Testing theories that predict time variation of fundamental constants

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ABSTRACT

We consider astronomical and local bounds on time variation of fundamental constants to test some generic Kaluza-Klein-like models and some particular cases of Beckenstein theory. Bounds on the free parameters of the different theories are obtained. Furthermore, we find that none of the proposed models, is able to explain recent results (Webb et al. 1999, 2000) claiming an observed variation of the fine structure constant from quasar absorption systems at redshifts $0.5 < z < 3$.

1. Introduction

The time variation of fundamental constants has motivated numerous theoretical and experimental research since the large number hypothesis (LNH) proposed by (Dirac 1937). The great predictive power of the LNH, induced a large number of research papers and suggested new sources of variation. Among them, the attempt to unify all fundamental interactions resulted in the development of multidimensional theories like Kaluza-Klein (Marciano 1987; Chodos and Detweiler 1980; Kaluza 1921; Klein 1926) and superstring ones (Damour and Polyakov 1994) which predict not only energy dependence of the fundamental constants but also dependence of their low-energy limits on cosmological time. In such theories, the temporal variation of fundamental constants is related with the variation of the extra compact dimensions.

Following a different path of research, Beckenstein (1982) proposed a theoretical framework to study the fine structure constant variability based on very general assumptions:

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covariance, gauge invariance, causality and time-reversal invariance of electromagnetism, as well as the idea that the Planck-Wheeler length ($10^{-33} \text{cm}$) is the shortest scale allowable in any theory.

Different versions of the theories mentioned above predict different time behaviours for the fundamental constants. Thus, experimental bounds on the variation of fundamental constants are an important tool to check the validity of such theories (Marciano 1987; Chodos and Detweiler 1980; Beckenstein 1982).

The experimental research can be grouped into astronomical and local methods. The latter ones include geophysical methods such as the natural nuclear reactor that operated about $1.8 \times 10^9$ in Oklo, Gabon (Damour and Dyson 1996), the analysis of natural long-lived $\beta$ decayers in geological minerals and meteorites (Sisterna and Vucetich 1990) and laboratory measurements such as comparisons of rates between clocks with different atomic number (Prestage, Toelker and Maleki 1995). The astronomical methods are based mainly in the analysis of spectra form high-redshift quasar absorption systems (Drinkwater et al. 1998; Webb et al 1999, 2000; Cowie and Songaila 1995; Bahcall, Sargent and Schmidt 1967). Besides, other constraints can be derived from primordial nucleosynthesis (Bernstein, Brown and Feinberg 1988) and the Cosmic Microwave Background (CMB) fluctuation spectrum (Battye, Crittenden and Weller 2001; Avelino et al 2000; Landau, Harari and Zaldarriaga 2001).

Although, most of the previous mentioned experimental data gave null results, (Webb et al 1999), reported a significantly different measurement of the time variation of the fine structure constant, which was confirmed recently (Webb et al 2000). This suggests an examination of the available experimental results in the context of typical theories predicting time variation of fundamental constants.

Thus, in this work, we consider several astronomical and local bounds on time variation of fundamental constants in the framework of two Kaluza-Klein-like late time solutions (Marciano 1987; Bailin and Love 1987) and some particular cases of Beckenstein theory (Beckenstein 1982). In particular we put bounds on the free parameters of the different models, the size of the extra dimensions in the first case, and the parameters $l$ and $\gamma$ of Beckenstein’s theory. Besides, the consistency of experimental data with a given family of theories can be checked.

The paper is organized as follows: In section II we describe briefly the models we want to test, in section III we describe the experimental constraints, we will use to check our models, in section IV we present our results and briefly discuss our conclusions.
2. Theoretical models predicting time variation of fundamental constants

2.1. Kaluza-Klein-like models

The basic idea of Kaluza-Klein theories is to enlarge space-time to \(4 + D\) dimensions in such a way that the \(D\) extra spatial dimensions form a very small compact manifold with mean radius \(R_{KK}\).

So, the metric in \(4 + D\) dimensions can be written:

\[
dS^2 = dt^2 - r^2(t) \ g_{mn} - R_{KK}^2(t) \ g_{uv} \tag{1}
\]

where \(g_{mn}\) is the metric of an \(S^3\) of unit radius, \(r(t)\) is the scale factor of the ordinary space, \(g_{uv}\) is the metric of an \(S^D\) of unit radius and \(R_{KK}(t)\) is the scale factor of the internal space.

In Kaluza-Klein theories, gauge fields of the Standard Model of Fundamental Interactions are related to the \(g_{\mu\nu}\) elements that connect the internal dimensions with the usual \(3 + 1\) space-time. The gauge coupling constants are related to the “internal” scale of the extra dimensions through one or more scalar fields (Weinberg 1983).

