Hamiltonian systems with boundaries

Maxim Zabzine\textsuperscript{1}

\textit{Institute of Theoretical Physics, University of Stockholm}
\textit{Box 6730, S-113 85 Stockholm SWEDEN}

\textbf{ABSTRACT}

Lately, to provide a solid ground for quantization of the open string theory with a constant \(B\)-field it has been proposed to treat the boundary conditions as hamiltonian constraints. It seems that this proposal is quite general and it should be applicable to a wide range of models defined on manifolds with boundaries. The goal of the present paper is to show how the boundary conditions can arise as constraints in a purely algebraic fashion within the Hamiltonian approach without any reference to the Lagrangian formulation of the theory. The construction of the boundary Dirac brackets is also given and some subtleties are pointed out. We consider four examples of field theories with boundaries: the topological sigma model, the open string theory with and without a constant \(B\)-field and electrodynamics with topological term. A curious result about electrodynamics on a manifold with boundaries is presented.

\textsuperscript{1}zabzin@physto.se
1 Introduction and motivation

Lately, there has been a renewed interest in field theories on manifolds with boundaries. In general one would expect a nontrivial relation between bulk and boundary dynamics. One such model is the open string theory with a constant $B$-field. It turns out that the bulk and boundary properties of this model are quite different [1]. Recently in attempt to provide a solid ground for the quantization of the model it has been proposed to treat boundary conditions as Hamiltonian constraints within the Dirac approach [2, 3, 4, 5]. This idea seems quite powerful and that it could be applied to a wide range of models.

Let us recall that for systems with boundaries traditionally the boundary conditions have to be imposed to define the functional derivatives in the theory properly. Specifically for the Hamiltonian treatment one needs the boundary conditions for the proper definition of the Poisson brackets (symplectic structure on the phase space). However one may take an alternative approach to the definition of symplectic structures, the algebraic one. Using three basic properties of the Poisson bracket (antisymmetry, Leibniz rule and Jacobi identity) and the canonical brackets for momenta and coordinates one can calculate any bracket. Of course now a function on the phase space should be understood as a formal power expansion in momenta and coordinates. This approach to the Poisson bracket is in the spirit of quantum mechanics where the algebraic definitions are the basic ones.

Thus applying the algebraic definition of Poisson bracket one does not have to impose the boundary conditions in order to do concrete calculations. As a matter of fact one sees that to define momentum, hamiltonian and primary constraints formally in many models there is no need to use the boundary conditions. Thus the natural question arises: what is the status of boundary conditions in the present framework. In this letter we try to answer this question. The main point which we are going to make is that the boundary conditions can arise in a purely algebraic fashion as Hamiltonian constraints localized on the boundary. As a result all the Dirac machinery may be applied\(^2\) to the boundary conditions vs the Hamiltonian boundary constraints. Indeed we need those boundary constraints to make the whole Hamiltonian treatment consistent.

\(^2\)That is true up to certain technical problems which, we believe, can be resolved.
Let us make a few technical remarks. The basic idea is rather naive. Since one can formally define the momentum, hamiltonian, the primary constraints and do calculations with the Poisson brackets without using the boundary conditions we may proceed in a formal way along Dirac’s lines using the canonical Poisson brackets [6]. However now we are not allowed to throw away the full derivative terms (the boundary terms). These boundary terms produce the corresponding constraints on the boundary. The usual consistency conditions have to be required for these constraints. To handle the technical side of the idea it is useful to work with constraints \( \Phi \) smeared with test functions \( N(x) \)

\[
\Phi[N] = \int d^d x \Phi(x) N(x).
\] (1.1)

This is often a convenient notation, especially when one wants to keep track of partial integration in a calculation. For the case of boundary constraint one assumes that the smearing function has support on the boundary only. (However those kind of assumptions do not effect the formal calculations anyhow.) In all calculations we will avoid the functional questions and will concentrate attention on the algebraic aspect of the computations. We will see that there is no ambiguity as soon as a calculation is done using test functions. However to define the Dirac brackets one has to do calculations without using the test functions (as far as author knows) and this may lead to trouble in some cases. To make sense of those boundary Dirac’s bracket in certain cases one should give a mathematically rigorous definitions of the relevant objects. We do not do this and just present the formal answer with short comments. We clarify this point by considering concrete examples.

