Non-locality and the Contextuality of Measurable Possibilities in the Models of Meyer, Kent and Clifton

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In a previous paper the claim of Meyer, Kent and Clifton (MKC), that finite precision measurement nullifies the Kochen Specker theorem, was criticised. In this paper a different, completely independent critical argument is presented. Mermin's variant of the GHZ set-up is analysed from the point of view of the MKC models. It is shown that the MKC models exhibit a novel kind of non-locality. It is further shown that they exhibit a novel kind of contextuality. It is a kind of contextuality which is arguably more radical than the kind which features in the Kochen-Specker theorem: for it means that the very existence of an observable, as something which can be measured, is, in general, context dependent.

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I. INTRODUCTION

Meyer\textsuperscript{1}, Kent\textsuperscript{2} and Clifton and Kent\textsuperscript{3} (to whom we will subsequently refer as MKC) have recently argued that, by taking proper account of the finite precision of any real measurement process, it is possible to nullify the Kochen-Specker theorem\textsuperscript{4-7}; from which they infer that “there is no truly compelling argument establishing that non-relativistic quantum mechanics describes classically inexplicable physics”\textsuperscript{3}. They suggest that this may have significant implications for quantum information theory and quantum computing.

In a previous paper\textsuperscript{8} we criticised the claims of MKC. Our argument was an elaboration of some of the points made in Mermin’s critique\textsuperscript{9} (for other critical remarks see Havlicek et al\textsuperscript{10}, Cabello\textsuperscript{11}, and Basu et al\textsuperscript{12}). Using the methods developed in Appleby\textsuperscript{13}, we analysed the predictions which the MKC models make regarding approximate joint measurements of non-commuting observables. We showed that a form of contextuality then re-emerges. Our conclusion was that, although MKC have nullified the Kochen-Specker theorem \textit{strictly so-called}, there are other, related propositions which are not nullified.

In this paper we present a different argument, which is independent of the one given in Appleby\textsuperscript{8}. The argument we gave previously rested on an analysis of the predictions which the MKC models make regarding a class of measurements which MKC do not themselves consider. In the argument presented here, by contrast, we confine ourselves to measurements which are of the same type as the ones discussed by MKC (\textit{i.e.} ideal measurements which are not precisely specified, in the terminology explained previously\textsuperscript{8}).

In the argument which follows we consider Mermin's variant\textsuperscript{6,14} of the GHZ set-up\textsuperscript{15,16} (see Fig. 1). We show that, on the assumptions of MKC, the possible alignments for one detector are not independent of the alignments of the other two detectors. This property represents a novel kind of non-locality (for other remarks concerning non-locality in the MKC models, see Havlicek et al\textsuperscript{10}) . It also represents—and this is the point which bears on the claim of MKC—a novel kind of contextuality.

The usual kind of contextual hidden variables theory—\textit{i.e.} the kind which features in the Kochen Specker theorem—has the property that the value which is assigned to one observable, as representing the value that would be recorded if that observable was measured, does, in general, depend on which other observables are jointly measured with it. This property may be described as the contextuality of \textit{measurable values}. The MKC models do not exhibit this property. However, they do exhibit another property which is, perhaps, even more striking, and which may be described as the contextuality of \textit{measurable possibilities}. In the usual kind of contextuality, it is only the value assigned to an observable which is context dependent. But in the kind of contextuality which features in the MKC models, it is the very existence of the observable being valued (its existence, that is, as something which can be measured) which is context dependent.

It should be stressed that, although the argument which follows involves some of the non-local features of the MKC models, the main point of the argument is, not this, but to show that the MKC models exhibit a form of contextuality.
II. THE GHZ SET-UP AND THE MODELS OF MEYER, CLIFTON AND KENT

Consider the arrangement (see fig. 1) which features in Mermin’s\textsuperscript{6,14} variant of the GHZ argument\textsuperscript{15,16} from the point of view of the MKC models. The system consists of three spin-1/2 particles. Let $\sigma^{(r)}$ denote the Pauli spin vector for particle $r$, and let $H^{(r)}$ be the 2-dimensional Hilbert space on which it acts. The spin state is

