Action of Singular Instantons of Hawking–Turok Type

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(May 11, 2000)

Abstract

Using Kaluza–Klein technique we show that the singularity of Hawking–Turok type has a fixed point (bolt) contribution to the action in addition to the usual boundary contribution. Interestingly by adding this contribution we can obtain a simple expression for the total action which is feasible for both regular and singular instantons. Our result casts doubt on the constraint proposed by Turok in the recent calculation in which Vilenkin’s instantons are regarded as a limit of certain constrained instantons.
1 Introduction

Hawking and Turok [1, 2] found that a family of gravitational instantons exist for a generic potential for the scalar field. In the Hartle–Hawking ‘no-boundary’ proposal [3] the universe is supposed to be created from a compact instanton, so that Hawking–Turok instantons have a significant importance. The novelty is the allowance of singularities of special kind (Hawking–Turok (HT)-singularity) in the curvature and the scalar field. Hawking and Turok argued that the use of such singular instantons is justified because of the finiteness of the action. In addition, the singularity is mild and behaves as a reflecting boundary for scalar and tensor cosmological perturbations [1, 4, 5, 6]. Hence the Cauchy problem seems to be well posed and the model is well suited for the quantisation of small perturbations and for comparison with observations. As another justification, Wu [7] argued that Hawking–Turok instantons may be regarded as ‘constrained gravitational instantons,’ whose action is required to be stationary under the constraint that the 3-geometry is given on the three-surface where quantum transition occurs.

On the other hand, there is a criticism by Vilenkin [8] that asymptotically flat HT-singular instantons would destabilise the flat spacetime. Among such instantons especially the ones which do not have the potential for the scalar field are called as Vilenkin’s instantons. To this criticism it was argued [9] that Vilenkin’s instanton can be taken as compactified five-dimensional Schwarzschild metric and the large extra dimension would metastabilises the flat spacetime. This mechanism was first discussed by Witten [10]. Turok [11] argued that with a careful definition of a constraint Vilenkin’s instantons possess no negative mode, so that they do not lead to the decay of flat spacetime.

With such arguments it is desirable to examine the contribution of the HT-singularity to the action with much care. Most simply it is calculated as follows [8, 12]: at first we consider a manifold $M'$ which consists of the whole spacetime $M$ except a four-sphere which contains the singular point and then make the excised four-sphere shrink. In the zero-volume limit of the excised four-sphere, the Gibbons–Hawking boundary term [13, 14] of the boundary of $M'$ would give the contribution of the singularity. Other than this simple method, there are several attempts which do not agree with each other. Garriga [4] calculated the contribution of the singularity by regularizing the singularity with a membrane which wraps the singularity.
and found that the contribution of the singularity is just 1/3 of the Gibbons–Hawking term. But the method was criticised by Bousso and Chamblin [15] because of allowing negative energy density. In ref. [15] it is considered to regularise the singularity with a membrane of a positive energy density which couples to four-form field strength. Garriga calculated in ref. [9] the action of Vilenkin’s instanton by showing that compactified five-dimensional Schwarzschild metric looks as Vilenkin’s instanton and obtained the value of the 1/3 of the Gibbons–Hawking term. In ref. [16] González-Díaz calculated the action of Vilenkin’s instanton which is replaced the singularity with an axionic wormhole and obtained the value zero. Turok [11] considered to define Vilenkin’s instantons as a limit of constrained instantons which are non-singular and have no boundary, and obtained the value of the whole Gibbons–Hawking term as the value of the action.

Although the HT-singularity has been often regularised with new matter in these calculations, the singularity at the transition surface from Euclidean to Lorentzian metric would be allowed if we take HT-singular instantons as constrained gravitational instantons. On the other hand the simplest method mentioned above to calculate the contribution of the HT-singularity would not be justified because the contribution of the excised spacetime is overlooked. For example, in the case of the Euclidean Schwarzschild metric the limiting value of Gibbons–Hawking term does not contribute to the action although it has a non-zero value. For that reason let us consider to calculate the contribution of the excised HT-singularity. To do this we review the case of the Euclidean Schwarzschild metric in the next section. Then we apply it to the case of the HT-singular instantons using Kaluza–Klein technique.