It is worthwhile to make some remarks concerning the status of boundary conditions also within the Lagrangian formalism. The presence of boundary can spoil some properties which hold in the situation without boundary. For instance, two classically equivalent actions (in the sense that they reproduce the same equations of motion) can give rise different boundary conditions. A nontrivial example of this situation is the relation between the Howe-Tucker and Nambu-Goto actions for the open branes. These actions give slightly different boundary conditions. To relate them is subtle task and it depends on dimension of background space-time\(^3\). This is another reason to look at the Hamiltonian treatment of the boundary conditions.

\(^3\)For instance, the strings in two dimensional space-time: The Nambu-Goto action
The main motivation behind the present work is to understand the status of the boundary conditions in a quantum theory especially an interacting one. The Hamiltonian approach has certain advantages when it comes to quantizing a theory. (At least in principle it is clear what one should do.) At the end we will comment on the possible applications of our results to the quantum theory.

Since the general idea by itself is a simple one and it is fairly difficult to state any general theorem, examples are quite helpful. Thus throughout the paper we consider four examples: the topological sigma model with boundary, the open string theory with and without a constant $B$-field and four dimensional $U(1)$ gauge theory on a manifold with boundary. There is a section for every example and at the end we summarize the results and discuss the problems. In the first section we consider the topological sigma model with boundaries. This example is rather simple and specific. It demonstrates that there is a difference between the functional and algebraic approaches to the Poisson bracket and the former approach loses some interesting information about the boundary. In next two sections we consider the open string theory. We show that the boundary constraints can arise in a purely algebraic fashion from the algebra of constraints. We hope that the discussion in the fourth section will clarify some points of earlier analyses of the problem [2, 3, 4, 5]. We also point out that some problems may arise in the definition of modified symplectic structure on the boundary. The last section is devoted to Euclidean electrodynamics with a topological term on a manifold with boundaries. In the spirit of open string theory with $B$-field we derive the modification of symplectic structure on the boundary.

2 Topological sigma model with boundaries

Let us consider the topological sigma model with boundaries defined on 2d-dimensional smooth manifold which admits a symplectic structure $\omega$

$$ S = \frac{1}{2} \int_{\Sigma} d^2\xi \omega_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}, $$

(2.2)

is linear and it gives rise only the Dirichlet boundary conditions. However the Howe-Tucker (Polyakov) action is quadratic and it might produce as well the Neumann boundary conditions.

$\omega_{\mu\nu} dX^\mu \wedge dX^\nu$ is a symplectic structure if $d\omega = 0$ and $\omega_{\mu\nu}$ is not degenerate.
where $\Sigma$ is a two-dimensional world-sheet with boundary. In the bulk this model is purely topological and has no local degrees of freedom \[7\]. For the present purposes we ignore the topological aspects of the model and thus assume that the background space-time manifold can be covered by one patch. It means that we can think of the symplectic form as an exact two form $\omega = dA$ and the action (2.2) becomes

$$S = \int_{\partial \Sigma} d\tau A_\mu (X) \dot{X}^\mu,$$

(2.3)

which describes the boundary dynamics. The boundary dynamics are trivial ($\dot{X}^\mu = 0$) and there is just a modification of the Poisson brackets for $X^\mu$ since there are 2d second class constraints.

Now we want to try to extract the information about boundary dynamics through the Hamiltonian treatment starting from the action (2.2). The variation of the action (2.2) gives

$$\delta S = \int_{\partial \Sigma} d\xi \alpha \omega_{\mu \nu} \delta X^\mu \partial_\alpha X^\nu + \frac{1}{2} \int_{\Sigma} d^2 \xi \left( d\omega \right)_{\mu \nu \rho} \partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon^{\alpha \beta} \delta X^\rho,$$

(2.4)

where the last term vanishes by itself (see footnote) and the boundary term should vanish as well. Since $\omega_{\mu \nu}$ is nondegenerate then one should impose the Dirichlet boundary condition

$$\delta X^\mu |_{\partial \Sigma} \partial_{\tau} X^\mu \delta \tau = 0,$$

(2.5)

which simply means that the functional derivative with respect $X$ is not defined on the boundary. Thus the functional approach cannot be used to find the boundary dynamics. Instead we may use an algebraic approach to the problem. The action (2.2) produces 2d constraints

$$\Phi_\mu [N^\mu] = \int d\sigma N^\mu (P_\mu - \omega_{\mu \nu}(X) X^\nu),$$

(2.6)

which give the following Poisson bracket algebra

$$\{ \Phi_\mu [N^\mu], \Phi_\nu [M^\nu] \} = \int d\sigma \left( (d\omega)_{\mu \nu \rho} N^\mu M^\nu X^\rho \right) - N^\mu M^\nu \omega_{\mu \nu} |_{\partial \Sigma},$$

