Loop Transfer Matrix and Loop Quantum Mechanics

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Abstract

\[ K_{Q_i, Q_f} \]

\[ \kappa \]

\[ d \]

\[ d \]

\[ d \]

\[ d \]

\[ d \]

\[ d \]
1 Introduction

Various models of random surfaces built out of triangles embedded into continuous space \( \mathbb{R}^d \) and surfaces built out of plaquettes embedded into Euclidean lattice \( \mathbb{Z}^d \) have been considered in the literature [1, 2, 15]. These models are based on area action and suffer the problem of non-scaling behaviour of the string tension and the dominance of branched polymers [3]. Several studies have analyzed the physical effects produced by rigidity of the surface introduced by adding dimension-less extrinsic curvature term to the area action [4]

\[
T \cdot S_{\text{area}} - F \frac{\text{extrinsic curvature}}{\alpha}, \\
\alpha \\
F \frac{\text{extrinsic curvature}}{\alpha} \propto
\]

where \( T \) is a string tension, \( \alpha \) is dimension-less coupling constant and \( F(\text{extrinsic curvature})/\alpha \) is dimension-less functional. Comprehensive review of work in this area up to 1997 can be found in [3].

In [5, 6] authors suggested so-called gonihedric model of random surfaces which is also based on the concept of extrinsic curvature, but it differs in two essential points from the models considered in the previous studies. First it claims that only extrinsic curvature term should be considered as the fundamental action of the theory

\[
S_{\text{gonihedric}} = m \cdot A_{\text{extrinsic curvature}},
\]

where \( m \) has dimension of mass and there is no area term in the action. Secondly it is required that the dependence on the extrinsic curvature should be such that the action will have dimension of length

\[
A_{\text{extrinsic curvature}} \propto \text{length}.
\]

This means that the action should measure the surfaces in terms of length and it should be proportional to the linear size of the surface, as it was for the path integral. The last property will guarantee that in the limit when the surface degenerates into a single world line the functional integral over surfaces will naturally transform into the Feynman path integral for point-like relativistic particle (see Figure 1,2)

\[
m A_{xy} \frac{\text{extrinsic curvature}}{\text{length}} \rightarrow m \int_{x}^{y} dl.
\]

The theory may consistently describe asymptotic freedom and confinement as it is expected to be the case in QCD.

Our aim is to find out what the continuum limit of this regularized theory of random surfaces is, and if indeed the continuum limit corresponds to nontrivial string theory, does it describe QCD phenomena. This is a complicated problem and one of the possible ways to handle the problem is to go further and to formulate the theory also on a lattice. An equivalent representation of the model on Euclidean lattice \( \mathbb{Z}^d \) has been formulated in [7]. This representation depends on dimension \( d \) of the embedding space. In three dimensions

\[1\]

This is in contrast with the previous proposals when the extrinsic curvature term is a dimension-less functional \( F(\text{extrinsic curvature}) \propto 1 \) and can not provide this property.
Figure 1: It is required that the action $A_{xy}$ should measure the surfaces in terms of length, as it was for the path integral. When a surface degenerates into a single world line we have natural transition to a path integral.

It is a spin model in which the interaction between spins is organized in a very specific way (see equation (8)). In high dimensions it is a gauge spin system which contains one, two and three plaquette interaction terms.

In this article I shall consider the above model of random surfaces embedded into 3d Euclidean lattice $\mathbb{Z}^3$. The reason to focus on that particular case is motivated by the fact that one can geometrically construct the corresponding transfer matrix [12] and find its exact spectrum [13]. The spectrum of the transfer matrix which depends only on symmetric difference of initial and final loops has been evaluated exactly in terms of correlation functions of the 2d Ising model in [13]. This transfer matrix has the form [12]

$$K(x, y) = \sum_{\text{paths}} e^{-A_{xy}} \sum_{\text{surfaces}} e^{-A_{xy}}$$

where $x$ and $y$ are closed polygon-loops on a two-dimensional lattice, $k(Q)$ is the curvature and $l(Q)$ is the length of the polygon-loop $Q$. This transfer matrix describes the propagation of the initial loop $Q_1$ to the final loop $Q_2$.