In some models, the “internal” dimensions are small compared to the large “ordinary” dimensions. However, at the Planck time, the characteristic size of both internal and external dimensions are likely to be the same. The cosmological evolution which determines the way in which the extra dimensions are compactified depends on how many extra dimensions are taken and on the energy-momentum tensor considered: radiation, monopoles, cosmological constant, etc.

The generalized Einstein equations can be written as follows (Kolb and Turner 1990):

\[
R_{MN} = 8\pi\tilde{G} \left[ T_{MN} - \frac{1}{D + 2} g_{MN} T^P_P - \frac{1}{D + 2} \frac{\tilde{\Lambda}}{8\pi\tilde{G}} g_{MN} \right] \tag{2}
\]

where \(\tilde{G}\) is the gravitational constant in \(4 + D\) dimensions and \(\tilde{\Lambda}\) is a cosmological constant in \(4 + D\) dimensions.

The evolution of the extra dimensions with cosmological time is related with the time variation of fundamental constants through the equation (Kaluza 1921; Klein 1926; Marciano 1987; Weinberg 1983):

\[
\alpha_i(M_{KK}) = \frac{K_i G}{R_{KK}^2} = K_i G M_{KK}^2 \tag{3}
\]

where \(\alpha_i(M_{KK}), i=1,2,3\) are the coupling constants of \(U(1), SU(2)\) and \(SU(3)\) for a typical energy \(R_{KK} = \frac{1}{M_{KK}}\). We assume as usual, the existence of a GUT energy scale \(\Lambda_{GUT}\) beyond
which all these constants merge in only one \( \alpha_i \). The \( K_i \) are numbers that depend on the \( D \) dimensional topology.

The expressions for the gauge coupling constants at different energies are related through the group renormalization equation (Marciano 1987):

\[
\alpha_i^{-1}(E_1) = \alpha_i^{-1}(E_2) - \frac{1}{\pi} \sum_j C_{ij} \left[ \left( \frac{E_2}{m_j} \right) + \theta(E_1 - m_j) \ln \left( \frac{m_j}{E_1} \right) \right]
\]  

(4)

So, we can find the low-energy limit for the gauge coupling constants using eq.(4) twice:

\[
\begin{align*}
E_1 &= \Lambda_{GUT} \\
E_2 &= M_K K
\end{align*}
\]  

(5)

Inserting eq.(3) we obtain:

\[
\begin{align*}
\alpha_1^{-1}(M_W) &= \frac{KG}{R_{KK}^2} - \frac{76}{6\pi} \ln \left( \frac{R_{KK}^{-1}}{\Lambda_{GUT}} \right) + \frac{2}{\pi} \ln \left( \frac{\Lambda_{GUT}}{M_W} \right) \\
\alpha_2^{-1}(M_W) &= \frac{KG}{R_{KK}^2} - \frac{76}{6\pi} \ln \left( \frac{R_{KK}^{-1}}{\Lambda_{GUT}} \right) - \frac{5}{3\pi} \ln \left( \frac{\Lambda_{GUT}}{M_W} \right) \\
\alpha_3^{-1}(M_W) &= \frac{KG}{R_{KK}^2} - \frac{76}{6\pi} \ln \left( \frac{R_{KK}^{-1}}{\Lambda_{GUT}} \right) - \frac{7}{2\pi} \ln \left( \frac{\Lambda_{GUT}}{M_W} \right)
\end{align*}
\]  

(6)

(7)

(8)

In this way we get expressions for the gauge coupling constants depending on \( R_{KK} \) and \( \Lambda_{GUT} \).

In order to compare equations (6), (7) and (8) with experimental and observational values, we still should calculate the adjustment for energies \( \sim 1 \) GeV. However, since this adjustment is very small, we will not consider it.

The gauge coupling constants are related with the fine structure constant \( \alpha \), the QCD energy scale \( \Lambda_{QCD} \) and the Fermi coupling constant \( G_F \) through the following equations:

\[
\alpha^{-1}(E) = \frac{5}{2} \alpha_1^{-1}(E) + \alpha_2^{-1}(E)
\]  

(9)

\[
\Lambda_{QCD} = E \exp \left[ -\frac{2\pi}{\ell} \alpha_3^{-1}(E) \right]
\]  

(10)
It has been shown that Kaluza-Klein equation are either non-integrable, or their solutions lack of physical interest (Helmi and Vucetich 1995). However, several non-exact solutions of eq.(2) have been analyzed in the literature (see Bailin and Love (1987); Kolb and Turner (1990) and references therein).