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|1, 1, 1\rangle - |1, -1, -1\rangle) \quad (1)$$

where $|s_1, s_2, s_3\rangle$ denotes the joint eigenstate of $\sigma_x^{(1)}$, $\sigma_y^{(2)}$, $\sigma_z^{(3)}$ with eigenvalues $s_1$, $s_2$, $s_3$. The particles emerge from a source and pass through three space-like separated detectors, as illustrated in Fig. 1. For each $r$ the corresponding detector measures one of the two target observables $\sigma_x^{(r)}$ or $\sigma_y^{(r)}$ (where we use the term “target observable” in the same sense as in Appleby\textsuperscript{8}, to refer to the observable which the apparatus is intended to measure). One has\textsuperscript{14}

$$\sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} |\psi\rangle = - |\psi\rangle \quad (2)$$

and

$$\sigma_x^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} |\psi\rangle = \sigma_y^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} |\psi\rangle = \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} |\psi\rangle = |\psi\rangle \quad (3)$$

Consequently, if the detectors are ideal, and if they are precisely set at the combination $xxx$, then the product of measured values must necessarily be $-1$. Similarly, if the detectors are ideal, and if they are precisely set at one of the combinations $xyy$, $yxy$, $yyx$, then the product of measured values must necessarily be $+1$.

In practice the detectors cannot be aligned with perfect precision. Consequently, if one attempts to measure the target observable $\sigma_x^{(r)}$ (respectively $\sigma_y^{(r)}$), then what one probably will in fact be measuring is the observable $\mathbf{n}_r \cdot \sigma^{(r)}$ where $\mathbf{n}_r$ is a unit vector close to, but not exactly coincident with the unit vector $\mathbf{e}_x$ (respectively $\mathbf{e}_y$) pointing along the $x$-axis (respectively $y$-axis). This being so there is a small probability that, in a measurement with the nominal settings $xxx$, the product of measured values will in fact be $+1$. Similarly, if the nominal settings are $xyy$, $yxy$ or $yyx$, then there is a small probability that the product of measured values will in fact be $-1$.

It should be noted that we follow MKC in confining ourselves to the case of ideal detectors whose alignments are not precisely specified\textsuperscript{8,9}. This is in contrast with the case of non-ideal detectors which has been widely discussed in the literature (see, e.g., Acacio de Barros and Suppes\textsuperscript{16}).

Let us now consider the situation from the point of view of the MKC models. Performing a joint measurement of the three target observables $\sigma_j^{(1)}$, $\sigma_j^{(2)}$, $\sigma_j^{(3)}$ (where $j_{123}$ is one of the combinations $xxx$, $xyy$, $yxy$, $yyx$) is equivalent to performing a joint measurement of the eight target projections

$$P_{s_1 s_2 s_3} = \frac{1}{8} (1 + s_j \sigma_j^{(1)})(1 + s_j \sigma_j^{(2)})(1 + s_j \sigma_j^{(3)}) \quad (4)$$

where, for each $r$, $s_r = \pm 1$. The unavoidable inaccuracies in the alignments of the three detectors mean that what is actually measured will be the neighbouring, but in general slightly different family of projections $P_{s_1 s_2 s_3}$. The argument of MKC rests on the assumption that $P_{s_1 s_2 s_3}$ is constrained to lie in a certain, carefully constructed dense subset of the set of all resolutions of the identity for the space $H^{(1)} \otimes H^{(2)} \otimes H^{(3)}$.

Define

$$\hat{P}_r^{(1)} = \hat{P}_{++} + \hat{P}_{+-} + \hat{P}_{-+} + \hat{P}_{--} \quad (5)$$

$$\hat{P}_r^{(2)} = \hat{P}_{++} + \hat{P}_{+-} + \hat{P}_{-+} + \hat{P}_{--} \quad (6)$$

$$\hat{P}_r^{(3)} = \hat{P}_{++} + \hat{P}_{+-} + \hat{P}_{-+} + \hat{P}_{--} \quad (7)$$
Then \( \hat{P}^{(r)} \), \((1 - \hat{P}^{(r)}) \) is the spectral resolution for the observable \( n_r \cdot \hat{\sigma}^{(r)} \) which is actually measured by detector \( r \) when attempting to measure the target observable \( \hat{\sigma}^{(r)} \). Consequently,