Our aim in this article is to extend these results in several directions. First to construct the transfer matrix for more general case of nonzero self-intersection coupling constant $K_{\kappa=0}(Q_1, Q_2) = K_{\kappa=0}(Q_1; Q_2) \exp\{-\beta \, k(Q_1 \Delta Q_2) \, l(Q_1) \, Q_2 \} \cdot \exp\{-\kappa \beta \, k_{\text{int}}(Q_1) \, l(Q_1 \cap Q_2) \, k_{\text{int}}(Q_2) \}$, where $k_{\text{int}}(Q)$ is number of self-intersection vertices of the loop $Q$. For $\kappa = 0$ this expression coincides with the previous one. In the limit $\kappa \to \infty$, one can see that on every time slice there will be no loops with self-intersections and we have propagation of nonsingular self-avoiding oriented loops.

We shall use the word "loop" for the "polygon-loop".

\[ Z^3 \]

\[ K_{\kappa=0} Q_1, Q_2 \exp\{-\beta \, k(Q_1 \Delta Q_2) \, l(Q_1) \, Q_2 \} \cdot \exp\{-\kappa \beta \, k_{\text{int}}(Q_1) \, l(Q_1 \cap Q_2) \, k_{\text{int}}(Q_2) \}, \]

\[ K_{\kappa=0} Q_1, Q_2 \exp\{-\kappa \beta \, k_{\text{int}}(Q_1) \, l(Q_1 \cap Q_2) \, k_{\text{int}}(Q_2) \}, \]

\[ k_{\text{int}} Q \]

\[ \kappa \to \infty, \]

\[ Q \]

\[ \kappa \]
Secondly, I will demonstrate that diagonalization of any transfer matrices
\( K(Q_1 \triangle Q_2) \), which depend only on symmetric difference of loops
\( Q_1 \triangle Q_2 \), can be performed by using a generalization of Fourier transformation in loop space. This transformation will be defined by using the loop eigenfunctions
\( \psi_P(Q) = e^{i \pi s(P \cap Q)} \)
which are numbered by the loop momentum \( P \) and \( s(Q) \) is the area of the region with boundary loop \( Q \). They are in a pure analogy with plane waves in quantum mechanics
\( p(x) = e^{ipx} \). We shall prove that any transfer matrix
\( K(Q_1 \triangle Q_2) \) can be diagonalized by using this loop Fourier transformation
\( K(P_1,P_2) = \sum_{\{Q_1,Q_2\}} K(Q_1 \triangle Q_2) e^{i \pi s(P_1 \cap Q_1) - i \pi s(P_2 \cap Q_2)} \delta(P_1,P_2) \).

Because the transfer matrix plays the role of the evolution operator, one can define a corresponding Hamiltonian operator as
\( H(P) = -\ln P \).

At this point one can talk about loop quantum mechanics which should be defined by using conjugate loop operators \( ^\ast Q \) and \( ^\ast P \) and the Hamiltonian \( H(\^P;\^Q) \).

I shall also consider the model of random three-dimensional manifolds which are build by gluing together tetrahedra in continuous space \( \mathbb{R}^4 \) and out of 3d cubes in 4d Euclidean lattice \( \mathbb{Z}^4 \). This system has very close nature with the gonihedric model because in both cases the action is proportional to the linear size of the manifold [12]. In [12] the authors were able to construct a transfer matrix for this theory as well. This transfer matrix describes the propagation of two-dimensional membrane \( M \) and our main result is that both theories can be solved exactly by using generalization of the Fourier transformation in loop and in membrane spaces. They represent two nontrivial theories in three and four dimensions respectively which can be solved exactly.

In the next sections I shall present background material, the definition of the system of random surfaces in continuous space as well as on Euclidean lattice and describe its basic properties. In the third section the construction of the transfer matrix for the loops will be extended to the case of nonzero self-intersection coupling constant. In the fourth main section we shall introduce the generalization of Fourier transformation in the loop space and demonstrate that loop Fourier transformation allows to diagonalize all transfer matrices which depend only on symmetric difference of loops. The net result is that all eigenvalues \( P \) of 3d transfer matrix are exactly equal to the loop correlation functions of the corresponding 2d system
\( \langle r_1 \cdots r_n \rangle_{2d} \).