For the purposes of this paper, though, we are interested in typical late time solutions since the data we work with belong to times later than nucleosynthesis. Thus, we consider models where the scale factor of the Universe behaves as in a flat Robertson-Walker spacetime with and without cosmological constant and the radius of the internal dimensions behaves as the following schematic solutions motivated in refs. (Marciano 1987; Bailin and Love 1987):

\[
R_{KK}(t) \sim R_0 + \Delta R \cos \left( \omega (t - t_0) \right)
\] (12)

\[
R_{KK}(t) \sim R_0 + \Delta R \left( \frac{t_0}{t} \right)^{3/4}
\] (13)

where \( R_0 = R_{KK}(t_{Planck}) \approx R_{Planck} \). We expect that typical solutions of Kaluza-Klein cosmologies behave asymptotically like eqs.(12) and (13) with \( \Delta R << R_0 \) and \( \omega \) depending on the details of the model. We will refer to solution 12 as generic model 1 and to solution 13 as generic model 2.

Thus, the free parameter in all Kaluza-Klein-like models will be : \( \frac{\Delta R}{R_0} \) and we will take as usual \( \Lambda_{GUT} = 10^{16} GeV \). Table 3 shows the cosmological model and the values of \( \omega \) considered for each particular model.

### 2.2. Beckenstein models

As we have mentioned above, Beckenstein (1982) proposed a framework for the fine structure constant \( \alpha \) variability based on very general assumptions such us: covariance, gauge invariance, causality and time-reversal invariance of electromagnetism , as well as the idea that the Planck-Wheeler length (10^{-33} cm) is the shortest scale allowable in any theory.

He obtained the following equation for the temporal variation of \( \alpha \):

\[
\left( \frac{a^3 \dot{\alpha}}{\alpha} \right) = -a (t)^3 \zeta \left( \frac{I^2}{\hbar c} \right) \rho_m c^4
\] (14)
where \( \varepsilon = \left( \frac{a}{a_{today}} \right)^{\frac{1}{2}} \), \( l \) is a length scale of the theory, \( \rho_m \) is the total rest mass density of matter, \( a(t) \) is the expansion scale factor and \( \varsigma \) is a dimensionless parameter which measures the fraction of mass in the form of Coulomb energy of an average nucleon, compared to the free proton mass (Beckenstein (Beckenstein 1982) assumed that \( \varsigma \) is constant and equal to \( 1.3 \times 10^{-2} \)).

In an expanding Universe where \( \rho_m = \frac{3H_0^2}{8\pi G} \left[ \frac{a(t_0)}{a(t)} \right]^3 \), we obtain:

\[
\frac{\dot{\varepsilon}}{\varepsilon} = -\varsigma \left( \frac{l^2c}{\hbar} \right) \rho_m (t - t_c)
\]

where \( t_c \) is an integration constant.

We consider a model with cosmological constant where the scale factor varies as:

\[
a(t) = a(t_0) \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^{\frac{1}{3}} \left[ \sinh \left( \frac{3}{2} \Omega_\Lambda^{1/2} H_0 t \right) \right]^{\frac{2}{3}}
\]

Next, integrating eq.(15), we obtain the time variation of the fine structure constant as follows:

\[
\frac{\Delta \alpha}{\alpha} = -\frac{3}{8} \varsigma \left( H_0 t_0^{-1} \right)^2 \left( \frac{l}{L_p} \right)^2 \left[ \beta \coth \beta - \frac{t}{t_0} \beta \coth \left( \frac{\beta}{t_0} \right) + \ln \left( \frac{\sinh \left( \frac{\beta}{t_0} \right)}{\sinh (\beta)} \right) \right]
\]

with

\[
\coth \beta = \Omega_\Lambda^{\frac{1}{2}}
\]

where \( t_c = \gamma t_0 \), \( L_p = \left( \frac{G\hbar}{c^3} \right)^{\frac{1}{2}} \). In all cases the integration constant is such that \( \varepsilon(t_0) = 1 \) and \( \Omega_m + \Omega_\Lambda = 1 \).

Table 4 shows the cosmological parameters for the models we use to test this theory.

The free parameters in this models are \( L = \frac{L}{L_p} \) and \( \gamma \)

### 3. Bounds from astronomical and geophysical data

In this section, we make a critical discussion of the rather heterogeneous data set we use to test our models.
3.1. The Oklo Phenomenon

One of the most stringent limits on time variation of fundamental constants follows from an analysis of isotope ratios of $^{149}\text{Sm}/^{147}\text{Sm}$ in the natural uranium fission reactor that took place $1.8 \times 10^9$ yr ago at the present day site of the Oklo mine in Gabon, Africa (Schlyakter 1976; Damour and Dyson 1996). From an analysis of nuclear and geochemical data, the operating conditions of the reactor could be reconstructed and the thermal neutron capture cross sections of several nuclear species measured. In particular, a shift in the lowest lying resonance level in $^{149}\text{Sm}$: $\Delta = E_r^{149(\text{Oklo})} - E_r^{149(\text{now})}$ can be derived from a shift in the neutron capture cross section of the same nucleus (Schlyakter 1976; Damour and Dyson 1996). From (Damour and Dyson 1996) we know that we can translate the shift in $\Delta$ into a bound on a possible difference between the values of $\alpha$ and $G_F$ during the Oklo phenomenon and their value now. In ref. (Damour and Dyson 1996) both bounds on $\alpha$ and $G_F$ were derived separately; here we consider both variations at the same time as follows:

$$\Delta = \alpha \frac{\partial E_r}{\partial \alpha} \Delta \alpha + G_F \frac{\partial E_r}{\partial G_F} \Delta G_F$$

