Cylindrically symmetric dust spacetime

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Abstract

We present an explicit exact solution of Einstein’s equations for an inhomogeneous dust universe with cylindrical symmetry. The spacetime is extremely simple but nonetheless it has new surprising features. The universe is “closed” in the sense that the dust expands from a big-bang singularity but recollapses to a big-crunch singularity. In fact, both singularities are connected so that the whole spacetime is “enclosed” within a single singularity of general character. The big-bang is not simultaneous for the dust, and in fact the age of the universe as measured by the dust particles depends on the spatial position, an effect due to the inhomogeneity, and their total lifetime has no non-zero lower limit. Part of the big-crunch singularity is naked. The metric depends on a parameter and contains flat spacetime as a non-singular particular case. For appropriate values of the parameter the spacetime is a small perturbation of Minkowski spacetime. This seems to indicate that flat spacetime may be unstable against some global non-vacuum perturbations.

Exact solutions of Einstein’s equations for an inhomogeneous dust with an Abelian $G_2$ on $S_2$ group of motions seem to be difficult to find, and very few of them are known, see [1] and references therein. Among them, there are some Petrov type-D known cases belonging to the general class found by Szekeres [2], some examples appear in [3, 4]. However, it seems that no exact solution for an inhomogeneous dust with cylindrical symmetry ([5] and references therein) has been singled out. In this short letter we prove that, actually, there is a very simple cylindrically symmetric dust universe included in the Szekeres family, which surprisingly has not been studied or considered hitherto. Some plausible reasons for this overlooking will also be analized: apparently, some

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minor errors in [3] led to a misunderstanding which has kept the solution “behind the screen”.

As a matter of fact, the solution was recently rederived as the only dust solution defined by the following separation Ansatz in half-null coordinates [6]:

\[
ds^2 = -2 dudv + [f(u) + g(v)]^2 \ dy^2 + [h(u) + k(v)]^2 \ dz^2,
\]

where none of the derivatives of the given functions vanish, and the coordinates are taken to be comoving, that is, the dust velocity vector field reads \( \vec{u} \propto \partial/\partial u + \partial/\partial v \). Changing to typical Lorentzian coordinates the line-element takes the strikingly simple form

\[
ds^2 = -d\tau^2 + \rho^2 d\varphi^2 + \left(1 - \frac{t^2 + \rho^2}{\alpha^2}\right)^2 \ dz^2, \quad (1)
\]

where \( \alpha \) is an arbitrary constant. The cylindrical symmetry is explicit, where \( \rho > 0 \) is the cylindrical radius (the axis is at \( \rho = 0 \)) and \( \varphi \) is the corresponding angular coordinate running from 0 to \( 2\pi \). It can be checked that there are no further isometries. The coordinate \( z \) is taken to run from \( -\infty \) to \( \infty \), while the coordinate \( t \) will be restricted by the singularity of the spacetime, as we will see presently.

The dust is comoving in the coordinates of (1), its energy density given by

\[ \varrho = 4 \left( \alpha^2 - t^2 - \rho^2 \right)^{-1}, \]

while the non-vanishing components of the kinematical quantities for \( \vec{u} = \partial/\partial t \) computed in the orthonormal co-basis given by \( \theta^\alpha \propto dx^\alpha \) are

\[ \theta = -2t \left( \alpha^2 - t^2 - \rho^2 \right)^{-1}, \]

\[ \sigma_{11} = \sigma_{22} = -\frac{\theta}{3}, \quad \sigma_{33} = \frac{2\theta}{3}. \]

Of course, the acceleration and vorticity vanish. The scalars of the Weyl tensor, computed in the null tetrad (see [7]) \( k = 2^{-1/2}(\theta^0 - \theta^1), \ l = 2^{-1/2}(\theta^0 + \theta^1), \ m = 2^{-1/2}(\theta^2 + i\theta^3) \), read

\[ \Psi_0 = \Psi_4 = 3\Psi_2 = -\frac{1}{4}\varrho, \quad \Psi_1 = \Psi_3 = 0, \]

and thus these solutions are of Petrov type D.

The expressions of the energy density and of the Weyl tensor show that a curvature singularity appears at \( t^2 + \rho^2 - \alpha^2 \to 0 \), which can be viewed as a connected hypersurface. The meaningful spacetime region (defined by \( \varrho > 0 \)) is clearly given by

\[ t^2 + \rho^2 < \alpha^2; \]

and therefore the singularity ‘wraps’ the entire manifold. This singularity has a general character, as it is spacelike in the regions given by \( t \in [-\alpha, -\alpha/\sqrt{2}] \cup [\alpha/\sqrt{2}, \alpha] \) (so
Figure 1: This diagram corresponds to the \((t, \rho)\) plane of the spacetime with line-element given by (1). As usual, null lines are at 45°. The whole spacetime is the product of this plane with the \((\varphi, z)\)-surfaces, which correspond to the orbits of the 2-dimensional isometry group defining the cylindrical symmetry. The thick curve represents the spacetime singularity. The dust particles move along the geodesics represented by arrowed vertical lines in the physical region where \(\varrho > 0\), whose “size” is measured by the radius \(\alpha\) of the semi-circle. In the limit with \(\alpha \to \infty\) the singularity disappears and the solution becomes flat Minkowski spacetime.

that \(\rho \in [0, \alpha/\sqrt{2})\), null in \(t = \pm \alpha/\sqrt{2}\) (where \(\rho = \alpha/\sqrt{2}\)), and timelike at the region \(t \in (-\alpha/\sqrt{2}, \alpha/\sqrt{2})\) (thus \(\rho \in (\alpha/\sqrt{2}, \alpha]\)). This is shown in Figure 1.