\[
\hat{P}^{(r)} = \frac{1}{2} (1 + n_r \cdot \hat{\sigma}^{(r)})
\]

from which it follows that

\[
\hat{P}_{s_1 s_2 s_3} = \frac{1}{8} (1 + s_1 n_1 \cdot \hat{\sigma}^{(1)}) (1 + s_2 n_2 \cdot \hat{\sigma}^{(2)}) (1 + s_3 n_3 \cdot \hat{\sigma}^{(3)})
\]

It should be noted that the construction given by Clifton and Kent does not ensure that \( \hat{P}_{s_1 s_2 s_3} \) is of the form specified by Eq. (9). It is not simply that Clifton and Kent leave open the possibility that the projections \( \hat{P}^{(r)} \) are non-local operators. It is not immediately apparent from their argument that it is even possible always to choose the operators \( \hat{P}_{s_1 s_2 s_3} \) in such a way that the operators \( \hat{P}^{(r)} \) are local. This is a defect in their argument; for it cannot reasonably be maintained that, merely in consequence of a detector not being aligned with perfect accuracy, what that detector measures is a non-local admixture of observables pertaining to more than one particle. In the following we will assume, for the sake of argument, that the defect can be and has been remedied.

It can be seen from Eq. (9) that the spectral family \( \hat{P}_{s_1 s_2 s_3} \), describing the measurement which is actually performed, is labelled by the vector triplet \((n_1, n_2, n_3) \in S_2 \times S_2 \times S_2 \), where \( S_2 \) denotes the unit 2-sphere \( \subset \mathbb{R}^3 \). The assumption of MKC may then be stated as follows: there exists a countable, dense subset \( S'_0 \subset S_2 \times S_2 \times S_2 \) such that the family \( \hat{P}_{s_1 s_2 s_3} \) is contained in the MKC set \( \mathcal{P}_d \) of physically measurable projections if and only if the corresponding triplet \((n_1, n_2, n_3) \in S'_0 \).

We will show that \( S'_0 \) is not a Cartesian product of the form \( S'^{(1)} \times S'^{(2)} \times S'^{(3)} \), where \( S'^{(1)}, S'^{(2)}, S'^{(3)} \subset S_2 \). The statements made in the Introduction are consequences of this proposition. We will begin by showing that \( S'_0 \) is not a Cartesian product in the case of the particular models described by Clifton and Kent. We will then go on to show that \( S'_0 \) is not a Cartesian product for any other model of the same general type.

Suppose that \( S'_0 \) is of the form \( S'^{(1)} \times S'^{(2)} \times S'^{(3)} \). Choose distinct \( n'_1, n''_1 \in S'^{(1)} \), and distinct \( n'_2, n''_2 \in S'^{(2)} \). Let \( n'_3 \in S'^{(3)} \). Define

\[
\hat{P}'_{s_1 s_2 s_3} = \frac{1}{8} (1 + s_1 n'_1 \cdot \hat{\sigma}^{(1)}) (1 + s_2 n'_2 \cdot \hat{\sigma}^{(2)}) (1 + s_3 n'_3 \cdot \hat{\sigma}^{(3)})
\]

\[
\hat{P}''_{s_1 s_2 s_3} = \frac{1}{8} (1 + s_1 n''_1 \cdot \hat{\sigma}^{(1)}) (1 + s_2 n''_2 \cdot \hat{\sigma}^{(2)}) (1 + s_3 n''_3 \cdot \hat{\sigma}^{(3)})
\]