(18)

where $\Delta \alpha = \alpha^{\text{Oklo}} - \alpha^{\text{now}}$ and $\Delta G_F = G_F^{\text{Oklo}} - G_F^{\text{now}}$. The value of $\Delta$ is shown in Table 1. Finally, using the values of $\Delta, \alpha \frac{\partial E_r}{\partial \alpha}, G_F \frac{\partial E_r}{\partial G_F}$ from ref. (Damour and Dyson 1996), we can relate $\Delta$ with $\frac{\Delta \alpha}{\alpha}$ and $\frac{\Delta G_F}{G_F}$ (see first entry in Table 2).

3.2. Long-lived $\beta$ decayers

The half-life of long-lived $\beta$ decayers such $^{187}\text{Re}$, $^{40}\text{K}$, $^{87}\text{Rb}$ has been determined either in laboratory measurements or by comparison with the age of meteorites, as found from $\alpha$ decay radioactivity analysis. Sisterna and Vucetich (1990) have derived a relation between the shift in the half-life of three long lived $\beta$ decayers and a possible variation between the values of the fundamental constants $\alpha, \Lambda_{QCD}$ and $G_F$ at the age of the meteorites and their value now (see entries 2,3 and 4 of Table 2).

The values of $\Delta \lambda$ for $^{187}\text{Re}$, $^{40}\text{K}$, $^{87}\text{Rb}$ are respectively shown in entries 2, 3, and 4 in Table 1 where $\Delta = \frac{\Delta \lambda}{\lambda}$ and $\Delta \lambda = \lambda(t = 5.535 \times 10^9) - \lambda(t = t_0 = 1.0035 \times 10^{10})$.

3.3. Laboratory experiments

The best limit on $\alpha$ variation, comes from a laboratory experiment (Prestage, Toelker and Maleki 1995); it is a limit on a present day variation of $\alpha$. The experiment is based on
a comparison of rates between clocks based on hyperfine transitions in atoms with different atomic number. H-maser and Hg+ clocks have a different dependence on $\alpha$ since their relativistic contributions are of order $(\alpha Z)^2$. The result of a 140 day clock day comparison between an ultrastable frequency standard based on Hg+ ions confined to a linear ion trap and a cavity tuned H maser (Prestage, Toelker and Maleki 1995) is shown in Table 1 where $\Delta = \frac{\Delta \alpha}{\alpha}$.

### 3.4. Quasar absorption systems

Quasar absorption systems present ideal laboratories in which to search for any temporal variation in the fundamental constants. The continuum spectrum of a quasar was formed at an epoch corresponding to the redshift $z$ of main emission details specified by the relationship $\lambda_{\text{obs}} = \lambda_{\text{lab}}(1+z)$. Quasar spectra of high redshift show the absorption resonance lines of the alkaline ions like CIV, MgII, FeII, SiIV and others, corresponding to the $S_{1/2} \rightarrow P_{3/2} (\lambda_1)$ and $S_{1/2} \rightarrow P_{1/2} (\lambda_2)$ transitions. The relative magnitude of the fine splitting of the corresponding resonance lines is proportional to the square of the fine structure constant $\alpha$ to lowest order in $\alpha$.

$$\frac{\Delta \lambda}{\lambda} = \frac{\lambda_1 - \lambda_2}{\lambda} \sim \alpha^2$$

(19)

Therefore, any change in $\alpha$ will result in a corresponding change in $\Delta \lambda$ in the separation of the doublets of the quasar as follows:

$$\frac{\Delta \alpha}{\alpha} = \frac{1}{2} \left[ \left( \frac{\Delta \lambda}{\lambda} \right)_{z} - \left( \frac{\Delta \lambda}{\lambda} \right)_{\text{now}} - 1 \right]$$

Cowie and Songaila (1995) and Varshalovich, Panchuk and Ivanchik (1996) have applied this method to SiIV doublets absorption lines systems at different redshifts ($2.8 < z < 3.33$) to find the values shown in entries 6 to 9 of table 1 where $\Delta = \frac{\Delta \alpha}{\alpha}$.