(2.7)

where $N^\mu$ and $M^\nu$ are test functions. The constraints (2.6) are first class in the bulk and second class constraints on the boundary. To simplify the
calculations we can do the following. Since we are working on one patch
one can assume that the symplectic structure $\omega_{\mu\nu}$ is a constant matrix of
a special form (due to the Darboux theorem in one patch there are always
special coordinates where the symplectic form can be brought to canonical
form). Thus because of (2.7) there is a suitable modification of the symplectic
structure on the boundary

$$\{X^\mu, X^\nu\}|_{\partial \Sigma} = \omega^{\mu\nu}$$  \hspace{1cm} (2.8)

where $\omega^{\mu\nu} \omega_{\nu\rho} = \delta^\mu_\rho$. Furthermore,

$$\{P_\mu, P_\nu\}|_{\partial \Sigma} = -\frac{1}{4} \omega_{\mu\nu}, \quad \{X^\mu, P_\nu\}|_{\partial \Sigma} = \frac{1}{2} \delta^\mu_\nu, \quad \{X^\mu, X^\nu\}|_{\partial \Sigma} = \frac{1}{2} \omega^{\mu\nu}.$$  \hspace{1cm} (2.9)

It is easy to check that all these brackets have the desired properties (every-
thing should have a trivial bracket with constraints on the boundary).
Proceeding along standard lines one finds that $\delta X^\mu = \dot{X}^\mu$ equals $N^\mu$ in the
bulk and zero on the boundary.

The present model is trivial, nevertheless it contains the essence of the
general situation of Hamiltonian models with boundaries. It shows that
there is a difference between the functional and the algebraic approaches
to the symplectic structure. In the algebraic approach the right boundary
conditions arise by itself in consistent way. In the next sections we consider
some less trivial examples of this situation.

### 3 Open string theory without a $B$-field

Let us consider the open string theory in a flat space-time ($\eta_{\mu\nu} = (-1, 1, \ldots, 1)$)
without antisymmetric background field. The model has the following action

$$S = -\frac{1}{2} \int d^2 \sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu},$$  \hspace{1cm} (3.10)

where $h^{\alpha\beta}$ is an auxiliary metric. The treatment of the theory is presented
in string theory textbooks (for instance [8]). To author’s knowledge the
canonical treatment of the open string was just given by Henneaux in his
lectures [9]. In this section we would like to have a new look at some well-
known facts about open strings. The action (3.10) produces the following
boundary condition

\[
(\sqrt{-h}h_{10} \dot{X}^\mu + \sqrt{-h}h_{11} \dot{X}^{\prime \mu})|_{0, \pi} = 0. \tag{3.11}
\]

If one starts from the Nambu-Goto action instead then the general boundary condition is \(\dot{X}^\mu \sim X^{\prime \mu}|_{0, \pi}\) which is equivalent to (3.11). The condition (3.11) can be rewritten in the phase space as follows

\[
(\eta_{\mu \nu} X^{\prime \nu} + \sqrt{-h}h^{01} P_\mu)|_{0, \pi} = 0, \tag{3.12}
\]

which states that \(\eta_{\mu \nu} X^{\prime \nu}\) and \(P_\mu\) are proportional to each other on the boundary.

Now let us turn to the Hamiltonian analysis of the system. For the model (3.10) the constraints are well known

\[
H_1[N] = \int_0^\pi d\sigma P_\mu X^{\prime \mu} N, \quad H[M] = \int_0^\pi d\sigma (P_\mu \eta^{\mu \nu} P_\nu + X^{\prime \mu} \eta_{\mu \nu} X^{\prime \nu}) M, \tag{3.13}
\]

and they hold at all points including boundary points. Since the system is generally covariant the naive Hamiltonian vanishes identically. Both constraints (3.13) are first class and they correspond to reparametrization of the two dimensional world sheet. The constraints obey the following Poisson bracket algebra

\[
\{H_1[N], H_1[M]\} = H_1[NM' - N'M], \tag{3.14}
\]

\[
\{H_1[N], H[M]\} = H[NM' - N'M] + NM(P_\mu \eta^{\mu \nu} P_\nu - X^{\prime \mu} \eta_{\mu \nu} X^{\prime \nu})|_0^\pi, \tag{3.15}
\]

\[
\{H[N], H[M]\} = H_1[4(NM' - N'M)]. \tag{3.16}
\]