2 The case of Euclidean Schwarzschild metric: A simple example

The Lorentzian Schwarzschild metric is given by

\[ ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 + r^2d\Omega_2^2. \]  

(1)
Euclideanising by $t = -i\tau$ and putting $x = 4M\sqrt{1 - 2MG/r}$, we obtain [17]

$$ds^2 = \left(\frac{x}{4MG}\right)^2 d\tau^2 + \left(\frac{r^2}{4M^2G^2}\right)^2 dx^2 + r^2 d\Omega^2_2.$$  \hfill (2)

This metric will be regular at $x = 0$, i.e. $r = 2MG$, if $\tau$ is regarded as an angular variable with period $8\pi MG$. The Euclidean Schwarzschild metric is defined on the manifold given by $x \geq 0$, $0 \leq \tau \leq 8\pi GM$. We denote this spacetime as $\mathcal{M}$ in this section. The submanifold $x = 0$ is called a bolt [18], which is the two-dimensional fixed point (FP) set of the periodic imaginary time isometry.

Let us consider to calculate the action of $\mathcal{M}$ by using a spacetime $\mathcal{M}'$ which has an outer boundary three-surface $\partial \mathcal{M}'_{\text{out}}$ at infinity and an inner boundary three-surface $\partial \mathcal{M}'_{\text{in}}$ at $x = x_\epsilon$ in the vanishing limit of $x_\epsilon$. Since the metric is a vacuum solution, the Ricci scalar is zero everywhere and the action $S(\mathcal{M}')$ comes entirely from the boundary terms. The Gibbons–Hawking boundary term [13, 14] is given by the integral over the boundary

$$S_{GH} = -\frac{1}{8\pi G} \int \! d^3\xi \sqrt{h} (K - K_0),$$  \hfill (3)

where $h$ is the determinant of the induced metric on the boundary, $K$ and $K_0$ are the trace of the extrinsic curvature of the boundary and the boundary imbedded in flat spacetime, respectively. The second term is included when we consider a non-compact spacetime. For calculation of the Gibbons–Hawking term it is convenient to use the formula

$$\int \! d^3\xi \sqrt{h} K = \partial_{\text{normal}}(\text{Volume of boundary}).$$ \hfill (4)

The boundary term of the outer boundary $\partial \mathcal{M}'_{\text{out}}$ at infinity gives the contribution $4\pi M^2G$. In addition, the boundary term of the inner boundary $\partial \mathcal{M}'_{\text{in}}$ is also non-vanishing:

$$S(\partial \mathcal{M}'_{\text{in}}) = \lim_{x_\epsilon \to 0} S_{GH} = 4\pi M^2G.$$ \hfill (5)

But this term is not expected to contribute to the action $S(\mathcal{M})$ of the whole spacetime, because the metric is regular at the $x = 0$ and so the spacetime
\( \mathcal{M} \) has no boundary there. Therefore we expect that the contribution of the excised spacetime (in this case, the bolt) will be non-zero and will compensate the contribution of the new boundary which is added by considering \( \mathcal{M}' \) instead of \( \mathcal{M} \).

To confirm this, we consider a four-sphere \( \mathcal{M}_B \) which is centered around the bolt and whose boundary three-surface is at \( x = x_\varepsilon \). We define \( S(\text{FP}) \) as the action of \( \mathcal{M}_B \) in the limit of \( x_\varepsilon \to 0 \). Because the Euclidean Schwarzschild metric is regular at the bolt, we can evaluate this term easily: In the vanishing limit of \( x_\varepsilon \) the bulk term vanishes and the action comes entirely from the Gibbons–Hawking term of \( \partial \mathcal{M}_B \), and we need not to calculate it because it is given by the Gibbons–Hawking term of \( \partial \mathcal{M}_{in} \) only with the reversal of the direction of the normal of the boundary. So we obtain

\[
S(\text{FP}) = \lim_{x_\varepsilon \to 0} S_{GH} = -S(\partial \mathcal{M}_{in}).
\]

(7)

The contribution of the bolt cancels with the inner boundary contribution of \( \mathcal{M}' \), as expected. Therefore using \( \mathcal{M}' \) we obtain the correct action of \( \mathcal{M} \) as

\[
S(\mathcal{M}) = S(\mathcal{M}') + S(\text{FP}) = 4\pi M^2 G.
\]

(8)

### 3 The case of singular instantons

In this section we calculate the fixed point contribution of HT-singularities. The method to calculate it presented in the previous section cannot be applied directly because the excised spacetime is not regular in this case. But we show that we can go around this obstruction by using Kaluza–Klein technique.