Webb et al (1999) have improved this method comparing transitions of different species, with widely differing atomic masses. As mentioned before, this is the only data consistent with a time varying fine structure constant. In turn, recent work (Webb et al 2000) including new optical data confirms their previous results. The values of $\frac{\Delta \alpha}{\alpha}$ at redshift $z = 1.2$, $z = 2.7$ and $z = 2.5$ are respectively shown in entries 10, 11 and 12 of Table 1.

Moreover, the ratio of frequencies of the hyperfine 21 cm absorption transition of neutral hydrogen $\nu_a$ to an optical resonance transition $\nu_b$ is proportional to $x = \alpha^2 g_p \frac{m_e}{m_p}$ where $g_p$ is the proton $g$ factor. Thus, a change of this quantity will result in a difference in the
measured redshift of 21 cm and optical absorption as follows:

$$\frac{\Delta x}{x} = \frac{z_{\text{opt}} - z_{21}}{(1 + z)} \quad (20)$$

So, combining the measurements of optical and radio redshift, a bound on \( x \) can be obtained.

The upper bounds on \( x \) obtained by Cowie and Songaila (1995) at redshift \( z = 1.776 \) are shown in Table 1 where \( \Delta = \frac{\Delta x}{x} \). The relationship between \( \frac{\Delta x}{x} \) and the variation of \( \alpha \), \( G_F \) and \( \Lambda_{\text{QCD}} \) is shown in Table 2.

Other bounds on \( x \) were obtained by Wolfe and Davis (1979) at redshift \( z = 0.69 \) (entry 14 of Table 1) and Wolfe, Brown and Roberts (1976) at redshift \( z = 0.52 \) (entry 15 of Table 1).

On the other hand, the ratio of the rotational transition frequencies of diatomic molecules such as CO to the 21 cm hyperfine transition in hydrogen is proportional to \( y = g_\rho \alpha^2 \). Thus, any variation in \( y \) would therefore be observed as a difference in the apparent redshifts:

$$\frac{\Delta y}{y} = \frac{z_{\text{mol}} - z_{21}}{(1 + z)} \quad (21)$$

Drinkwater et al. (1998) have placed upper limit on \( y \) at redshift \( z = 0.25 \) and at redshift \( z = 0.68 \). The observed values are shown in entries 16 and 17 of Table 1, where \( \Delta = \frac{\Delta y}{y} \). Entries 16 and 17 of Table 2 relate \( \frac{\Delta y}{y} \) with the variation of \( \alpha \).

Finally, observations of molecular hydrogen in quasar absorption systems can be used to set bounds on the evolution of \( \mu = \frac{m_e}{m_p} \). The most stringent bounds established by Pothekin et al (1998) are shown in entry 18 of Table 2.

### 3.5. Nucleosynthesis

Primordial nucleosynthesis also provides a bound on the variation of fundamental constants. A didactical analysis of \(^4\text{He} \) production can be found in Bernstein, Brown and Feinberg (1988). At the conclusion of the big-bang nucleosynthesis the \(^4\text{He} \) mass fraction of the total baryonic mass is given by (Bernstein, Brown and Feinberg 1988):

$$Y = 2 \exp[-\frac{t_c}{\tau}] \cdot X(t_F) \quad (22)$$

where \( t_c \) is the neutron capture time, \( \tau \) is the neutron mean life and \( X(t_F) \) is ratio of the neutron to total baryon number at the time where the baryons become uncoupled from the leptons (freeze-out time).
In appendix I, we derive the following expression for the change in the helium abundance $\Delta Y$ brought about by changes in the fundamental constants:

$$\frac{\Delta Y}{Y} = 0.74 \frac{\Delta R_{KK}}{R_{KK}} + 0.64 \frac{\Delta G_F}{G_F} + 1.78 \frac{\Delta \alpha}{\alpha} - 0.3 \frac{\Delta \Lambda_{QCD}}{\Lambda_{QCD}}$$  \hspace{1cm} (23)

### 3.6. Cosmic Microwave Background

Any variation of the fine structure constant $\alpha$ alters the physical conditions at recombination and therefore changes the cosmic microwave background (CMB) fluctuation spectrum. Moreover, the fluctuation spectrum of CMB is sensitive to many cosmological parameters such as the density of barionic and dark matter, the Hubble constant and the index of primordial spectral fluctuations. Recently, different independent analyses (Battye, Crittenden and Weller 2001; Avelino et al 2000; Landau, Harari and Zaldarriaga 2001) showed that the recent published data of Boomerang and Maxima are better fitted with a varying fine structure constant and a density of baryonic matter closer to nucleosynthesis bounds. The same authors established a bound on $\alpha$ variation at the epoch at which neutral hydrogen formed (see entry 20 in Table 1).

### 4. Results and Discussion

Taking the data of the last section, we have performed a statistical analysis working on $\chi^2$ function with MINUIT to compute the best-fit parameter values and uncertainties including correlations between parameters.