Then \( \hat{P}'_{s_1 s_2 s_3} \), \( \hat{P}''_{s_1 s_2 s_3} \) are distinct resolutions of the identity contained in the MKC set \( \mathcal{P}_d \). In the case of the models described by Clifton and Kent such resolutions are (in the terminology explained in their paper) totally incompatible. On the other hand

\[
\frac{1}{8} (1 + n'_3 \cdot \hat{\sigma}^{(3)}) = \hat{P}'_{+++} + \hat{P}'_{++-} + \hat{P}'_{+-+} + \hat{P}'_{-++} = \hat{P}''_{+++} + \hat{P}''_{++-} + \hat{P}''_{+-+} + \hat{P}''_{-++}
\]

from which it follows that the families \( \hat{P}'_{s_1 s_2 s_3} \), \( \hat{P}''_{s_1 s_2 s_3} \) are not totally incompatible. We conclude that \( S'_0 \) is not of the form \( S'^{(1)} \times S'^{(2)} \times S'^{(3)} \) if the model is of the particular kind described by Clifton and Kent.

However, this argument is not conclusive because it exploits a feature of Clifton and Kent’s construction which is not evidently essential to the MKC program. We wish to show that \( S'_0 \) cannot be of the form \( S'^{(1)} \times S'^{(2)} \times S'^{(3)} \) even in the case of models for which it is not true that the resolutions of the identity which are contained in \( \mathcal{P}_d \) are pairwise totally incompatible. We will do this by using a modified version of the GHZ argument.

As before we will begin by assuming that \( S'_0 \) is of the form \( S'^{(1)} \times S'^{(2)} \times S'^{(3)} \), and then show that this assumption leads to a contradiction.

For each \( r \) let \( n_{rx}, n_{ry} \) be any pair of vectors \( \in S'^{(r)} \) such that \( n_{rx} \) (respectively \( n_{ry} \)) is close to \( e_x \) (respectively \( e_y \)), the unit vector in the \( x \) (respectively \( y \)) direction. Then

\[
\langle \psi | (n_{rx} \cdot \hat{\sigma}^{(1)}) (n_{rx} \cdot \hat{\sigma}^{(2)}) (n_{rx} \cdot \hat{\sigma}^{(3)}) | \psi \rangle = -(1 - \epsilon_0)
\]

\[
\langle \psi | (n_{rx} \cdot \hat{\sigma}^{(1)}) (n_{ry} \cdot \hat{\sigma}^{(2)}) (n_{ry} \cdot \hat{\sigma}^{(3)}) | \psi \rangle = (1 - \epsilon_1)
\]

\[
\langle \psi | (n_{ry} \cdot \hat{\sigma}^{(1)}) (n_{rx} \cdot \hat{\sigma}^{(2)}) (n_{rx} \cdot \hat{\sigma}^{(3)}) | \psi \rangle = (1 - \epsilon_2)
\]

\[
\langle \psi | (n_{ry} \cdot \hat{\sigma}^{(1)}) (n_{ry} \cdot \hat{\sigma}^{(2)}) (n_{rx} \cdot \hat{\sigma}^{(3)}) | \psi \rangle = (1 - \epsilon_3)
\]
where, for each \( a, \epsilon_a \geq 0 \). Let \( \epsilon = \max(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \). It follows from Eqs. (2) and (3), and from the continuity of the expectation values when regarded as functions of the vectors \( \mathbf{n}_{rj} \), that \( \epsilon \to 0 \) as \( \mathbf{n}_{rx} \to \mathbf{e}_x, \mathbf{n}_{ry} \to \mathbf{e}_y \) for \( r = 1, 2, 3 \). The fact that \( S'_6 \) is dense in \( S_2 \times S_2 \times S_2 \) means that \( S_2^{(r)} \) is dense in \( S_2 \) for \( r = 1, 2, 3 \). Consequently, the vectors \( \mathbf{n}_{rj} \) may be chosen so as to make \( \epsilon \) arbitrarily small.