All the causal geodesics start and end at the singularity. As a particular case, the timelike geodesics defined by the dust flow begin at the part of the singularity where \(t < 0\) with \(\theta \to \infty\), go on expanding up to \(t = 0\), where the dust starts to contract and eventually die in the future part of the singularity \((t > 0)\) where \(\theta \to -\infty\). Actually, all endless causal curves passing through any given point of the spacetime reach the singularity both in their past and future for a finite value of their generalized affine parameter. Thus, the singularity is past and future universal and can be termed as a big-bang/big-crunch singularity with a general character (see [8] for definitions).

Some interesting features can be deduced from the diagram of Figure 1. To start with, the big-bang and the big-crunch are not simultaneous, something which happens in some other well-known cases, see e.g. [9]. The total proper lifetime of any dust particle is always finite and depends on its position (on \(\rho\)), so that it is smaller for bigger \(\rho\). This total proper time has no lower bound, so that one can find dust particles with as short a lifetime as desired. On the other hand, the maximum lifetime for a dust particle is reached at the axis, and is given by \(t_{\text{max}} = 2\alpha\). This increases with \(\alpha\). Notice that for very big \(\alpha\) the metric approaches flat spacetime, and in the limit \(\alpha \to \infty\) the spacetime is exactly Minkowskian (and the singularity disappears!).
Actually, for small values of $1/\alpha$ the metric can be considered as a small perturbation of the flat Minkowski metric. As $\alpha$ is a free parameter, the smallness of the perturbation can be chosen at will. Notice also that the spacetime is obviously globally hyperbolic and, furthermore, the hypersurface $t = 0$ is maximal in the sense that its second fundamental form vanishes. Thus, this spacetime seem to indicate that the classical results on the global non-linear stability of Minkowski spacetime [10, 11] may not be generalizable to the case of *non-vacuum* perturbations (or to non-asymptotically-flat perturbations). It is interesting to see how this metric shows that the appearance of an arbitrarily small quantity of matter *everywhere* in flat spacetime destroys its global structure and encloses the matter within a strong curvature singularity.

Another curious property is that there are causal curves (and geodesics) which never cross the $t = 0$ hypersurface, starting and ending at the part of the singularity with $t < 0$. Thus, a part of the singularity with $t < 0$ is actually the future singularity for some physical particles (not for the dust, though). In fact, some dust particles (and other particles as well) can communicate with each other at their respective birth times. In other words, some dust particles can send causal signals (even from their bang starting time) which reach other dust particles at their corresponding birth times. Thus, for instance, a dust particle could tell another dust particle (with a big $\rho$) about its fate, so that the latter may decide to leave the dust motion and travel to smaller values of $\rho$ in order to live longer. Similarly, a dust particle dying in the big-crunch can send a signal at that very moment showing its fate to other dust particles which could see how their end looks like. In other words, the big-crunch singularity is partly “naked” and visible for the dust. In general, and as the metric is time-symmetric, all statements for the future have their corresponding past counterpart.

The line-element (1) is a dust solution with Petrov type D, so that it belongs to the family of the so-called Szekeres spacetimes [2, 9, 12], which were later generalized to a bigger family of Petrov D spacetimes [13, 14, 9] known as the Szafron-Szekeres solutions. The Szekeres solutions do not contain any isometry in general [15], but particular families of this kind of inhomogeneous solutions with a $G_2$ group of isometries were given in [3] and [4]. More precisely, the metric (1) is in fact contained in the case PII of the parabolic models in the classification of the Szekeres solutions presented by Bonnor and Tomimura in [3] (page 86) where the evolution of these models was studied.

Nevertheless, this solution was not under the scope of the aforementioned references mainly due to the fact that interest was focused on the study of the solutions containing only an initial spacelike singularity. Indeed, in [3] it was already established that the Szekeres solutions contain two possible singularities, corresponding to the vanishing of two functions appearing in the metric: the so-called Friedmann function $R(t)$ and another arbitrary function $Q(t, x, y, z)$ (in what follows we are using the notation of [3]). The singularity coming from $R = 0$ occurs on a spacelike hypersurface $t = 0$, whereas that from $Q = 0$ admit a great variety of behaviours, and thus in [3] the assumption that $t = 0$ were reached before $Q = 0$ was tacitly demanded. The arbitrary functions ought then to be chosen such that $Q > 0$ in the spacetime in order to have $\rho > 0$. 4
However, in the PII class one has $R = 1$ so that the only singularity corresponds to $Q = 0$. As we have seen, the line-element (1) explicitly reaches $Q = 0$, which now is simply $Q = 1 - (t^2 + r^2)/\alpha^2$. The rest of the functions as they appear in [3] are given as follows: $\mu = \nu = \sigma = 0$, $\omega = 1$, $\beta = \text{const.}(= -1/\alpha^2)$. In fact, it was argued in [3] that the PII class would always contain a region of the spacetime with a negative energy density, but this is not the case as we have explicitly proved for the solution under consideration: for the solution (1) the condition $Q > 0$ is enough to define a spacetime with $\varrho > 0$ everywhere.

Regarding [16], the class PII was again dismissed because it has no non-vacuum Friedmann-Lemaître-Robertson-Walker (FLRW) specialization, so that its singularity structure, as well as the study of its evolution, cannot be performed with a reformulated line-element in terms of increasing and decreasing density perturbation modes of FLRW dust models, which was the technique used.

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