For the Kaluza-Klein like models, results within 99% of confidence level ($3\sigma$) are shown in table 3.. For the models derived from Beckenstein’s proposal we obtain results with 90% of confidence level (see table 4). The contours of the likelihood functions for Beckenstein’s models in regions of 90 % and 70 % of confidence level are shown in figures 1 and 2.

The values of the free parameters obtained are coincident within uncertainties for the Kaluza-Klein like models (table3) and for Beckenstein’s models (table 4). Besides, the values obtained are consistent with theoretical supposition $\Delta R << R_0$ for Kaluza-Klein like models, but they disagree with the supposition $l > L_p$ implied in Beckenstein’s framework.

Thus, the present available data set, considered within Bekenstein’s framework, is capable to rule out $\alpha$ variability, while the original paper had to recourse to Eötvös-like experiments to achieve the same result. However, it should be noted that Beckenstein’s framework is very similar to the dilatonic sector of string theory. And it has been pointed out in the
context of string theories (Bachas 2000; Antoniadis and Pioline 1999), that there is no need for an universal relation between the Planck and the string scale.

Finally, our results are consistent with no time variation of fundamental constants over cosmological time in agreement most of the experimental results. Indeed, excluding the Webb et al data point from our fits does not change significantly the values of the adjusted constants. Thus, this rather large class of theories cannot explain this discrepant result.

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A. Appendix I

Following (Bernstein, Brown and Feinberg 1988) and eq. 22, the change in the helium abundance is given by:

\[
\frac{\Delta Y}{Y} = \frac{t_c}{\tau} \left( \frac{\Delta \tau}{\tau} - \frac{\Delta t_c}{t_c} \right) + \frac{\Delta X (t_F)}{X (t_F)}
\]  

(A1)

where

\[
\frac{\Delta X (t_F)}{X (t_F)} = -0.52 \frac{\Delta b}{b}
\]  

(A2)

and

\[
b = 255 \left( \frac{45}{4\pi N} \right)^{1/2} \frac{M_{pl}}{\tau Q^2}
\]  

(A3)

\[
Q = \Delta m = m_n - m_p
\]  

(A4)

being \(N\) the number of neutrino types.

Since \(\tau = Q^5 G_F^2\), we find for the ratio of neutron to total baryon number at the freeze-out time:
\[
\frac{\Delta X(t_F)}{X(t_F)} = -0.52 \left[ \frac{\Delta M_{\text{pl}}}{M_{\text{pl}}} - 2 \frac{\Delta G_F}{G_F} - 7 \frac{\Delta Q}{Q} \right]
\]  

(A5)

Next, also from (Bernstein, Brown and Feinberg 1988) we take the following expression for the neutron time capture:

\[
t_c = \left( \frac{45}{16\pi N} \right)^{1/2} \left( \frac{11}{4} \right)^{2/3} \frac{M_{\text{pl}}}{T_{\gamma,c}^2} + t_0
\]

where \( t_0 \) is an integration constant, \( T_{\gamma,c} \) is the temperature of the photon at the neutron capture time. Thus, the last equation yields:

\[
\frac{\Delta t_c}{t_c} = \frac{\Delta M_{\text{pl}}}{M_{\text{pl}}} - 2 \frac{\Delta T_{\gamma,c}}{T_{\gamma,c}}
\]

(A7)

Writing \( T_{\gamma,c} = \frac{\varepsilon_D}{z_c} \) with \( \varepsilon_D = m_n + m_p - m_D \) and \( z_c = \frac{\varepsilon_D}{T_{\gamma,c}} \) we obtain:

\[
\frac{\Delta T_{\gamma,c}}{T_{\gamma,c}} = \frac{\Delta \varepsilon_D}{\varepsilon_D} \quad \frac{\Delta z_c}{z_c} = \frac{\Delta \Lambda_{\text{QCD}}}{\Lambda_{\text{QCD}}} \quad \frac{\Delta z_c}{z_c}
\]

(A8)

Since at the neutron capture time, the neutrons are essentially all converted into helium, we may identify the temperature \( T_{\gamma,c} \) at which neutrons are captured, or equivalently the redshift \( z_c = \frac{\varepsilon_D}{T_{\gamma,c}} \), by the condition:

\[
\left( \frac{dX_D}{dz} \right)_{z=z_c} = 0
\]

(A9)

where \( X_D \) is the ratio of deuterons to total baryon number.