Introduction of an extra dimension in HT-singular instantons was first discussed by Garriga [6, 9]. He showed that Vilenkin’s instanton may be reinterpreted à la Kaluza–Klein as the Euclidean Schwarzschild solution of five-dimensional gravity, and calculated the action of Vilenkin’s instanton as the action of the Euclidean Schwarzschild solution. He argued that if the fifth dimension is physical and we can interpret the HT-singularity as such, then the large extra dimension as in the \( \mathcal{M} \)-theory metastabilises the flat
space\textsuperscript{time}. But at the same time he also argued that the five-dimensional interpretation would not work if a general potential term is present, because HT-singular instantons can be represented as compactified five-dimensional regular metrics only when the initial value of the scalar field takes a particular value. He exemplified this by using a model of five-dimensional gravity with cosmological constant and argued that this may indicate that the family of HT-singular instantons would not exist.

But if we take the HT-singular instantons as ‘constrained gravitational instantons [7],’ the singularity does not need the justification by five-dimensional regularity. Hence in this paper we use the five-dimensional theory as a mere trick only to calculate the contribution of HT-singularity. Then the difficulty mentioned above disappears (see also the discussions in the final section). We show that near HT-singularity any HT-singular instantons can be approximated as five-dimensional regular metrics on compactification. It turns out that the HT-singularity corresponds to a bolt in five dimensions. Thus we can calculate the fixed point contribution of HT-singularities by evaluating the bolt contribution in five dimensions. The use of an extra dimension may be seen as a method to constrain HT-singularities.

First we review the behaviour of HT-singular instantons. The action of the HT-singular instantons is given by

\begin{equation}
S = \int d^4x \sqrt{g} \left( -\frac{R}{16\pi G} + \frac{1}{2}(\partial\phi)^2 + V(\phi) \right) + B, \tag{11}
\end{equation}

where $B$ represents the boundary contribution $S(\text{boundary})$ and the fixed point contribution of the HT-singularity $S(\text{FP})$. We seek solutions of $O(4)$ symmetric metric [1]

\begin{equation}
ds^2 = d\sigma^2 + b^2(\sigma) d\Omega_3^2 = d\sigma^2 + b^2(\sigma) \left( d\psi^2 + \sin^2(\psi) d\Omega_3^2 \right). \tag{12}
\end{equation}

The equations of motion read

\begin{equation}
\phi'' + 3 \frac{b'}{b} \phi' = \frac{\partial V}{\partial \phi}, \quad b'' = \frac{8\pi G}{3} b \left( \phi'^2 + V \right), \tag{13}
\end{equation}

where primes denote derivatives with respect to $\sigma$. There are two choices of boundary conditions:

1) Hawking–Turok’s case [1]. The metric and the scalar field are non-singular at $\sigma = 0$, i.e.

\begin{equation}
b(\sigma) \simeq \sigma \quad (\sigma \simeq 0), \quad \phi'(0) = 0. \tag{14}
\end{equation}
We can set the value \( \phi(0) = \phi_0 \) arbitrarily except values which correspond to regular instanton solutions.

2) Asymptotically flat case (including Vilenkin's case [8]).

\[
b(\sigma) \simeq \sigma, \quad \phi(\sigma) \to \phi_\infty \quad (\sigma \to \infty),
\]

where \( \phi_\infty \) represents the value which the field \( \phi \) takes at the extremum of the potential, \( \frac{\partial V}{\partial \phi}(\phi_\infty) = 0 \) and \( V(\phi_\infty) = 0 \).

Because \( b \) goes to zero as we approach the singularity, the (anti-)damping term \( 3b'\phi' / b \) in the equation of motion overwhelms the potential term \( \partial V / \partial \phi \) near singularity, if the potential is not too steep there. Hence HT-singular instantons have the similar behaviours near the singularity \( \sigma = \sigma_* \), regardless of the potential and the choice of the boundary conditions, as

\[
\begin{align*}
  b^3(\sigma) & \simeq \kappa C |\sigma_* - \sigma| \\
  \phi(\sigma) & \simeq \pm \frac{1}{\kappa} \log |\sigma_* - \sigma| + \text{const.},
\end{align*}
\]

where \( \kappa = \sqrt{12\pi G} \) and \( C \) is a constant. We see that the approximation we used is consistent if the potential is

\[
\text{not so steep as } V(\phi) \sim e^{2\kappa|\phi|} \text{ near singularity.}
\]

Using Eq.(16), (17), (18) and the relation \( \mathcal{R} = 6(-bb'' - b^2 + 1)/b^2 \), we can check that the bulk terms of the action Eq.(11) do not contribute at the singularity.