The bracket between \(H_1\) and \(H\) gives rise the boundary term which should be set to zero to make the Hamiltonian treatment consistent. Since \(H_1\) and \(H\) hold everywhere we must require the following constraints on the boundary

\[
P_\mu \eta^{\mu \nu} P_\nu|_{0, \pi} = 0, \quad X^{\prime \mu} \eta_{\mu \nu} X^{\prime \nu}|_{0, \pi} = 0, \quad P_\mu X^{\prime \mu}|_{0, \pi} = 0. \tag{3.17}
\]

One might call them the boundary constraints. The next step should be to check whether these the algebra of new constraints is closed or not. As we said before all calculations can be done in a formal way in order to avoid questions
of regularization. For example let us introduce the following notation for the boundary constraints

\[ \phi_1[N] = \int_0^\pi d\sigma N P_\mu \eta^{\mu\nu} X_\nu, \quad \phi_2[M] = \int_0^\pi d\sigma M X''^\mu \eta_{\mu\nu} X''^\nu \]

(3.18)

where \( N \) and \( M \) might be thought as test functions localized on the boundary (or around boundary if there is some regularization assumed). This kind of assumptions does not effect the formal calculations. For instance we calculate the following brackets

\[ \{\phi_1[N], \phi_2[M]\} = 4 \int_0^\pi d\sigma \mathcal{N}P_\mu \eta^{\mu\nu} X_\nu + 4 \int_0^\pi d\sigma \mathcal{N}M_\mu X''^\mu - 4 \mathcal{N}M_\mu X''^\mu|_0^\pi, \]

(3.19)

and see that the secondary constraints arise. However the constraints (3.17) can be resolved since \( P \) and \( X' \) are null vectors on the boundary and they are orthogonal to each other there is a proportionality relation on the boundary

\[ (\alpha P_\mu + \eta_{\mu\nu} X''^\nu)_{0,\pi} = 0, \]

(3.20)

where \( \alpha \) is some proportionality constant which is subject to gauge condition (since it relates world-sheet density to the world-sheet vector). The conditions (3.20) give us the same information as one would get from the Lagrangian formalism (3.12). Hence the whole system can be described as two first class constraints \( H_1, H \) plus set of second class boundary constraints (3.20). The constraints (3.20) are second class because of the non vanishing brackets

\[ \{\Phi_{\mu}[N^\mu], \Phi_{\nu}[M^\nu]\} = \alpha \int_0^\pi d\sigma [N''^\nu M''^\mu - N''^\mu M''^\nu] \eta_{\mu\nu}, \]

(3.21)

where \( \Phi_{\mu}[N^\mu] \) is (3.20) smeared with the test function \( N^\mu \). Proceeding formally for the second class constraints (3.20) we define the corresponding Dirac brackets

\[ \{X'^\mu(\sigma), X'^\nu(\sigma')\} = \frac{\alpha}{2} \eta^{\mu\nu} \frac{1}{\partial_\sigma} \delta(\sigma - \sigma'), \]

(3.22)

as well as the brackets

\[ \{X'^\mu(\sigma), P_\nu(\sigma')\} = \frac{1}{2} \delta^{\mu\nu} \delta(\sigma - \sigma'), \quad \{X''^\mu(\sigma), X''^\nu(\sigma')\} = \frac{\alpha}{2} \eta^{\mu\nu} \delta(\sigma - \sigma'). \]

(3.23)
We are interested in the restriction of these brackets to the boundary and it is rather unclear how to do it, especially for the non-local bracket (3.22). The point is that this question can not be answered unless our description of the model is supplemented with a certain amount of additional information. The extra information should concern the restrictions on the behaviour of the fields in order to make operator $\partial_\sigma$ invertable (in general there is a constant zero mode for this operator). Therefore to make further progress one should have more insight into the model. It would be interesting to quantize the free open string theory in the nonconformal gauge (where $\alpha \neq 0$) and calculate the commutators (3.22), (3.23) explicitly on the boundary. Resolving this kind of questions can lead to the proper understanding of the foundations of Witten’s open string field theory [10] where the noncommutative of the ends of string plays crucial role.