Let \( \Lambda \) be the set of hidden variables, and for each \( \lambda \in \Lambda \) let \( t_\lambda \) be the corresponding truth function on the MKC set \( \mathcal{P}_A \). Let \( s_{rj}(\lambda) \) be the valuation of \( \mathbf{n}_{rj} \cdot \hat{\sigma}^{(r)} \) which \( t_\lambda \) induces. Then \( s_{rj}(\lambda) = \pm 1 \) for all \( r, j \). Define

\[
\begin{align*}
   f_0(\lambda) &= -s_{1x}(\lambda)s_{2x}(\lambda)s_{3x}(\lambda) \\
   f_1(\lambda) &= s_{1x}(\lambda)s_{2y}(\lambda)s_{3y}(\lambda) \\
   f_2(\lambda) &= s_{1y}(\lambda)s_{2x}(\lambda)s_{3y}(\lambda) \\
   f_3(\lambda) &= s_{1y}(\lambda)s_{2y}(\lambda)s_{3x}(\lambda)
\end{align*}
\]

Then \( f_a(\lambda) = \pm 1 \) for all \( a, \lambda \). Also

\[
f_0(\lambda)f_1(\lambda)f_2(\lambda)f_3(\lambda) = -(s_{1x}(\lambda)s_{2x}(\lambda)s_{3x}(\lambda)s_{1y}(\lambda)s_{2y}(\lambda)s_{3y}(\lambda))^2 = -1
\]

for all \( \lambda \).

Let \( \mu \) be the probability measure on \( \Lambda \) which corresponds to the state \( |\psi\rangle \). Then it follows from Eqs. (13)–(16) that

\[
1 - \epsilon \leq 1 - \epsilon_a = \int d\mu f_a(\lambda) \leq 1
\]

for all \( a \). For each \( a \) let \( A_a \) be the set

\[
A_a = \{ \lambda \in \Lambda : f_a(\lambda) = 1 \}
\]

Then it follows from Inequality (22) that

\[
1 - \epsilon \leq \int d\mu f_a(\lambda) \leq 2\mu(A_a) - 1
\]

for \( a = 0, 1, 2, 3 \). Hence \( \mu(A_a) \geq 1 - \epsilon/2 \) for all \( a \), which implies that \( \mu(A_0 \cap A_1 \cap A_2 \cap A_3) \geq 1 - 2\epsilon \). It was shown above that, with a suitable choice of the vectors \( \mathbf{n}_{rj} \), \( \epsilon \) can be made arbitrarily small. It follows that there exist vectors \( \mathbf{n}_{rj} \) such that \( \mu(A_0 \cap A_1 \cap A_2 \cap A_3) > 0 \) (in fact, there exist vectors for which \( \mu(A_0 \cap A_1 \cap A_2 \cap A_3) \approx 1 \)). On the other hand, it follows from Eq. (21) that \( \mu(A_0 \cap A_1 \cap A_2 \cap A_3) = 0 \) for every choice of the vectors \( \mathbf{n}_{rj} \); which is a contradiction.

We have thus shown that the set \( S'_6 \) does not have the form of a Cartesian product for any model of MKC type. This has important consequences. In the first place, it implies that it must, in general, happen that changing the alignment of one detector forces a change in the alignment of at least one of the other two detectors: which represents a novel form of non-locality. In the second place, and for our purposes more significantly, it implies that, in order to know what is a possible alignment for one detector, it is generally necessary to know how the other two detectors are aligned. This represents a novel form of contextuality. It is a particularly striking form of contextuality: for it is not simply the value assigned to an observable which is context dependent, but the very existence of that observable, as something which can be measured, which is so dependent.

### III. CONTEXTUALITY OF APPROXIMATE MEASUREMENTS

We now briefly discuss the bearing which the results proved in this paper have on those proved in Appleby\(^8\), concerning the contextuality of approximate measurements. In our previous paper we considered Kochen and Specker’s original example, of a spin 1 particle, with angular momentum \( \mathbf{L} \). We considered joint measurements of the projections \( (\mathbf{e}_r \cdot \mathbf{L})^2 \), for \( r = 1, 2, 3 \), in the case when the three unit vectors \( \mathbf{e}_r \) are not precisely orthogonal, so that the measurements are only approximate. The measurements may be performed sequentially, by three different detectors, or they may be performed “all at once”, by a single device. In the latter case we may continue to speak of “detectors”, in the plural, provided it is understood that by a “detector” is meant one of the parts, or one of the functions of a single, composite piece of apparatus. We showed that in such a case one of two alternatives must apply: namely, it must either happen that the outcome of an approximate measurement is, in general strongly dependent on the particular manner in which the measurement is carried out (which represents a form of contextuality); or else it must happen
that the fluctuations in one detector are not independent of the fluctuations in the others (which may also be regarded as a form of contextuality). Moreover, we stressed that, although it is conceivable that there exist models in which the fluctuations in the different detectors are correlated in just such a way as to make the outcome of an approximate measurement, with high probability, approximately independent of the manner in which it is performed, it would not be easy actually to demonstrate that this is the case.

