otimes \Delta$ and $P_E = \sum h_{p,q} \otimes ([U_p] \times [S_p])$ with $h_{p,q} : E_p \to E_q$ given by parallel translation along the broken flow line. Thus, $h_p$ corresponds to $\text{Id} : E_p \to E_p$ under the canonical trivializations $E|_{U_p} \cong U_p \times E_p$ and $E|_{S_p} \cong S_p \times E_p$. We obtain the operator equation

$$\delta \circ T_E + T_E \circ \delta = I - P_E$$

(10.1)

where $P_E$ maps onto the finite complex

$$S_E \overset{\text{def}}{=} \bigoplus_{p \in \text{Cr}(f)} E_p \otimes [S_p]$$

by integration of forms over the unstable manifolds. The restriction of $\delta$ to $S_E$ is given as in (4.3) by $d(e \otimes [S_p]) = \sum h_{p,q} \otimes [S_q]$ where $h_{p,q} = (-1)^{\lambda} \sum h_\gamma$ and $h_\gamma : E_p \to E_q$ is parallel translation along $\gamma \in \Gamma_{p,q}$. By (10.1) the complex $(S_E, d)$ computes $H^*(X; E)$.

Reversing time in the flow shows that the complex $\mathcal{U}_{\hat{E}} = \bigoplus_p E^*_p \otimes [U_p]$ with differential defined as above computes $H^*(X; E^*)$. As in §5 the dual pairing of these complexes establishes Poincaré duality. Furthermore, this extends to integral currents twisted by representations of $\pi_1(X)$ in $\text{GL}_n(\mathbb{Z})$ or $\text{GL}_n(\mathbb{Z}/p\mathbb{Z})$ and gives duality with local coefficient systems.

§11. Cohomology operations. Our method has many extensions. For example, consider the triple diagonal $\Delta_3 \subset X \times X \times X$ as the kernel of the wedge-product operator $\wedge : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X)$. Let $f$ and $f'$ be functions with Morse-Stokes flows $\varphi_t$ and $\varphi'_t$ respectively. Assume that for all $(p, p') \in \text{Cr}(f) \times \text{Cr}(f')$ the stable manifolds $S_p$ and $S'_{p'}$ intersect transversely in a manifold of finite volume, and similarly for the unstable manifolds $U_p$ and $U'_{p'}$. Degenerating $\Delta_3$ gives a kernel

$$T \equiv \{(\varphi_t(x), \varphi'_t(x), x) \in X \times X \times X : x \in X \text{ and } 0 \leq t < \infty\}$$

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and a corresponding operator $T : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ of degree -1. One calculates that $\partial T = \Delta_3 \cdot M$ where $M = \sum_{(p,p')} [U_p] \times [U_{p'}] \times [S_p \cap S_{p'}]$. The corresponding operator $M : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ is given by

\begin{equation}
M(\alpha, \beta) = \sum_{(p,p') \in \text{Cr}(f) \times \text{Cr}(f')} \left( \int_{U_p} \alpha \right) \left( \int_{U_{p'}} \beta \right) [S_p \cap S_{p'}].
\end{equation}

The arguments of §§1-3 adapt to prove the following.

**Theorem 11.1.** There is an equation of operators $\wedge \cdot M = d \circ T + T \circ d$ from $\mathcal{E}^*(X \times X)$ to $\mathcal{E}^*(X)$ (where $\wedge$ denotes restriction to the diagonal). In particular for $\alpha, \beta \in \mathcal{E}^*(X)$ we have the chain homotopy

\begin{equation}
\alpha \wedge \beta - M(\alpha, \beta) = dT(\alpha, \beta) + T(d\alpha, \beta) + (\text{deg} \alpha) T(\alpha, d\beta).
\end{equation}

Note that the operator $M$ has range in the finite dimensional vector space $M \overset{\text{def}}{=} \text{span}_R \{ [S_p \cap S_{p'}] \}_{(p,p')}$. It converts a pair of smooth forms $\alpha, \beta$ into a linear combination of the pairwise intersections of the stable manifolds $[S_p]$ and $[S_{p'}]$. If $d\alpha = d\beta = 0$, then $M(\alpha, \beta) = \alpha \wedge \beta - dT(\alpha, \beta)$, and so $M(\alpha, \beta)$ is a cycle homologous to the wedge product $\alpha \wedge \beta$. This operator maps onto the subspace $M$, and for forms $\alpha, \beta \in \mathcal{E}^*(X)$, it satisfies the equation

\begin{equation}
dM(\alpha, \beta) = M(d\alpha, \beta) + (\text{deg} \alpha) M(\alpha, d\beta)
\end{equation}

It follows that $d(M) \subset M$. Furthermore, for $(p,p') \in \text{Cr}(f) \times \text{Cr}(f')$ one has

\begin{equation}
d[S_p \cap S_{p'}] = \sum_{q \in \text{Cr}(f)} n_{pq} [S_q \cap U_{p'}] + (\text{deg} - 1) \sum_{q' \in \text{Cr}(f')} n_{p'q'} [S_p \cap U'_{q'}]
\end{equation}

where the $n_{pq}$ are defined as in §4. Thus we retrieve the cup product over the integers in the Morse complex.

A basic example of a pair satisfying these hypotheses is simply $f, -f$ where the gradient flow is Morse-Stokes. Here $U_p = S_p$ and $U'_{p'} = U_{p'}$ for all $p \in \text{Cr}(f)$, and we have:

**Proposition 11.2.** Suppose the gradient flow of $f$ is Morse-Stokes and $Y$ is a cycle in $X$ which is transversal to $U_p$ and $S_p$ for all $p \in \text{Cr}(f)$. Then for any closed forms $\alpha, \beta$

\[ \int_Y \alpha \wedge \beta = \sum_{p,p' \in \text{Cr}(f)} \left( \int_{U_p} \alpha \right) \left( \int_{S_{p'}} \beta \right) [S_p \cap U_{p'} \cap Y]. \]

Analogous formulas are derived by the same method for the comultiplication operator.

Following ideas in [BC] one can extend these constructions to other cohomology operations.

**§12. Knot invariants.** From the finite volume flow 1.7 our methods construct an $(n+1)$-current $T$ on $S^n \times S^n$ with the property that $\partial T = S^n \times \{*\} + \{*\} \times S^n$. This is a singular analogue of the form used by Bott and Taubes [BT] to study knot invariants. Employing it in their fashion leads to interesting formulas for such invariants.
REFERENCES