4 Open string theory with a constant $B$-field

Now let us turn to the open string theory with a constant $B$-field. The model has the following action

$$S = -\frac{1}{2} \int d^2 \sigma \left( \sqrt{-h} h^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu \nu} - \epsilon^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu \nu} \right). \quad (4.24)$$

This system attracted much attention recently because of the noncommutative properties of the end points of string. The treatment of the model has been given in [1] (also see [11] for the quite full list of references). Let us just recall that in the Lagrangian formalism one should impose the boundary conditions

$$(\sqrt{-h} h^{10} \eta_{\mu \nu} \dot{X}^\nu + \sqrt{-h} h^{11} \eta_{\mu \nu} X^\nu + B_{\mu \nu} \dot{X}^\nu)|_{0, \pi} = 0, \quad (4.25)$$

which have the following form in the phase space

$$(B_{\mu \nu} P_\nu + G_{\mu \nu} X^\nu + \sqrt{-h} h^{01} P_\mu)|_{0, \pi} = 0, \quad (4.26)$$

where $G_{\mu \nu} = \eta_{\mu \nu} - B_{\mu \sigma} B^{\sigma \nu}$. For the sake of simplicity we assume that $B$ is a non-degenerate matrix (for the degenerate case one can easily generalize all the following arguments).
Now we look at the Hamiltonian formalism. In the usual fashion the constraints are

\[ H_1[N] = \int_0^\pi d\sigma P_\mu X'^\mu N, \]  
\[ H[M] = \int_0^\pi d\sigma (P_\mu \eta^{\mu\nu} P_\nu - 2P_\mu B_\mu X'^\nu + X'^\mu G_{\mu\nu} X'^\nu) M. \]  

These are first class constraints and they hold everywhere including at the boundary points. Next we calculate the algebra keeping track of the boundary terms. The constraints obey the following Poisson bracket algebra

\[ \{H_1[N], H_1[M]\} = H_1[NM' - N'M], \]  
\[ \{H_1[N], H[M]\} = H[NM' - N'M] + NM(P_\mu \eta^{\mu\nu} P_\nu - X'^\mu G_{\mu\nu} X'^\nu)|_0,\pi, \]  
\[ \{H[N], H[M]\} = H[4(NM' - N'M)]. \]  

To make the theory consistent one should set the boundary term to zero. Since \( H_1 \) and \( H \) hold everywhere there is a boundary constraint

\[ X'^\mu (B_\mu P_\nu + G_{\mu\nu} X'^\nu)|_{0,\pi} = 0, \]  

which is the difference between \( H_1 \) and the boundary term in (4.30). One cannot solve the system in as simple way as before. Therefore we proceed along Dirac’s lines [6]. We look at possible secondary and tertiary constraints and then try to separate them into first and second class constraints. Sometimes before separating them into different classes it might be helpful to solve some of the constraints.

We thus have to calculate brackets of all constraints including the boundary one and see if any new constraints arise. We will do the calculations in a formal way and introduce the following notation for the boundary constraint

\[ \Phi[N] = \int_0^\pi d\sigma N X'^\mu (B_\mu P_\nu + G_{\mu\nu} X'^\nu), \]  

where \( N \) is a test function. As a result of the computations there will be some new constraints. Let us look at some of them to see the pattern

\[ \{\Phi[N], \Phi[M]\} = \int_0^\pi d\sigma [NM' - N'M]X'^\mu B_\mu B_\rho (B_\rho P_\nu + G_{\rho\nu} X'^\nu). \]
Introducing the following notation for the new constraint

\[ \Phi_1[N] = \pi \int_0^\pi d\sigma \, N X^{\mu} B_\mu^\rho (B_\rho^\nu P_\nu + G_{\rho\nu} X^{\nu}), \]  

(4.35)

we get

\[ \{ \Phi_1[N], \Phi_1[M] \} = \pi \int_0^\pi d\sigma \, [NM' - N'M] X^{\mu} B_\mu^\delta B_\delta^\sigma B_\sigma^\rho (B_\rho^\nu P_\nu + G_{\rho\nu} X^{\nu}), \]  

(4.36)

and

\[ \{ \Phi_1[N], \Phi[M] \} = \pi \int_0^\pi d\sigma \, [NM' - N'M] X^{\mu} B_\mu^\delta B_\delta^\rho (B_\rho^\nu P_\nu + G_{\rho\nu} X^{\nu}), \]  

