Width and Partial Widths of Unstable Particles

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Abstract

In the gauge theory context, a definition of branching ratios and partial widths of unstable particles is proposed that satisfies the basic principles of additivity and gauge independence. The difference between the new, gauge-independent formulation and the conventional one reflects the fact that unstable particles are not asymptotic states. Aside from restoring the crucial property of gauge independence, the new formulation avoids well-known pitfalls of the conventional approach.

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The mass, width, and partial widths of unstable particles rank among the basic concepts in particle physics. In fact, most fundamental particles of nature are unstable, and their masses, widths, and partial widths are some of their crucial defining properties. Yet, the precise and consistent definitions of these concepts have been notoriously difficult and elusive over a period spanning several decades. The reason is that unstable particles are not asymptotic states and, consequently, they lie somewhat outside the traditional formulation of Quantum Field Theory.

The traditional definitions of mass and width are

\[ M^2 = M_0^2 + \text{Re} A(M^2), \]
\[ M \Gamma = -\frac{\text{Im} A(M^2)}{1 - \text{Re} A'(M^2)}, \]

where \( M_0 \) is the bare mass and \( A(s) \) is the self-energy in the case of scalar bosons and the transverse self-energy in the case of vector bosons. The partial widths are then defined by decomposing the numerator of Eq. (2) into a sum of contributions involving distinct sets of final-state physical particles. Most calculations of partial and total decay rates are based on Eqs. (1) and (2). We will refer to \( M \) as the on-shell mass and to Eqs. (1) and (2) as the conventional on-shell formulation.

The emergence of gauge theories has brought into the discussion a new and powerful element, namely the requirement of gauge independence of physical observables. It was shown in Ref. [1] that, in a gauge theory, Eqs. (1) and (2) become gauge dependent in \( O(g^4) \) and \( O(g^6) \), respectively, where \( g \) is a generic gauge coupling. As the leading contributions to \( M^2 \) and \( \Gamma \) are of order \( O(g^0) \) and \( O(g^2) \), respectively, we see that in both cases the problem arises in the next-to-next-to-leading order (NNLO). In the same paper, it was proposed that the way of solving this predicament is to base the definitions of mass and width on the complex-valued position of the propagator’s pole:

\[ \bar{s} = M_0^2 + A(\bar{s}), \]

an idea that goes back to well known tenets of S-matrix theory [2]. A frequently employed parameterization is \( \bar{s} = m_2^2 - im_2 \Gamma_2 \), where we use the notation of Ref. [1]. Identifying \( m_2 \) and \( \Gamma_2 \) with the gauge-independent definitions of mass and width of the unstable particle, it follows from Eq. (3) that

\[ m_2 \Gamma_2 = -\text{Im} A(\bar{s}). \]

Other frequently employed parameterizations of \( \bar{s} \) will be mentioned later on. Over the last several years, a number of authors have advocated the use of \( \bar{s} \) as the basis for the definition of mass and width [3], the conclusions of Ref. [1] have been confirmed by later studies [4–8], and it has been shown that, in the case of a heavy Higgs boson, the gauge dependences of \( M \) and \( \Gamma \) are numerically large [6,7]. It has also been emphasized that the on-shell definition of width (Eq. (2)) leads to severe problems if \( A(s) \) is not analytic in the neighborhood of \( M^2 \). This occurs, for instance, when the mass of the decaying particle
lies very close to a physical threshold \[9\] or, in the resonance region, when the unstable particle is coupled to massless quanta, as in the cases of the \(W\) boson and the unstable quarks \[10\]. Several of these developments are reviewed in Ref. \[11\]. Important progress has also been achieved in the treatment of unstable particles in the Pinch Technique framework \[12,13\].