In this formula the loop momentum \( P \) is defined by the vectors \( r_1, \ldots, r_n \) (see Figure 8). In its form the last relation is very close with the ones discussed in [16, 17, 18].

In the last section I shall consider the transfer matrix for membranes. This matrix also can be diagonalized by Fourier transformation and all its eigenvalues are equal to the correlation functions of the corresponding 3d system. In particular we shall see that free energy of the membrane system is equal to the free energy of gonihedric system of loops and is equal to the free energy of 2d Ising model.
2 Random Surfaces and Path Integral

Surfaces in Continuous Space.

\[ m \sum_{\langle ij \rangle} \lambda_{ij} \cdot |\pi - \alpha_{ij}|^\zeta \quad \zeta \leq d - \frac{2}{d}, \]

\[ \lambda_{ij} < ij > \quad M \quad \alpha_{ij} < ij > \]

\[ \kappa m \sum_{\langle ij \rangle} \lambda_{ij} \cdot |\pi - \alpha_{ij}^1|^\zeta \quad \ldots \quad |\pi - \alpha_{ij}^r|^\zeta. \]

\[ A M \quad m \sum_{\langle ij \rangle} \lambda_{ij} \cdot |\pi - \alpha_{ij}|^\zeta \quad m \kappa \sum_{\langle ij \rangle} \lambda_{ij} \cdot |\pi - \alpha_{ij}^1|^\zeta \quad \ldots \quad |\pi - \alpha_{ij}^r|^\zeta. \]

\(^3\)The angular factor defines the rigidity of the random surfaces and for \( \zeta \leq (d - 2)/d \) it increases sufficiently fast near angles \( \alpha = \pi \) to suppress transverse fluctuations [6, 10].
\[
\sum_{<ij>} \lambda_{ij} \left[ (\pi - \alpha)^\xi_{ij} + (\pi - \beta)^\xi_{ij} + (\pi - \gamma)^\xi_{ij} \right]
\]

\[
\alpha \quad \beta \quad \gamma \quad \pi
\]

\[
\kappa
\]

4

\[
\sum_{<ij>} \lambda_{ij} \cdot |\pi - \alpha|^\xi \leftrightarrow \sum_{<ij>} \lambda_{ij}.
\]

\[
\sigma_{\text{quantum}} \frac{d}{a^2} - \ln \frac{d}{\beta}, \quad d \quad \zeta \quad \frac{d-2}{d} \quad \beta \rightarrow \beta_c \quad d/e
\]

\[
\sigma_{\text{classical}} A M \rightarrow m R T.
\]

\[
A N \quad m \sum_{<i,j>} \lambda_{ij} \cdot |\pi - \omega_{ij}|^\zeta
\]

\[
\lambda_{ij} < ij > \quad \omega_{ij} < ij >
\]

\[
\pi - \omega_{ij} < ij >
\]

\[4\] This property of the gonihedric action guarantees that spike instability does not appear here because the action is proportional to the total length of the spikes and suppresses the corresponding fluctuations [5, 6].
Equivalent spin system on a lattice [7].

\[ M \]

\[ \pi/ \]

\[ \kappa n_4 \quad n_4 M \]

\[ \kappa \quad \text{singular part} \]

\[ \text{regular part} \quad M \]

\[ Z \beta \sum_{\{\text{surfaces } M\}} e^{-2\beta A(M)}, \]

\[ A(M) \]

\[ A_{gonihedric} M \quad n_2 M \quad \kappa n_4 M. \]

\[ H_{gonihedric}^{3d} = \kappa \sum_{\vec{r}, \vec{\alpha}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha}} - \kappa \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha} + \vec{\beta}} - \kappa \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha}} \sigma_{\vec{r} + \vec{\alpha} + \vec{\beta}} \sigma_{\vec{r} + \vec{\beta}}, \]

\[ \vec{r}, \vec{\alpha}, \vec{\beta} \]

\[ Z^3 \]

\[ H \quad \kappa \]

\[ \kappa/ \]

\[ \kappa \quad \kappa \quad / \]

\[ N \quad N^3 \quad Z^3 \quad \kappa \quad \kappa \]

all states, including the ground state are exponentially degenerate

degenerate

one can flip the spins on arbitrary layers, even on intersecting ones

\[ k Q \]

\[ l Q \]

\[ \text{degenerate} \]

\[ k Q \]
Figure 4: The geometrical theorem provides an equivalent representation of the action $A(M)$ in terms of total curvature $k(E)$ of the curves $Q_1(E), \ldots, Q_k(E)$ which appear in the intersection of the $(d-1)$-dimensional plane $E$ with the given two-dimensional surface $M$.