Secondly let us calculate the boundary contribution \( S(\text{boundary}) \) at the singularity. From the whole spacetime \( \mathcal{M} \) we excise a four-sphere which is centered around the singularity and of coordinate radius \( \sigma_\varepsilon \) and call the remaining spacetime as \( \mathcal{M}' \). We consider to take the limit of \( \sigma_\varepsilon \to 0 \). Hereafter we only consider the boundary at singularity. The contribution of the boundary at infinity is restored, if any, in the end of the calculation, Eq.(34). Then the action contribution of the boundary of \( \mathcal{M}' \) becomes as

\[
S(\partial \mathcal{M}') = \lim_{\sigma_\varepsilon \to 0} S_{GH}
\]

\[
= -\frac{1}{8\pi G} \left( \epsilon \frac{\partial}{\partial \sigma} 2\pi^2 b^3(\sigma) \right)_{\sigma \to \sigma_*}
\]

\[
= \sqrt{\frac{3\pi^3}{4G}} C,
\]
where $\partial \mathcal{M}'$ denotes the boundary of $\mathcal{M}'$ and $\epsilon$ is defined by

$$
\epsilon = \begin{cases} 
+1 & \text{Hawking–Turok’s case} \\
-1 & \text{asymptotically flat case.}
\end{cases}
$$

We see that the constant in the behaviour of $\phi$ (Eq.(17)) is unimportant for the boundary contribution. This is because the Gibbons–Hawking term involves only $b$, and only $\phi'$ and $b$ are important in the equations of motion near singularity.

Next, we calculate the contribution of the excised spacetime. In order to find it, we at first calculate the action contribution of bolts in five dimensions, and then we show the correspondence of the bolts in five-dimensions to HT-singularities in four dimensions. The five-dimensional action is given by

$$
S = \int d^5x \sqrt{\tilde{g}} \left( - \frac{1}{16\pi G} \tilde{R} + \text{(other terms)} \right) - \frac{1}{8\pi G} \int d^4\xi \sqrt{\tilde{h}} \tilde{K},
$$

where the tildes distinguish five-dimensional quantities from their four-dimensional counterparts. We consider $O(4) \times U(1)$ symmetric solutions. As it turns out below, the submanifold where the fifth dimension closes becomes the HT-singularity in four dimensions. So we approximate the five-dimensional regular metric there as

$$
d\tilde{s}^2 \simeq d\tau^2 + R_0^2 d\Omega_3^2 + \tau^2 d\theta^2 \quad (\tau \to 0),
$$

where $R_0$ is a constant and the regularity at $\tau = 0$ requires that $\theta$ is identified with period $2\pi$. The submanifold $\tau = 0$ is the bolt in five dimensions (three-dimensional fixed point set). Its contribution $S(\text{FP})$ becomes

$$
S(\text{FP}) = \lim_{\tau \to 0} S_{\text{GH}}
$$

Then we show the correspondence of the bolts in five dimensions to HT-singularities in four dimensions. We take fields to be independent of the fifth coordinate $\theta$ and use the ansatz

$$
\tilde{g}_{AB} = e^{\frac{2\kappa}{3}(\pm\phi+a)} \left( \begin{array}{cc} g_{\mu\nu} & 0 \\ 0 & e^{-2\kappa(\pm\phi+a)} \end{array} \right) = \left( \begin{array}{cc} e^{\frac{2\kappa}{3}(\pm\phi+a)} g_{\mu\nu} & 0 \\ 0 & e^{-\frac{2\kappa}{3}(\pm\phi+a)} \end{array} \right),
$$

(28)
where \( a \) is a constant. The addition of \( a \) to \( \pm \phi \) is a mere shift of the zero point of \( \phi \) and does not affect the calculation of the contribution of singularity to the action, but it is added for clarity. We obtain the four-dimensional action which has appropriate coefficients

\[
S = \int d^4x \sqrt{g} \left( -\frac{R}{16\pi G} + \frac{1}{2}(\partial \phi)^2 + \text{(other terms)} \right) - \frac{1}{8\pi G} \int d^5\xi \sqrt{h}K. \tag{29}
\]