In our previous paper we assumed that the set of possible alignments for one detector is independent of the way in which the other two detectors are aligned. This assumption is valid in the case of the model presented in Meyer’s original paper\(^1\). However, it appears from the discussion in the previous section that there may be other models for which it ceases to be valid. For such models, the fluctuations in the detectors would not generally be independent. However, it would not follow just from this that the outcomes of approximate measurements were, with high probability, approximately independent of context. In order to establish that the statistical fluctuations in the detector alignments are correlated in just that particular manner which is needed to ensure the non-contextuality of approximate measurements, it would probably be necessary to carry out a detailed analysis of the dynamics of the system+apparatus+environment composite. The models described by MKC are incomplete in this respect, since they do not include a specification of the dynamics.

IV. FURTHER REMARKS

The problem with the models described by MKC is that they are incomplete. MKC present us with the sketch, or the outline of a physical theory, rather than that theory itself. The criticisms given here, and in our previous paper, depended on developing the theory in directions which MKC do not themselves consider. It may be that further attempts to elaborate the theory would expose further difficulties.

Peres\(^18\) has remarked that MKC, besides nullifying the Kochen-Specker theorem, also nullify the principle of superposition. It might be thought that this is a purely mathematical point, which invites the question: how is it empirically known that the principle of superposition is strictly and exactly valid? It appears to us that this would be a mistake. It is true that Peres’ point is not, by itself, decisive. However, it is also true that MKC, by sacrificing the principle of superposition, inflict significant damage on the basic mathematical integrity of quantum mechanics; and this may reasonably be regarded as an indication that serious problems may appear in the course of trying to elaborate the detailed physical consequences of the theory. The argument presented in Section II may, perhaps, serve to illustrate this point: for the argument depended on some of the peculiarities in the way in which MKC treat the tensor product structure underlying the quantum mechanical description of composite systems.

We may discern another possible difficulty connected with the description of composite systems if we consider the process of measurement. MKC’s discussion of measurements is somewhat formal. They simply postulate that certain observables can be measured, while others cannot. They do not consider the details of the measurement process itself.

In order to measure an observable \(A\), pertaining to a given physical system, it is usually necessary to couple the system to a measuring apparatus. The result of the measurement is given by a pointer observable \(\hat{a}\), pertaining to the apparatus. The apparatus is itself a physical system: and the process of reading the pointer is itself a measurement. MKC assume, that to each physical system, there is an associated set of physically measurable observables; and they assume that this set is strictly smaller than the set of all self-adjoint operators. Let \(\mathcal{P}_{d,sy}\) be the set of measurable system observables, and let \(\mathcal{P}_{d,ap}\) be the set of measurable apparatus observables. The question then arises: are the definitions of these sets mutually consistent?

It seems clear, that if \(\hat{a} \in \mathcal{P}_{d,ap}\), then its value can be recorded, and so the observable \(\hat{A}\) is measurable. In other words, if it is true that \(\hat{a} \in \mathcal{P}_{d,ap}\), then it should follow that \(A \in \mathcal{P}_{d,sy}\). The arguments given by MKC do not establish that this is always the case, for every possible measurement interaction.

V. CONCLUSION

In conclusion we would stress that, notwithstanding the critical character of this paper, it appears to us that the constructions of MKC, besides being ingenious, and deeply interesting, are also important, and very valuable. Meyer, Kent and Clifton have taken a question which was previously thought to be well understood, and shown that in fact it was not so well understood. They have thereby significantly deepened our understanding of the conceptual implications of quantum mechanics.
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17 N.D. Mermin, personal communication.
18 A. Peres, personal communication.
FIG. 1. Set-up considered in Mermin's variant of the GHZ argument.