The curvature $k(E)$ should then be integrated over all planes $E$ intersecting the surface $M$.

Approximations to the full transfer matrix confirm strong connection with the 2d Ising model [13].

Below I shall remind the derivation of the corresponding transfer matrix for the case $\kappa = 0$, which is based on geometrical theorem proven in [11] and then I shall derive the transfer matrix for more general case $\kappa \neq 0$. This will allow to study physical picture of string propagation which follows from the transfer matrix approach.

3 Loop Transfer Matrix


$$H_{gonihedric}^{3d} \kappa \kappa$$

$$\ldots \sum_{\bar{r}, \bar{a}, \bar{\beta}} \sigma_{\bar{r}} \sigma_{\bar{r} + \bar{a}} \bar{a} \sigma_{\bar{r} + \bar{a} + \bar{\beta}} \sigma_{\bar{r} + \bar{\beta}},$$

$$\kappa /$$

$A \ M$

$\begin{array}{c}
\kappa \\
E \\
\kappa \\
M
\end{array}$

$\begin{array}{c}
\kappa \\
E \\
\kappa \\
M
\end{array}$

$\begin{array}{c}
\kappa \\
E \\
\kappa \\
M
\end{array}$

$Q_1 \ E, \ldots, Q_k \ E$
\[ \kappa(E) = \sum_{\langle i,j \rangle} |\pi - \alpha_{ij}^E| \]

\[ k \sum_{i=1}^{k} k Q_i - \sum_{\langle i,j \rangle} |\pi - \alpha_{ij}^E|, \]

\[ \alpha_{ij}^E \]

\[ \sum_{1}^{k} E \]

\[ Q_1 E, ..., Q_k E \]

\[ Z \]

\[ \sum_{\{E\}} \exp\left\{-\beta \sum_{\{E\}} k E \right\} \]

\[ \text{Geometrical Theorem on a lattice [12].} \]
\[
\exp\left\{-\beta \sum_{\{E\}} k \cdot E \right\} \prod_{\{E\}} e^{-2\beta k(E)} \prod_{\{E_z\}} e^{-2\beta k(E_z)} \prod_{\{E_y\}} e^{-2\beta k(E_y)} \prod_{\{E_x\}} e^{-2\beta k(E_x)}.
\]

Transfer Matrix for \( \kappa \) [12].

\[
\begin{array}{c}
k \cdot E_y \\
\{E_y\} \quad \{E_x\}
\end{array}
\quad
\begin{array}{c}
k \cdot E_x \\
\{E_x\}
\end{array}
\quad
\begin{array}{c}
M \\
E^i_z \quad E^{i+1}_z
\end{array}
\quad
\begin{array}{c}
Q_i \\
E_i \quad E_{i+1}
\end{array}
\quad
\begin{array}{c}
M \\
k \cdot E_x \\
k \cdot E_y
\end{array}
\quad
\begin{array}{c}
Q_{i+1} \\
E_z \quad E^{i+1}_z
\end{array}
\quad
\begin{array}{c}
M
\end{array}
\]

\[
l \cdot Q_i \quad l \cdot Q_{i+1} = l \cdot Q_i \cap Q_{i+1} \quad l \cdot Q_i \triangle Q_{i+1},
\]

\[
Q_1 \triangle Q_2 \equiv Q_1 \cup Q_2 \setminus Q_1 \cap Q_2 \quad Q_1 \cup Q_2
\]

\[
A \cdot M \sum_{\{E\}} k \cdot E \sum_{\{E_z\}} k \cdot Q_i \quad l \cdot Q_i \triangle Q_{i+1}.
\]

\[
Z \beta \sum_{\{Q_1, Q_2, \ldots, Q_N\}} K_\beta Q_1, Q_2 \cdots K_\beta Q_N, Q_1 \quad \text{tr} K_\beta^N,
\]