An important issue that arises at this stage is the following: if Eq. (4) provides a consistent definition of width, what is the definition of partial widths? It must clearly satisfy two important properties: additivity, \(i.e.\) the sum of the partial widths must equal the total width (Eq. (4)), and gauge independence. The fact that \(\text{Im} \, A(\bar{s})\) in Eq. (4) is not the analytic continuation of \(\text{Im} \, A(s)\), but rather the imaginary part of the analytic continuation of \(A(s)\) into the second Riemann sheet, hampers the decomposition of Eq. (4) into partial contributions corresponding to distinct physical processes. In order to circumvent this problem, we introduce the functions

\[
\begin{align*}
R(s) & \equiv \text{Re} \, A(s), \\
I(s) & \equiv \text{Im} \, A(s),
\end{align*}
\]

where \(s\) is the real, invariant four-momentum squared of the unstable particle. One readily finds the relation

\[
A(\bar{s}) = R(\bar{s}) + iI(\bar{s}),
\]

from which it follows that

\[
-\frac{m^2}{2} \Gamma_2 = \text{Im} \, A(\bar{s}) = \text{Im} \, R(\bar{s}) + \text{Re} \, I(\bar{s}),
\]

\[
\text{Re} \, A(\bar{s}) = \text{Re} \, R(\bar{s}) - \text{Im} \, I(\bar{s}).
\]

The function \(I(s)\) can be expressed as a sum of cut contributions according to the Cutkosky rules:

\[
I(s) = \sum_f I_f(s) + \sum_g I_g(s),
\]

where \(f\) and \(g\) represent physical and unphysical intermediate states, respectively.\(^1\) The latter include contributions involving unphysical degrees of freedom of the gauge bosons, as well as would-be Goldstone bosons and Faddeev-Popov ghosts. Eq. (9) suggests that the branching ratio into the physical final state \(f\) is given by

\[
B_f \equiv \frac{\Gamma_{2,f}}{\Gamma} = \frac{I_f(m_2^2)}{\sum_f I_f(m_2^2)},
\]

which is a natural generalization of the conventional expression

\[
\frac{\Gamma_f}{\Gamma} = \frac{I_f(M^2)}{I(M^2)}.
\]

\(^1\)A good illustration of the decomposition of Eq. (9) is provided, in the Higgs boson case, by Eq. (6) of Ref. [6], where the first and second terms correspond to the one-loop functions \(I_f(s)\) and \(I_g(s)\), respectively.
The important differences between Eqs. (10) and (11) will be discussed later on. By construction, Eq. (10) satisfies the property of additivity: \( \sum_f B_f = 1 \). We now discuss the issue of whether Eq. (10) satisfies the crucial requirement of gauge independence. Our analysis is based on the judicious application of Nielsen identities \([8,14,15]\), which describe the gauge dependence of Green functions. Specifically, we consider the Nielsen identity for the inverse propagator of the unstable particle:

\[
\frac{\partial}{\partial \xi} \Pi(s) = \Lambda(s)\Pi(s),
\]

where

\[
\Pi(s) = s - m_2^2 - R(s) + \text{Re} A(s) - iI(s),
\]

\( \xi \) is a generic gauge parameter, and \( \Lambda(s) \) is a Green function associated with relevant sources and fields \([8,15,16]\). \( \Lambda(s) \) involves unphysical degrees of freedom, it is complex, of leading order \( O(g^2) \), and finite over the real axis.\(^2\) The imaginary part of Eq. (12) is given by

\[
\frac{\partial}{\partial \xi} I(s) = R_A(s)I(s) - I_A(s)\text{Re} \Pi(s),
\]

where \( R_A(s) \equiv \text{Re} \Lambda(s) \) and \( I_A(s) \equiv \text{Im} \Lambda(s) \). Next, we insert Eq. (9) into Eq. (14). We note that the various terms \( I_f(s) \) in Eq. (9) involve distinct cut singularities associated with specific sets of intermediate particles with physical degrees of freedom, while \( I_g(s) \) and \( I_A(s) \) arise from unphysical intermediate states. As a consequence, Eq. (14) can be decomposed as

\[
\frac{\partial}{\partial \xi} I_f(s) = R_A(s)I_f(s),
\]

\[
\frac{\partial}{\partial \xi} G(s) = R_A(s)G(s) - I_A(s)\text{Re} \Pi(s),
\]

where \( G(s) \equiv \sum_g I_g(s) \). In this way, the l.h.s. and r.h.s. of Eq. (15) involve the same physical cut singularities, while both sides of Eq. (16) contain unphysical intermediate states.