(4.37)

and so on. There seems to be implications for the following conditions on the boundary

\[ X^{\mu} M_\mu^\sigma (B_\sigma^\nu P_\nu + G_{\sigma\nu} X^{\nu})|_{0,\pi} = 0, \]  

(4.38)

where \( M \) is some power of \( B \). Since \( B \) is nondegenerate and antisymmetric all these conditions can be replaced by the following one

\[ (B_\mu^\nu P_\nu + G_{\mu\nu} X^{\nu} + \beta P_\mu)|_{0,\pi} = 0, \]  

(4.39)

where \( \beta \) is the coefficient of proportionality which is subject to the gauge condition (like \( \alpha \) in the previous section). We will see that (4.39) are second class constraints. Introducing the notation

\[ \mathcal{K}_\mu[N^\mu] = \pi \int_0^\pi d\sigma \, N^\mu (B_\mu^\nu P_\nu + G_{\mu\nu} X^{\nu} + \beta P_\mu) \]  

(4.40)

it is easy to check the brackets

\[ \{ \mathcal{K}_\mu[N^\mu], \mathcal{K}_\nu[M^\nu] \} = \pi \int_0^\pi d\sigma \, [N^\mu M^{\nu} - N^{\mu} M^\nu](B_\mu G_{\rho\nu} + \beta G_{\mu\nu}) \]  

(4.41)

where at the right-hand side we have nondegenerate matrix. Therefore we conclude that in order to make the whole Hamiltonian treatment consistent
one must impose the boundary conditions (4.26) which play the role of second class constraint on the boundary. Otherwise the algebra (4.31) would not be closed. Therefore the bracket algebra should be modified on the boundary. The Poisson bracket has to be replaced by the Dirac bracket on the boundary. For example on the boundary the coordinates have the following bracket on the boundary

$$\{X^\mu, X^\nu\}_\partial = -B^\mu_\sigma (G^{-1})^{\sigma\nu} + \beta (\text{nonlocal part}) \quad (4.42)$$

where the nonlocal part has the same structure like in the previous section. For the case of $\beta = 0$ (for instance, conformal gauge or static gauge) the brackets (4.42) are well defined. The discussion of the modified brackets is given in [2, 3, 4, 5].

5 Electrodynamics with topological term

As the last example we consider a different kind of theory which has a non-vanishing Hamiltonian. We will take a look at four dimensional Euclidean electrodynamics with a topological term. The action is defined by the following expression

$$S = \frac{1}{2g^2} \int_M F \wedge *F + \frac{i\theta}{4\pi^2} \int_M F \wedge F, \quad (5.43)$$

where we use differential form notation or, equivalently, in components

$$S = \frac{1}{4g^2} \int_M d^4 x F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{4\pi^2} \int_M d^4 x \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. \quad (5.44)$$

The theory is defined on the manifold $\mathcal{M}$ with non empty boundary $\partial \mathcal{M}$. Since we are interested in the Hamiltonian treatment we assume that $\mathcal{M} = R \times \Sigma$ where $\Sigma$ is a spatial manifold. For the sake of simplicity we suppose that $\Sigma$ is closed set in $R^3$ and thus it carries a flat metric. This assumption is not essential and the whole logic can be generalized to the general curved case.

Before going into Hamiltonian formalism we take a brief look at the Lagrangian formalism. To the author’s knowledge this system has not been
specially studied anywhere except in [13]. The action (5.43) gives the following equations of motion

\[ d \ast F = 0, \]

which should be supplemented by the boundary condition

\[ \int_{\delta M} \delta A \wedge \left( \frac{1}{g^2} \ast F + \frac{i \theta}{4 \pi^2} F \right) = 0. \]  

To proceed further let us write (5.46) in components

\[ - \int dt \int d^2 s \left[ n^a \left( \frac{1}{g^2} E_a + \frac{i \theta}{2 \pi^2} B_a \right) \delta A^0 - n_b \epsilon^{abc} \delta A_c \left( \frac{1}{g^2} B_a + \frac{i \theta}{2 \pi^2} E_a \right) \right], \]

where we introduce the standard conventions \( E_a \equiv F_{0a} \) and \( B_a = \frac{1}{2} \epsilon^{abc} F_{bc} \) and \( n^a \) is normal vector to \( \partial \Sigma \). Now it is straightforward to read off the boundary conditions

\[ n^a \left( \frac{1}{g^2} E_a + \frac{i \theta}{2 \pi^2} B_a \right)|_{\partial M} = 0 \quad \text{or} \quad \delta A^0|_{\partial M} = 0, \]

\[ n_b \epsilon^{abc} \left( \frac{1}{g^2} B_a + \frac{i \theta}{2 \pi^2} E_a \right)|_{\partial M} = 0 \quad \text{or} \quad \delta A_c|_{\partial M} = 0. \]