\( G \) and \( \tilde{G} \) are related by \( \tilde{G} = G \int d\theta = 2\pi G \). From Eq. (28), we see that the four-dimensional fields diverge when the fifth dimension closes. Namely the submanifold where the fifth dimension closes is identified as the singularity in four dimensions. The five-dimensional metric of Eq.(24) becomes

\[
\tau^3 \approx \frac{9}{4} (\sigma_* - \sigma)^2 \tag{30}
\]

\[
b^3(\sigma) \approx \frac{3R_0^3}{2} |\sigma_* - \sigma| \tag{31}
\]

\[
\phi(\sigma) \approx \mp \frac{1}{\kappa} \log |\sigma_* - \sigma| \mp \frac{1}{\kappa} \log \frac{3e^{\kappa a}}{2}, \tag{32}
\]

near the bolt, \( \tau = 0 \) i.e. \( \sigma = \sigma_* \). We see that any HT-singular behaviours can be obtained by choosing \( R_0 \) and \( a \) properly.

Comparing Eq.(16) and Eq.(31), we have that \( 3R_0^3/2 = \kappa C \), so the bolt contribution \( S(\text{FP}) \) can be rewritten in terms of the four-dimensional quantities as \( S(\text{FP}) = -\sqrt{\pi^3 C^2 / 3G} = -\frac{2}{3} S(\partial M') \). Hence the total action \( S(M) \) becomes

\[
S(M) = S(M') + S(\text{FP}) \tag{33}
\]

\[
= S(\text{bulk}) + \frac{1}{3} S(\partial M') \left( +S(\text{boundary at infinity}) \right). \tag{34}
\]

Thus the contribution of the HT-singularity to the action can be calculated as the one third of the Gibbons–Hawking term of the three-surface which wraps the singularity.

4 The Total Action

We can obtain a very simple formula for the total action of the system considered above. Interestingly, we have the same formula for both the regular and HT-singular instantons only if we use the above results, Eq.(34).
From the Einstein equations

\[ b'' = -\frac{8\pi G}{3} b \left( \phi'' + V \right) \]  
\[ b'{}^2 = \frac{8\pi G}{3} b^2 \left( \frac{1}{2} \phi'' - V \right) + 1, \]  
we have

\[ (b^3)'' = 3 \left( -8\pi G b^3 V + 2b \right). \]

By integration, we obtain the relation

\[ \int d\sigma b^3 V = \frac{1}{4\pi G} \int d\sigma b - \frac{1}{3} \left[ \frac{1}{8\pi G} (b^3)' \right]. \]

On the other hand, the trace of the Einstein equation reads

\[ \mathcal{R} = 8\pi G \left( (\partial \phi)^2 + 4V(\phi) \right). \]

Hence the total action Eq.(11) can be rewritten as

\[ S = -\int d^4 x \sqrt{-g} V(\phi) + \mathcal{B} \]
\[ = -2\pi^2 \int d\sigma b^3(\sigma) V(\phi) + \mathcal{B} \]
\[ = -\frac{\pi}{2G} \int d\sigma b(\sigma) + \frac{1}{3} \left[ \frac{1}{8\pi G} (2\pi^2 b^3)' \right] + \mathcal{B}. \]

In the case of compact regular instantons, the second and third terms vanish, so we obtain a simple formula

\[ S = -\frac{\pi}{2G} \int d\sigma b. \]

We observe that the total action is given by the area of the circle plotted by \( b(\sigma) \) in the \((\sigma, b)\)-plane. In the case of Hawking-Turok instantons, the second term is non-vanishing at the singularity, but it is exactly cancelled by the third term

\[ \mathcal{B} = \frac{1}{3} \left( -\frac{1}{8\pi G} (2\pi^2 b^3)' \right)_{\sigma \to \sigma_s}. \]

Therefore we have the same expression for regular and HT-singular instantons. Here the coefficient 1/3 in front of the Gibbons-Hawking term which
originates from the addition of $S$ (FP) is crucial. It is to be noted that by changing $\phi_0$ we can obtain both regular and HT-singular instantons when the potential $V(\phi)$ has additional extrema.

In the case of HT-singular asymptotically flat instantons, the total action becomes

$$S = \lim_{s \to \infty} \left\{ -\frac{\pi}{2G} \int_{\sigma_*}^{s} d\sigma b(\sigma) + \frac{\pi}{12G} (b^3)'(s) \right\}. \quad (45)$$

Here we have dropped the boundary contributions at $s$ in the $B$ term because they cancel in the limit of $s \to \infty$ on account of the asymptotically flat condition.

5 Discussions

In conclusion, we showed in this paper that the HT-singularity has a fixed point contribution to the action and it is given by $-2/3$ of the Gibbons-Hawking term of the three-surface which wraps the singularity and whose normal vectors point to the singularity, in the zero-volume limit of the three-surface. By adding the contribution to the action, we also obtained a simple formula for the action which is feasible for both regular and HT-singular instantons.