\[
K_\gamma Q_1, Q_2 \quad \gamma \times \gamma
\]

\[
K_{\kappa=0}^{\text{gonahedric}} Q_1, Q_2 \quad \exp\left\{-\beta \cdot k \cdot Q_1 \quad l \cdot Q_1 \triangle Q_2 \quad k \cdot Q_2 \right\},
\]

\[
Q_1 \quad N^2 \quad Q_2
\]

\[
\begin{array}{c}
T^2 \\
Q_2
\end{array}
\quad
\begin{array}{c}
T^2 \\
Q_1
\end{array}
\quad
\begin{array}{c}
Q \quad \delta \\
Q_1 \triangle Q_2 \quad Q_1 \quad Q_2
\end{array}
\]

\[
l \cdot Q \quad l \cdot Q_1 \triangle Q_2 \quad Q_1 \triangle Q_2 \quad Q_1 \triangle Q_2
\]

\[
l \cdot Q_1 \triangle Q_2 \quad Q_1 \triangle Q_2 \quad Q_1 \triangle Q_2 \quad Q_1 \triangle Q_2
\]

---

\(^5\)Layer-to-layer transfer matrices for three-dimensional statistical systems have been considered in the literature [20, 21]. Using Yang-Baxter and Tetrahedron equations one can compute the spectrum of the transfer matrix in a number of cases [21]. In the given case the transfer matrix has geometrical interpretation which helps to compute the spectrum.

\(^6\)Note that the operations \( \cup \) and \( \cap \) do not have this property. These operations acting on a polygon-loops can produce link configurations which do not belong to \( \Pi \). The symmetric difference of sets \( Q_1 \triangle Q_2 \) is an important concept in functional analysis [22].
Figure 6: The set of planes \( f_x, f_y, f_z \) perpendicular to \( x, y, z \) axis intersect a given surface \( M \) in the middle of the links. On each of these planes we shall have an image of the surface \( M \). Every such image is represented by closed polygon-loops \( P(E) \) appearing in the intersection of the plane with surface \( M \). The energy of the surface \( M \) is equal to the sum of the total curvature \( k(E) \) on all these planes as it is given by the formula (12).

For the surface on the picture this sum is equal to \( A = 16 + 20 + 20 = 56 \) times \( \frac{\pi}{2} \).

Figure 7: The transfer matrix (18) can be viewed as describing the propagation of the polygon-loop \( P_1 \) at time \( t \) to another polygon-loop \( P_2 \) at time \( t+1 \). For the surface on the picture we have

\[
\begin{align*}
k(E_1) &= 4; \quad k(E_2) = 12; \\
\ell(P_1 \triangle \phi) &= 12; \\
\ell(P_2 \triangle \phi) &= 16.
\end{align*}
\]

Summing all these quantities in accordance with the formula (16) we shall get \( A(M) = 4 + 12 + 12 + 16 + 12 = 56 \) times \( \frac{\pi}{2} \). This, as it should, coincides with the previous result.
The eigenvalues of the transfer matrix \( K_{Q_1, Q_2} \) define all statistical properties of the system and can be found as a solution of the following integral equation in the loop space \([13]\):

\[
\int f_{Q_2} K_{Q_1, Q_2} \Psi(Q_2) = \Psi(Q_1),
\]

where \( \Psi(Q) \) is a function on loop space. The Hilbert space of complex functions \( \Psi(Q) \) on \( \mathbb{L}^2 \) will be denoted as \( H = L^2(\mathbb{L}) \). The eigenvalues define the partition function \( Z(3d) = N_0 + \cdots + N_\gamma - 1 \), \( (20) \).

In the thermodynamical limit the free energy is equal to

\[
- f(3d) = \lim_{N \to \infty} \frac{1}{N^2} \ln Z(3d); \quad (21)
\]

Finite time propagation amplitude of an initial loop \( Q_i \) to a final loop \( Q_f \) for the time interval \( t = M \) can be defined as

\[
K_{Q_i, Q_f} = \sum_{(Q_1, Q_2, \ldots, Q_{M-1})} K_{Q_i, Q_1} \cdots K_{Q_{M-1}, Q_f} \exp \left( -\beta \int_{E_{Q_i}} \kappa \cdot \sum_{\{E_z\}} k_{int} Q_i \right),
\]

where we have introduced natural normalization to the biggest eigenvalue \( \delta = 0 \).