From (Bernstein, Brown and Feinberg 1988) it is easy to see that the last equation is equivalent to the following:

\[
f(z_c) = \ln(C_0) + \frac{4}{3} \ln \left( \frac{\varepsilon_D}{m_p} \right) + \ln \left( \frac{M_{\text{pl}}}{m_p} \right) + \frac{4}{3} \ln(\alpha) - \frac{17}{6} \ln(z_c) + z_c - 5.11 \frac{\alpha^{\frac{1}{4}} z_c^{\frac{1}{4}}}{\left( \frac{\varepsilon_D}{m_p} \right)^{\frac{1}{3}}} = 0
\]

(A10)

where \( C_0 \) is a constant and \( z_c = 26 \)

Assuming:

\[
\delta f = \left( \frac{\partial f}{\partial z} \right)_{z=z_c} \delta z + \left( \frac{\partial f}{\partial \alpha} \right)_{z=z_c} \delta \alpha + \left( \frac{\partial f}{\partial M_{\text{pl}}} \right)_{z=z_c} \delta M_{\text{pl}} + \left( \frac{\partial f}{\partial \varepsilon_D} \right)_{z=z_c} \delta \varepsilon_D = 0
\]

(A11)
where $\alpha_i = \alpha_{i0}$ means $\alpha = \alpha_{today}$ and $\Lambda_{QCD} = \Lambda_{QCDtoday}$ we obtain the following expression:

$$\frac{\Delta z_c}{z_c} = - \left[ \left( \frac{\partial f}{\partial \alpha} \right)_{\alpha = \alpha_0} \frac{\Delta \alpha}{\alpha} + \left( \frac{\partial f}{\partial M_{pl}} \right)_{\alpha = \alpha_0} \frac{\Delta M_{pl}}{M_{pl}} + \left( \frac{\partial f}{\partial \varepsilon_D} \right)_{\alpha = \alpha_0} \frac{\Delta \varepsilon_D}{\varepsilon_D} \right] \left( \frac{\partial f}{\partial z} \right)^{-1} \right] \right]

(A12)

Evaluating eq.A12,yields:

$$\frac{\Delta z_c}{z_c} = -0.13 \frac{\Delta \alpha}{\alpha} + 0.046 \frac{\Delta M_{pl}}{M_{pl}} + 0.26 \frac{\Delta \Lambda_{QCD}}{\Lambda_{QCD}}$$

(A13)

Thus, from eqs. (A1), (A5), (A7), (A8), (A13) and as $\frac{\Delta Q}{Q} = \frac{\Delta \alpha}{\alpha}$, the final expression yields:

$$\frac{\Delta Y}{Y} = 0.74 \frac{\Delta R_{KK}}{R_{KK}} + 0.64 \frac{\Delta G_F}{G_F} + 1.76 \frac{\Delta \alpha}{\alpha} - 0.3 \frac{\Delta \Lambda_{QCD}}{\Lambda_{QCD}}$$

(A14)

where we have used the equality $R_{KK}(t_{pl}) \simeq R_{pl} = \frac{1}{M_{pl}}$

REFERENCES


Bachas C.P. 2000, Class.Quant.Grav., 17, 951


Chodos, A. and Detweiler, S., Phys. Rev. D, 21, 2167


Dirac, P. A. M. 1937, Nature, 139, 323
Klein, O. 1926, Z. Phys. 1926, 37, 895
Okada Y. 1985, Phys. Lett. B, 150, 103
Table 1: Observational Data. The columns show the data number (correlated with the respective equation in Table 2), the method considered, the time interval for which the variation was measured in units of $10^9$ yr, computed for models with and without cosmological constant, the observed value, the standard deviation and the corresponding reference.