Two comments are in order.

At the singularity the equations of motion are not satisfied and so the action is not stationary. But the probability of quantum transition is calculated from the path integral which is integrated over instantons which is constrained by a given 3-metric on the considering three-surface. Hence the singularity at the constrained surface would be allowed. Motivated from this observation by Wu [7], it seems plausible that the HT-singularity in the cases of the creation of open universe [1] and decay of flat spacetime [8] would be allowed. In these cases the HT-singularity is on the quantum transition surface, so that the parameter $C$ (in other words, $\phi_0$ or $\sigma_*$) is constrained and the dependence of the action on it would be justified. (However, in ref. [7] it was suggested to throw away the hemisphere which contains the HT-singularity and the whole manifold should be made by joining the remaining regular hemisphere and its oriented reversal in order to avoid the trouble caused by the HT-singularity.) Recently, Turok [11] introduced a
constraint and argued that Vilenkin’s instantons may be defined as a limit of constrained instantons which are non-singular (albeit with discontinuous first derivatives) and have no boundary. Using such instantons it was argued that the action of Vilenkin’s instantons is given by the same value calculated by Vilenkin [8], i.e. the whole Gibbons–Hawking term. This differs with our result, 1/3 of the Gibbons–Hawking term, and so it seems doubtful that the obtained instantons are Vilenkin’s instantons. The suggested constraint may be an unappropriate one so that HT-singularity does not appear from the limit of such instantons.

As a second comment, we examine the five-dimensional theory consisting of gravity and a cosmological constant $\Lambda$ to show how the difficulty mentioned in Sec. III disappears. In ref. [9] it was stated that the regular instantons exist in this theory only when $\phi_0$ is zero. However, careful examination reveals that this is not the case. The five-dimensional metric with $O(4) \times U(1)$ symmetry is

$$
\begin{align*}
\frac{ds^2}{d\sigma^2} &= d\tau^2 + \hat{A}^2 \sin^2 \left(\frac{\tau}{\hat{A}}\right) d\Omega^2_3 + \hat{A}^2 \cos^2 \left(\frac{\tau}{\hat{A}}\right) d\theta^2,
\end{align*}
$$

where $\hat{A} = \sqrt{6/\Lambda}$. The coordinate $\tau$ runs from 0 to $\hat{A}\pi/2$. This metric is regular everywhere for any $\hat{A}$. The HT-singularity appears at $\tau = \hat{A}\pi/2$ on compactification. The behaviour of the scalar field near $\sigma = 0$ (in this case it corresponds to $\tau = 0$) becomes

$$
\pm \phi(\sigma) \simeq -\frac{3}{2\kappa} \log \hat{A} + \frac{3\sigma^2}{4\kappa\hat{A}^3} - a,
$$

and the potential becomes

$$
V(\phi) = \frac{\hat{A}}{8\pi G} e^{\frac{2\pi}{\Lambda} (\pm \phi + a)} = \frac{3}{4\pi G} e^{(-\frac{3}{2\kappa} \log \hat{A} + \frac{3\sigma^2}{4\kappa\hat{A}^3})} e^{\pm \frac{2\pi}{\Lambda} \phi}.
$$

From these equations we see that we can have any $\phi_0$ without changing the four-dimensional potential term by choosing $a$ and $\hat{A}$ appropriately.

We may interpret this calculation in terms of constraints on instantons as follows. Four-dimensional singular instantons are one parameter family in constrast to regular instantons and the action depends on the parameter. This parameter enters the instanton solution as an integration constant of the equations of motion. We saw in Sec. III that regular instantons in five
dimensions whose fifth dimension closes show the HT-singular behaviour on compactification. In general there would be only a finite number of such regular instantons in a five-dimensional theory. They have one-to-one correspondence to four-dimensional HT-singular instantons of different parameter, so that we have a constraint on the parameter of the HT-singular instantons. If we choose one five-dimensional theory (i.e. the value of $\Lambda$, in the above example), then one or some of HT-singular instantons are picked out from the family of HT-singular instantons. The parameter of the HT-singular instantons can be later integrated over in path integral by integrating over the appropriate parameter space of the constraining five-dimensional theory.

Acknowledgements

I am grateful to Ken-Iti Izawa and Hiroaki Terashima for reading the manuscript and for useful comments.
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