Thus one of the condition from these two sets should be imposed to make the Lagrangian treatment consistent\(^5\).

Let us rewrite the boundary conditions (5.48), (5.49) in the phase space. The momentum is defined as follows

\[ \pi_a = \frac{1}{g^2} E_a + \frac{i \theta}{2 \pi^2} B_a, \]

and there is the usual constraint \( \pi_0 = 0 \) which we will discuss later on. Using (5.50) the boundary conditions (5.48), (5.49) become

\[ n^a \pi_a|_{\partial M} = 0 \quad \text{or} \quad \delta A^0|_{\partial M} = 0, \]

\(^5\)Apart from this one can think about other physical requirements such as absence of momentum flow through the boundary. We will ignore this kind of questions.
There are one normal condition (on the left handside (5.51)) and two tangential conditions (on the left handside (5.52)). We will keep in mind these expressions. We hope to get them as the boundary constraints which must be required in order to make the whole Hamiltonian treatment consistent. Let us assume that one can choose such a coordinate system that the normal vector has the form \( \vec{n} = (1, 0, 0) \).

Let us turn to the Hamiltonian treatment. Using (5.43) and (5.50) one defines the Hamiltonian

\[
H = \int_{\Sigma} d^3x \left[ \frac{g^2}{2} \pi_a \pi^a - \frac{i \theta}{2 \pi^2} g^2 \pi_a B^a - \frac{1}{2} \left( \frac{1}{g^2} - \frac{i \theta}{2 \pi^2} g^2 \right) B_a B^a + (\partial_a A_0) \pi^a \right].
\]

(5.53)

Thus one defines the Hamiltonian \( H \) and primary constraint \( \pi_0 \) without using the boundary conditions. Introducing the notation

\[
\Pi[\Lambda] = \int_{\Sigma} d^3x \Lambda(x) \pi^0(x),
\]

(5.54)

one has the following bracket

\[
\{\Pi[\Lambda], H\} = \mathcal{G}[\Lambda] - \int_{\partial \Sigma} d^2s \Lambda(n^a \pi_a),
\]

(5.55)

where \( \mathcal{G}[\Lambda] \) is the Gauss law constraint

\[
\mathcal{G}[\Lambda] = \int_{\Sigma} d^3x \Lambda(x) \partial_a \pi^a(x).
\]

(5.56)

Consistency then implies that the right hand of (5.55) must be equal to zero. Thus we are getting the standard Gauss law and as well the boundary constraint \( n^a \pi_a = 0 \) (or if one assumes that \( A^0 \) is zero at boundary then there is no boundary term in (5.55)).

Following the standard prescription one should look at the time evolution of the Gauss law (5.56) and the new boundary constraint

\[
\Phi_1[\Lambda] = \int_{\Sigma} d^3x \Lambda n^a \pi_a,
\]

(5.57)
where $\Lambda$ can be thought as test function with support on $\partial \Sigma$. The formal computation gives us the following result

$$
\{G[\Lambda], H\} = \int_{\partial \Sigma} d^2 s \left[ n_b \partial_c \Lambda \epsilon^{abc} \right] \left( \frac{i\theta}{2\pi^2} g^2 \pi^a + \left( \frac{1}{g^2} - \left( \frac{i\theta}{2\pi^2} \right)^2 g^2 \right) B^a \right),
$$

(5.58)

$$
\{\Phi_1[\Lambda], H\} = \int_{\Sigma} d^3 x \left[ \partial_c (\Lambda n_b) \epsilon^{abc} \right] \left( \frac{i\theta}{2\pi^2} g^2 \pi^a + \left( \frac{1}{g^2} - \left( \frac{i\theta}{2\pi^2} \right)^2 g^2 \right) B^a \right).
$$

(5.59)