Loop Transfer Matrix for \( \kappa / \).
Now let us consider oriented polygon-loops on two planes \( E_i \) and \( E_{i+1} \). Then one should select that bonds on these polygon-loops which are common. Only bonds with the opposite orientation should be counted. The number of common bonds with opposite orientation we shall denote as \( l(Q_i \cap \cap Q_j) \). The notation \( \cap \) is used to denote the common bonds, the symbol \( \cap \) on the top of it to denote that they should have opposite orientations.

Therefore the contribution from horizontal edges can be expressed as

\[
4 \sum_{E_i} f(E) g(l(Q_i \cap \cap Q_j)) \quad (25)
\]

and the contribution from all self-intersections as

\[
4 \sum_{E_i \cap E_j} (k \text{int}(Q_i) + 4 l(Q_i \cap \cap Q_{i+1})) + 4 l(Q_i \cap \cap Q_{i+1}) \quad (26)
\]

The total action (7) now takes the form:

\[
A(M) = \sum_{E_i} f(E) g(l(Q_i \cap \cap Q_j)) + \kappa \cdot \sum_{E_i} k \text{int}(Q_i) l(Q_i \cap \cap Q_{i+1}) \quad (27)
\]

The partition function (13) can be now represented in the same form (17) where \( K(Q_1, Q_2) \) is again the transfer matrix of size \( \gamma \times \gamma \), defined as

\[
K(Q_1, Q_2) = \exp\left\{-\beta k(Q_1 l(Q_2) \Delta Q_2) - \kappa \cdot \text{int}(Q_1 l(Q_1 \cap \cap Q_2) \cap \cap Q_2)\right\},
\]

\[
N \times N \quad \gamma \times \gamma \quad \gamma \times \gamma \quad T^2 \quad T^2 \quad T^2 \quad T^2
\]

\[
\kappa \rightarrow \infty,
\]

4 Loop Fourier transformation

\[
K(Q_1, Q_2)
\]

\[
k(Q_1 \cup Q_2)
\]

\[
K(Q_1 \Delta Q_2) \exp\{-\beta k(Q_1 \Delta Q_2) l(Q_1 \Delta Q_2)\},
\]
\[ K_{Q_1 \triangle Q_2} \exp\{-\beta \int_{Q_1 \triangle Q_2} \}. \]

\[ \sum_{\{Q_2\}} K_{\beta} Q_1 \triangle Q_2 \quad Q_2 \quad Q \quad Q_1. \]

\[ \sum_{\{Q\}} K_{\beta} Q \quad Q \triangle Q_1 \quad \beta \quad Q_1. \]

\[ \int K_{\beta} x - y \, \psi(x) \, \psi(y) \, dx \quad \lambda \beta \psi(y), \quad K_{\beta} x - y \quad \exp\{-\beta \, x - y \}^2 \]

\[ \psi(x) \quad e^{ipx}, \quad \lambda_p x \quad e^{-p^2/4\beta}. \]

\[ \emptyset \quad 2 \quad Q \]

\[ Q \quad \emptyset \quad \emptyset \]

\[ 2 \quad Q \quad \Rightarrow \quad Q \quad \pm. \]
This suggests the following form of the eigenfunctions \( P \Psi \); the number of functions in this orthogonal set is equal therefore to the number of closed loops which are numbered by the loop momentum \( s \).

Because the expression \( \exp \sum_{Q} g_{Q} \Psi_{P} \) should interpret the loop \( s \), the solutions are in pure analogy with plane \( 2d \) Ising: \( \sum_{\{Q\}} e^{i\pi s(P \cap Q)} = 0 \) if \( P \) is nonempty, \( \sum_{\{Q\}} e^{i\pi s(P \cap Q)} = 1 \) if \( P \) is empty.