<table>
<thead>
<tr>
<th>Method</th>
<th>$t - t_0$</th>
<th>$t - t_0$</th>
<th>$\Delta$</th>
<th>$\sigma (\Delta)$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Omega_\Lambda = 0$</td>
<td>$\Omega_\Lambda = 0.75$</td>
<td>$\times 10^{-6}$</td>
<td>$\times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>1 Oklo reactor</td>
<td>1.8</td>
<td>1.8</td>
<td>-15000</td>
<td>1050000</td>
<td>1</td>
</tr>
<tr>
<td>2 Long lived $\beta$ decayers</td>
<td>4.5</td>
<td>4.5</td>
<td>0</td>
<td>6700</td>
<td>2</td>
</tr>
<tr>
<td>3 Long lived $\beta$ decayers</td>
<td>4.5</td>
<td>4.5</td>
<td>0</td>
<td>13000</td>
<td>2</td>
</tr>
<tr>
<td>4 Long lived $\beta$ decayers</td>
<td>4.5</td>
<td>4.5</td>
<td>0</td>
<td>13000</td>
<td>2</td>
</tr>
<tr>
<td>5 Laboratory bounds</td>
<td>$3.8 \times 10^{-10}$</td>
<td>$3.8 \times 10^{-10}$</td>
<td>0</td>
<td>$10^{-8}$</td>
<td>3</td>
</tr>
<tr>
<td>6 Quasar absorption systems</td>
<td>8.7</td>
<td>13</td>
<td>0</td>
<td>350</td>
<td>4</td>
</tr>
<tr>
<td>7 Quasar absorption systems</td>
<td>8.9</td>
<td>13</td>
<td>0</td>
<td>350</td>
<td>4</td>
</tr>
<tr>
<td>8 Quasar absorption systems</td>
<td>8.7</td>
<td>12.8</td>
<td>0</td>
<td>83</td>
<td>5</td>
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<tr>
<td>9 Quasar absorption systems</td>
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<td>12.5</td>
<td>0</td>
<td>80</td>
<td>5</td>
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<td>9.17</td>
<td>-7</td>
<td>2.3</td>
<td>6</td>
</tr>
<tr>
<td>11 Quasar absorption systems</td>
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<td>12.4</td>
<td>-7.6</td>
<td>2.8</td>
<td>6</td>
</tr>
<tr>
<td>12 Quasar absorption systems</td>
<td>6.5</td>
<td>8.5</td>
<td>-5</td>
<td>1.3</td>
<td>6</td>
</tr>
<tr>
<td>13 Quasar absorption systems</td>
<td>7.8</td>
<td>11</td>
<td>7</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>14 Quasar absorption systems</td>
<td>5.5</td>
<td>6.9</td>
<td>0</td>
<td>120</td>
<td>7</td>
</tr>
<tr>
<td>15 Quasar absorption systems</td>
<td>4.7</td>
<td>5.7</td>
<td>0</td>
<td>280</td>
<td>8</td>
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<tr>
<td>16 Quasar absorption systems</td>
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<td>3.2</td>
<td>0</td>
<td>5</td>
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<td>17 Quasar absorption systems</td>
<td>5.4</td>
<td>6.8</td>
<td>0</td>
<td>5</td>
<td>9</td>
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<tr>
<td>18 Quasar absorption systems</td>
<td>8.65</td>
<td>12.6</td>
<td>0</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>19 Nucleosynthesis</td>
<td>10</td>
<td>15</td>
<td>0</td>
<td>43000</td>
<td>11</td>
</tr>
<tr>
<td>20 CMB</td>
<td>10</td>
<td>15</td>
<td>0</td>
<td>10000</td>
<td>12,13,14</td>
</tr>
</tbody>
</table>

Table 2: The equation: \( \Delta = a \frac{\Delta \alpha}{\alpha} + b \frac{\Delta G}{G_F} + c \frac{\Delta \Lambda_{QCD}}{\Lambda_{QCD}} \) relates the observed value (\( \Delta \) of table 1) with the relative variation of fundamental constants. In this table we show the coefficients of this equation for each data considered in table 1.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(10^6)</td>
<td>5.6</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(2.16 \times 10^4)</td>
<td>2</td>
<td>(5.62 \times 10^3)</td>
</tr>
<tr>
<td>3</td>
<td>(4.6 \times 10)</td>
<td>2</td>
<td>(1.7 \times 10)</td>
</tr>
<tr>
<td>4</td>
<td>(1.07 \times 10^3)</td>
<td>2</td>
<td>2.71</td>
</tr>
</tbody>
</table>
Table 3: Results for the Kaluza-Klein like models. The columns show the number of particular model considered, the number of generic model, the cosmological parameters and the free parameters of the theory taken as constant in this work, the best fit parameter value and standard deviation in units of $10^{-14}$. $t_{01} = 1.0 \times 10^{10}$ yr is the age of the universe for models without cosmological constant; $t_{02} = 1.5 \times 10^{10}$ yr is the age of the universe for models with cosmological constant. For all models $H_0 = 65 \ km \times seg^{-1} \times Mpc^{-1}$

<table>
<thead>
<tr>
<th>$\Omega_m$</th>
<th>$\Omega_\Lambda$</th>
<th>$\omega$</th>
<th>$\frac{\Delta R_{KK}}{R_{KK}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\frac{2\pi}{t_{01}}$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.25</td>
<td>$\frac{2\pi}{t_{02}}$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>--</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.25</td>
<td>--</td>
</tr>
</tbody>
</table>

Table 4: Results for the Beckenstein’s type models. The columns show the number of particular model, the cosmological parameters, the value and standard deviation of the best fit parameters and the correlation coefficient. For all models $H_0 = 65 \ km \times seg^{-1} \times Mpc^{-1}$

<table>
<thead>
<tr>
<th>$\Omega_m$</th>
<th>$\Omega_\Lambda$</th>
<th>$L$</th>
<th>$\gamma$</th>
<th>$\rho(L, \gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$0.031^{+0.038}_{-0.030}$</td>
<td>$1.5^{+1.08}_{-0.93}$</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.75</td>
<td>$0.03^{+0.049}_{-0.012}$</td>
<td>$2^{+1.4}_{-1.4}$</td>
</tr>
</tbody>
</table>
Fig. 1.— Contours for Beckenstein’s models