The bracket (5.58) is localized on the boundary and it gives us the tangential boundary constraints which exactly coincide with the boundary conditions (5.52). The same boundary constraint is given by the bracket (5.59) and it is localized on the boundary since $\Lambda$ has support on the boundary only. Therefore we introduce the new boundary constraint

$$
\Phi_a[N^a] = \int d^3 x N_a \left[ \frac{i\theta}{2\pi^2} g^2 \pi^a + \left( \frac{1}{g^2} - \left( \frac{i\theta}{2\pi^2} \right)^2 g^2 \right) B^a \right],
$$

(5.60)

where it is assumed that $N^a = (0, N_2, N_3)$. To decide on the status of boundary constraints $\Phi_1, \Phi_2$ and $\Phi_3$ one should calculate the following brackets

$$
\{\Phi_1[\Lambda], \Phi_b[N^b]\} = \left( \frac{1}{g^2} - \left( \frac{i\theta}{2\pi^2} \right)^2 g^2 \right) \int_{\Sigma} d^3 x \Lambda n^a \partial^b N^c \epsilon_{abc},
$$

(5.61)

$$
\{\Phi_a[N^a], \Phi_b[M^b]\} = \frac{i\theta}{2\pi^2} g^2 \left( \frac{1}{g^2} - \left( \frac{i\theta}{2\pi^2} \right)^2 g^2 \right) \int_{\Sigma} d^3 x \partial^a (N^b M^c) \epsilon_{abc}.
$$

(5.62)

One notices that the brackets (5.62) of $\Phi_a$ ($a = 2, 3$) are non-zero because of the boundary term on the right hand side of (5.62). In the bulk such brackets are zero since the constraints $\Phi_a$ are generalization of the chiral condition for two forms [12]. Therefore the brackets of all boundary constraints give a field independent antisymmetric matrix with rank 2. It turns out that there is one first class boundary constraint $n^a \pi_a$ which makes us able to gauge away the normal component of the connection on the boundary. The boundary constraints $\Phi_2$ and $\Phi_3$ are second class constraints which lead to the the following Dirac bracket on the boundary

$$
\{A_2(x), A_3(y)\}|_{\partial \Sigma} = \frac{i\theta}{2\pi^2} g^2 \left( \frac{1}{g^2} + \frac{\theta^2}{4\pi^2} g^2 \right)^{-1} \delta^{(2)}(x - y).
$$

(5.63)
In analogy with the models considered in the previous section we see that at the boundary the Poisson brackets should be replaced by the corresponding Dirac bracket. In general the boundary Dirac bracket will depend on the geometry of the boundary. We do not discuss this subject in the present work but return to the discussion elsewhere. The physical interpretation of (5.63) is unclear. It seems that one will have problems with localizing of photons on the boundary. Certainly this subject deserves an independent study and we go no further in the analysis of the boundary theory.

6 Discussion and problems

In this paper we made an attempt to understand the status of the boundary conditions within the Hamiltonian formalism motivated by the quantum theory. We have shown that the boundary conditions can arise in a purely algebraic fashion as Hamiltonian boundary constraints. Their existence is necessary to make the whole Hamiltonian treatment consistent. Our arguments were based on the four examples: the topological sigma model, the open string theory with and without a B-field and electrodynamics with a topological term. For some systems it is important to motivate that the boundary conditions can be treated as Hamiltonian constraints. This type of systems has non trivial boundary conditions which mix momenta and coordinates. Such boundary conditions change the canonical brackets on the boundary drastically and therefore they are very important for the quantization of the system as whole. However as we saw in some instances (e.g.,(3.22)) problems can arise with the definition of the Dirac bracket on the boundary. To resolve those problems one needs more insight into the models. In other cases there is no ambiguity in defining the modified symplectic structure (e.g., (2.8), local part of (4.42) and (5.63)).

As can be seen from the last example the boundary conditions give rise not only to second class constraints but also to first class constraint. It is unclear how this kind of boundary constraint should be applied in the quantum theory. We hope to return to this question and do explicit calculations for this model in the presence of a simple boundary.

It would be interesting to study an interacting theory like Yang-Mills theory and gravity systems in this context. We hope to treat these questions elsewhere.
Acknowledgments

It is pleasure to thank Ingemar Bengtsson, who has promoted my interest in the subject and who has helped a lot during the preparation of this work. I am grateful to Ingemar Bengtsson and Ulf Lindström for reading and commenting on the manuscript.
References