In what follows we shall consider the transfer matrix (30). Analogous formulas are valid for the transfer matrix (29) with curvature term \( k(Q) \).
identically equal to the spin correlation functions of the 2d Ising model

\[ P(\vec{r}_1, \ldots, \vec{r}_n) = \langle \sigma_{\vec{r}_1} \cdots \sigma_{\vec{r}_n} \rangle_{2dIsing} \]

where the loop momentum \( P \) is defined by the vectors \( \vec{r}_1, \ldots, \vec{r}_n \). In particular for the smallest (one-box) loop momentum \( P = 2 \) we shall have

\[ \langle \vec{r} \rangle_{2dIsing} = \langle \vec{r} \rangle = \langle \vec{r}_{r+1} \rangle = -u \beta \]

which coincides with the magnetization of 2d Ising model. Because it does not depend on position of the loop on the lattice, this eigenvalue is \( N \) degenerate. For the two-box loop momentum \( P = \leq \) consisting of two neighboring boxes we shall have

\[ \langle \vec{r} \rangle_{2dIsing} = \langle \vec{r} \rangle = \langle \vec{r}_{r+1} \rangle = -u \beta \]

which is \( 2N \) degenerate and after summation over all \( 2N \) bonds of the lattice we shall get the internal energy of the 2d Ising model.

We can also use loop wave functions to define generalized loop-Fourier transformation as

\[ \Psi(P) = \frac{1}{2N} \sum_{\{Q\}} e^{i\pi s(P \cap Q)} \]

and then compute the loop Fourier transformation of the transfer matrices (29) and (30) as:

\[ K(P_1, P_2) = \sum_{\{Q_1, Q_2\}} K(Q_1 \triangle Q_2) e^{i\pi s(P_1 \cap Q_1) - i\pi s(P_2 \cap Q_2)} \]

\[ t = \frac{M}{\beta} \]

\[ K(P_i, P_f) = \frac{P_1}{M} \delta P_i, P_f e^{M \ln \left( \frac{P}{\beta} \right)} \delta P_i, P_f e^{-M H(P_i)} \delta P_i, P_f \]

\[ H = -\ln \frac{P}{P_0} \]

Spin system which corresponds to the matrix \( K Q_1 \triangle Q_{i+1} \) (30).
Figure 8: The loop momentum $P \sim r_1; \ldots; r_n$ is defined by the vectors $r_1; \ldots; r_n$ and the corresponding eigenvalue $P_{r_1, \ldots, r_n}$ is equal to the correlation function of the $n$ spins $\langle r_1 \cdots r_n \rangle$. The contribution coming from the edges which are parallel to the $x$ and $y$ direction are left and their contribution is equal to (15):

$$X f E g_l (Q_1 + Q_{i+1})$$

(54)

In terms of spin variables the original action can be expressed as a sum of energies coming from all edges of the surface (see formulas (5a) and (7) in [7])

$$H = \sum_{\text{over all edges}} H_{\text{edge}},$$

(55)

$$H_{\text{edge}} = U_1 U_{-1} U_1 U_{-1} \sigma_1 \sigma_2 \sigma_{-1} \sigma_{-2}.$$}

(56)

is a four-spin interaction term with spins distributed around the edge. Because we ignore the edges in the $z$ direction, we have to sum in (55) only over $A_{\text{edges}}$ in $x$ and $y$ direction. This corresponds to the summation over four-spin interaction terms (56) which lie only on the planes $E_x$ and $E_y$. These planes are normal to the $x$ and $y$ direction. Therefore our approximation corresponds to the 3d lattice spin system in which four-spin interactions take place only on vertical planes $E_x$ and $E_y$:

$$H Q_1 \Delta Q_2 \sum_{E_x, E_y} \sigma \sigma \sigma \sigma.$$}

(57)

It is obvious that this spin system cannot be factored into non-interacting two-dimensional subsystems. We can ask what is the effective interaction between spins which lie on a given two-dimensional $E_z$ plane? Let us label spins which are on that plane as $\lambda$, spins which are on the previous plane as $\mu$, and spins which are on the next plane as $\nu$. The part of the Hamiltonian which contains only $\lambda$ spins can be written in the form:

$$\sum_i \sigma_i \sigma_{i+1} \lambda_i \lambda_{i+1} \sigma_i \sigma_{i+1} \mu_i \mu_{i+1} \sigma_i \sigma_{i+1} \sum_i J_{\text{eff}} \sigma_i \sigma_{i+1},$$

(58)
where the effective coupling $J_{\text{eff}}$ depends on spins on the neighboring planes. It takes the values $-2, 0, 2$ with probabilities $1/6, 1/3, 1/6$ simply because $J_{\text{eff}} = \sum_i J_{ii} + 1 + \sum_i J_{i+1}$. For the observer who lives on the plane $E_z$ spin interactions look like 2d Ising model with randomly distributed coupling constant $J_{\text{eff}}$.