The proportionality of \( \partial I_f(s)/\partial \xi \) and \( I_f(s) \) in Eq. (15) implies the gauge independence of \( I_f(s)/\sum_f I_f(s) \) and, since \( m_2 \) is also gauge independent, that of Eq. (10). This important property is not shared by the conventional expression of Eq. (11), since \( M \) is gauge dependent in \( O(g^4) \). There is a second, subtle difference between Eqs. (11) and (10). In the conventional formulation, it is assumed that \( I(M^2) \) can be expressed as a sum of contributions involving solely physical intermediate states, namely \( I(M^2) = \sum_f I_f(M^2) \). The argument invokes the unitarity of the S-matrix and it would, in fact, be valid if \( I(M^2) \)

\(^2\) Using Eq. (12), one can also show that the conventional definition of width (Eq. (2)) is gauge dependent in \( O(g^6) \), in agreement with the conclusions of Refs. [1,6,7].
were an S-matrix amplitude. However, as the unstable particle is not an asymptotic state, this is not the case, and the above relation must be viewed as an approximation. In fact, as discussed later on, the contribution \( G(M^2) \) to \( I(M^2) \) [and analogously, that of \( G(m_2^2) \) to \( I(m_2^2) \)] is not zero, but rather of \( O(g^6) \). Because of these two facts, Eq. (11) differs from Eq. (10) in gauge-dependent terms of relative order \( O(g^4) \), i.e. in NNLO.

It is also important to obtain a gauge-independent counterpart of the conventional expression of the total width (Eq. (2)). A simple way of achieving this is to rewrite Eq. (4) in the form

\[
m_2 \Gamma_2 = -\frac{X}{1 + [\text{Im}\, A(s) - X]/m_2 \Gamma_2},
\]

(17)

where \( X \) is an arbitrary amplitude. Solving Eq. (17) for \( m_2 \Gamma_2 \), one recovers Eq. (4), independently of \( X \), so that the two expressions are equivalent. A convenient choice for \( X \) is \( I(m_2^2) \), in which case Eq. (17) reduces to

\[
m_2 \Gamma_2 = -\frac{I(m_2^2)}{1 + [\text{Im}\, A(s) - I(m_2^2)]/m_2 \Gamma_2}.
\]

(18)

In the perturbative regime in which \( \Gamma_2/m_2 = O(g^2) \ll 1 \), we may expand Eq. (18) in powers of \( m_2 \Gamma_2 \). The leading terms are

\[
m_2 \Gamma_2 = -\frac{I(m_2^2) - m_2^2 \Gamma_2 I''(m_2^2)/2}{1 - R'(m_2^2)} + O(g^8),
\]

(19)

where the primes indicate differentiation with respect to the \( s \) variable. Noting that

\[
I(M^2) = I(m_2^2) + (M^2 - m_2^2) I'(m_2^2) + O(g^8) = I(m_2^2) - [I'(m_2^2)]^2 m_2 \Gamma_2 + O(g^8),
\]

we see that the leading terms in the difference between Eqs. (2) and (18) are given by \([I'(m_2^2)]^2 m_2 \Gamma_2 - m_2^2 \Gamma_2 I''(m_2^2)/2 + O(g^8)\). In some cases, such as that of a heavy Higgs boson, this difference is a sensitive function of the gauge parameter and becomes numerically very large as we approach the unitary gauge [6,7].