3d Ising Transfer Matrix. Similar expression for the transfer matrix has been derived in [12] for the 3d Ising model

$$K_{3d} = \text{Tr} K_N e^{-\beta f_{3d}(\beta) N^3},$$

$$p \sum_{\{Q\}} e^{-i \pi s(P \cap Q) - \beta [l(Q) + 2s(Q)]} = \sum_{\{Q\}} e^{-i \pi s(P \cap Q) - 2s(Q)}.$$

\section{Matrices depending only on symmetric difference}

$$Q_1 \triangle Q_2$$

$$Z^{3d} = \text{Tr} K_N e^{-\beta f_{3d}(\beta) N^3},$$

$$p \sum_{\{Q\}} e^{-i \pi s(P \cap Q)} K Q \sum_{\{Q\}} e^{-i \pi s(P \cap Q) - \beta \Omega(Q)}.$$
The eigenvalues \( P \) of three-dimensional system are exactly equal to the correlation functions of the two-dimensional system with the partition function

\[
Z^{2d} \beta \sum_{\{Q\}} e^{-\beta \Omega(Q)} e^{-\beta f_{2d}(\beta) \cdot N^2} \lambda_0^N \cdots \lambda_{2N-1}^N
\]

\[
f_{2d} \beta \lambda_i \quad \text{and} \quad 0
\]

\[
\sum_{\{Q\}} e^{-\beta \Omega(Q)} \equiv Z^{2d} \beta \lambda_0^N \quad \lambda_0^N \cdots \lambda_{2N-1}^N
\]

\[
\text{N} \rightarrow \infty \ln Z^{2d} \beta \text{N} \rightarrow \infty \ln \{N \cdots\} - \beta f_{2d} \beta
\]

\[
\sum_{\{Q\}} e^{-i \pi s(P \cap Q) - \beta \Omega(Q) / Z^{2d} \beta} < e^{-i \pi s(P \cap Q)}> Q < \sigma_{r_1} \cdots \sigma_{r_n}>_{2d}
\]

\[
\frac{P_{r_1 \cdots r_n}}{\emptyset} < \sigma_{r_1} \cdots \sigma_{r_n}>_{2d}
\]

\[
\chi M \sum_{\{Q\}} \pi - \omega_i
\]

**Transfer Matrix for Membranes**

\[
K_{\beta} M_1, M_2 \exp\{-\beta \chi M_1 \Delta M_2 \chi M_2\}
\]

\[
\chi M \sum_{\{Q\}} \pi - \omega_i
\]
\[ \chi_{P} M = e^{i\pi v(P \cap M)}, \]

\[ v M \]

\[ K M_1, M_2 \exp\{-\beta \chi M_1 \triangle M_2 A M_1 \triangle M_2\}, K M_1, M_2 \exp\{-\beta A M_1 \triangle M_2\}, \]

\[ P \sum_{\{M\}} e^{-i\pi v(P \cap M) - \beta \chi(M) + 2A(M)}, P \sum_{\{M\}} e^{-i\pi v(P \cap M) - 2\beta A(M)}. \]

\[ P \emptyset \sum_{\{M\}} e^{-2\beta A(M)}, P \emptyset \sum_{\{M\}} e^{-2\beta A(M)}. \]

\[ f_{4d} \beta \quad f_{3d} \beta \quad f_{2d - Ising} \beta \]

\[ \frac{P}{\emptyset} 4d Correlation functions^{3d}. \]

6 Discussion

We have seen that transfer matrix approach allows to solve large class of statistical systems when transfer matrix depends on symmetric difference of propagating loops, membranes or p-branes. The eigenvalues are equal to the correlation functions of the corresponding statistical system, which has smaller dimension than the original one. This dimensional reduction expresses the hierarchical structure in which more complicated high-dimensional systems are a superposition of lower-dimensional ones [12]. This structure is especially visible when one uses generalization of Fourier transformation in loop space or in p-brane spaces.

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References