Eqs. (10) and (18) can be combined to provide a gauge-independent and additive expression for the partial widths:

\[
m_2 \Gamma_{2,f} = -\frac{I_f(m_2^2)}{F(m_2^2)} \frac{I(m_2^2)}{1 + [\text{Im}\, A(s) - I(m_2^2)]/m_2 \Gamma_2},
\]

(20)

where \( F(s) \equiv \sum_f I_f(s) \). Comparison with Eq. (19) shows that, in NNLO, this becomes

\[
m_2 \Gamma_{2,f} = -\frac{I_f(m_2^2)}{1 - R'(m_2^2)} \left[ 1 - \frac{G(m_2^2)}{m_2 \Gamma_2} + \frac{1}{2} m_2 \Gamma_2 I''(m_2^2) \right] + O(g^8).
\]

(21)

Using Eqs. (13) and (16), and the fact that \( G(m_2^2) \) vanishes in the unitary gauge, we find that, in leading order,

\[
\frac{G(m_2^2)}{m_2 \Gamma_2} = G^{(1)'}(m_2^2) \left[ F^{(1)'}(m_2^2) + \frac{1}{2} G^{(1)'}(m_2^2) \right] + O(g^8),
\]

(22)
where the superscripts refer to one-loop self-energies. The second and the third terms within the square brackets of Eq. (21) are, therefore, of $O(g^4)$. Like $R'(m_2^2)$, they represent universal, gauge-dependent corrections that cancel the gauge dependence of $I_f(m_2^2)$. However, unlike $R'(m_2^2)$, they occur in NNLO. In this approximation, they are easily calculated, since they are expressed in terms of one-loop self-energies.

In summary, the proposed definitions of branching ratios, total widths, and partial widths are given by Eqs. (10), (18), and (20). They satisfy the crucial requirements of gauge independence and additivity. Their differences with the conventional expressions reflect the fact that unstable particles are not asymptotic states. It is important to note that, in the denominator of Eqs. (18) and (20), the familiar factor $1 - R'(M^2)$, associated with the wave function renormalization, is replaced by $1 + [\text{Im } A(s) - I(m_2^2)]/m_2 \Gamma_2$. Although the two expressions coincide in the limit $\Gamma_2 \to 0$, for finite $\Gamma_2$ this modification removes the well-known pitfalls of the conventional approach when $M^2$ is very close to a physical threshold [9].

Other parameterizations of $\bar{s}$, with specific advantages, have been advocated in the literature. Examples are $\bar{s} = (m_1^2 - i m_1 \Gamma_1)/(1 + \Gamma_1^2/m_1^2)$ [1] and $\bar{s} = (m_3 - i \Gamma_3/2)^2$ [9,17]. As $m_1$, $\Gamma_1$, $m_3$, and $\Gamma_3$ are gauge independent, they can also be used in definitions of branching ratios and partial widths. On the other hand, Eqs. (10), (18) and (20) are particularly simple, as they involve only $\bar{s}$, the zero of $\Pi(s)$, and its real and imaginary parts, $m_2^2$ and $m_2 \Gamma_2$, respectively.

A few comments on the amplitudes $I_f(m_2^2)$ in Eqs. (10) and (20) are in order. It is understood that the physical final states $f$ are fully exclusive, i.e. all the particles must be identified. In phenomenological applications, one often considers semi-inclusive partial widths, e.g. $Z \to b\bar{b} + X$, where $X$ means unobserved or unidentified particles. Then, care must be exercised in order not to violate the additivity of the partial widths by multiply counting final states. For instance, $Z \to b\bar{b}c\bar{c}$ should not be included in both $Z \to c\bar{c} + X$ and $Z \to b\bar{b} + X$. Another caveat refers to the presence of hadrons in the final state. The decomposition in Eqs. (9) and (20) is formulated in the language of Feynman diagrams and, therefore, it is based on the validity of perturbation theory. Our considerations are thus restricted to the parton level of quarks and gluons, i.e. we are led to ignore the effects of confinement. However, these caveats apply equally well to the conventional approach.

In the past, the concepts of width and partial widths have often been discussed outside the realm of gauge theories [2,18], in which case the strong constraints arising from the requirement of gauge independence do not apply. However, it seems natural to generalize to such cases the expressions obtained in the gauge theory context. Alternatively, one may argue that all the current realistic theories of particle physics are gauge theories, so that the requirement of gauge independence of physical observables is of paramount importance.

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