TELEOR CALCULUS FOR SUPERGRAVITY

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ABSTRACT

The multiplication rule for multiplets of Poincaré and conformal supergravity is obtained. It is shown how to construct locally supersymmetric densities and hence actions out of these multiplets. Various examples illustrate these general results.

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In global supersymmetry a tensor calculus for the multiplication of multiplets exists, provided one includes the auxiliary fields needed to close the global algebra \(^1\). Recently, the minimal set of auxiliary fields needed to close the gauge algebra of Poincaré supergravity \(^2\),\(^3\) was found, and it was also found that the gauge algebra of conformal supergravity \(^4\),\(^5\) closes \(^4\). In this letter we formulate a tensor calculus for the multiplication of scalar multiplets of Poincaré and conformal supergravity with or without internal indices, and show how to construct supersymmetric scalar densities and hence actions out of these multiplets \(^\ast\).

In order to put our results in perspective, we note that the introduction of global multiplets and a multiplication rule for them obviates the necessity for manipulations with individual components in order to obtain invariants. Similarly, by introducing the multiplets of local supersymmetry and a multiplication rule for these local multiplets, we can now obtain actions without the laborious order-by-order-in-\(\lambda\) Noether construction \(^6\).

We begin by recalling that two scalar multiplets of global supersymmetry can be multiplied to yield again a scalar multiplet as follows \(^1\)

\[
(A,B,X,F,G) \odot (a,b,\chi,\xi,\eta) = (Aa - Bb + Ab + aB, \\
(A+i\chi_5 B)\chi + (a+i\xi_5) X, A\xi_B + aF - B\eta_B - G - \overline{X} \chi, \\
A \eta + aG + B\xi + b\eta + i \overline{X} \chi_5 \chi)
\]

Examples are the canonical scalar multiplet \(\Sigma\) with two real scalar fields \(A\) and \(F\), two real pseudoscalar fields \(B\) and \(G\) and a Majorana spinor \(\chi\), the kinetic multiplet \(T(\Sigma) = [\mathcal{P}, -G, \chi, \chi^\dagger, \mathcal{D}, -\mathcal{D}^\dagger]\) and the vector multiplet \(\Phi\)

\[
\mathcal{W}_\alpha = \left[ \mathcal{X}_\alpha, -i(\chi_5)_{\alpha\beta}, (-F + i\chi_5 \mathcal{D})_{\beta\alpha}, -\mathcal{V}_\alpha, (\chi_{\beta} \alpha), i \mathcal{T}_\alpha (\chi_{\beta} \chi_5)_{\alpha}\right].
\]

\(^\ast\) Formulæ which look very similar to our formulæ were obtained some time ago for two-dimensional supergravity (spinning string) by one of us (S.F.) and B. Zumino (unpublished). However, these results could not be used in four dimensions since the auxiliary field structure is different and since we needed recent methods based on conformal supergravity.
In the last case the internal spinorial indices are contracted with the charge conjugation matrix $c^{\alpha\gamma'}$. The global supersymmetry transformation rules of a scalar multiplet with weight $\lambda$ are given by

$$
\begin{align*}
\delta A &= \bar{\epsilon}_Q \chi, \quad \delta B = -i \bar{\epsilon}_Q \gamma_5 \chi \\
\delta \chi &= D(A - i \gamma_5 B) \epsilon_Q + (F + i G \gamma_5) \epsilon_Q + \lambda (A + i \gamma_5 B) \epsilon_S \\
\delta F &= \bar{\epsilon}_Q \gamma_5 \chi + (1 - \lambda) \chi \epsilon_S, \quad \delta G = i \bar{\epsilon}_Q \gamma_5 \chi - i (1 - \lambda) \Gamma \chi \epsilon_S
\end{align*}
$$

(2)

and the product of two multiplets with weights $\lambda_1$ and $\lambda_2$ has weight $\lambda_1 + \lambda_2$. The parameters $\epsilon_Q$, $\epsilon_S$ are the constant spinorial parameters of global $Q$ and $S$ supersymmetry transformations of the superconformal group $^1$).

For conformal supergravity we choose the same scalar multiplets as basic multiplets but with transformation rules

$$
\begin{align*}
\delta^c Q A &= \bar{\epsilon} \chi, \quad \delta^c Q B = -i \bar{\epsilon} \gamma_5 \chi \\
\delta^c Q \chi &= \bar{\Delta}^c (A - i \gamma_5 B) \epsilon + (F + i G \gamma_5) \epsilon \\
\delta^c Q F &= \bar{\epsilon} \bar{\Delta}^c \chi, \quad \delta^c Q G = i \bar{\epsilon} \gamma_5 \bar{\Delta}^c \chi
\end{align*}
$$

(3)

while the superconformal gauge fields transform as $^4$)

$$
\begin{align*}
\delta^c Q e^a \gamma\rho &= \bar{\epsilon} \gamma^a \gamma^\rho, \quad \delta^c Q A^a = 3 i \bar{\epsilon} \gamma_5 \gamma^a \gamma^\rho
\end{align*}
$$

$$
\begin{align*}
\delta^c Q \gamma^a &= i A^a \gamma_5 + b^a \\
\delta^c Q \gamma^a &= [2 \gamma^a + (\gamma_{a\beta}(e, b) + \gamma_{a\beta}(e, \gamma)) \gamma^b + i A^a \gamma_5 + b^a] \epsilon
\end{align*}
$$

(4)

*) The flat $S$ supersymmetry transformation is of the form as in Eq. (2) for variations of the fields at $x = 0$. This is sufficient to induce the transformation at an arbitrary space-time point $x$. 
The derivatives $\tilde{\gamma}_\mu^\nu$ are covariantized with respect to the full superconformal group. For a multiplet with weight $\lambda$ one thus has

$$
\overline{D}_\mu^c A = \lambda A - \frac{1}{2} \overline{\psi}_\nu \chi - \frac{1}{3} A^\mu B - \lambda b^\mu A + \text{hvem} - \overline{D}_\nu^c B \chi = \left[ \lambda + \frac{1}{2} (\omega_{\alpha} + \kappa_{\alpha}) e_\mu \right] \overline{\psi}^a (\frac{1}{2} - \frac{1}{3} A^\mu \gamma_5) \psi^a - \frac{1}{2} (F^{\mu \nu} \gamma_5) \psi^a - \lambda (A^{\mu} \gamma_5) \psi^a
$$

(5)

Since matter fields (as well as the superconformal gauge fields $e_\mu^a$, $f_\mu^a$, and $A_\mu$) are inert under proper conformal boosts $\mathcal{K}$, no $\mathcal{K}$ connection is present. Moreover, 4),

$$
\omega_{ab}(e, b) = \omega_{ab}(e) + (e_\mu b^\mu_a - e_\mu b^\mu_b)
$$

$$
\overline{D}_\mu \psi^a = \frac{1}{2} \gamma^\nu \left( \overline{D}_\nu \psi^a - \overline{D}_\mu \psi^a + \frac{1}{2} e_\nu \epsilon_{\sigma \tau} \gamma_5 \overline{D}_\sigma \psi^a \right)
$$

$$
\overline{D}_\nu \psi^a = \left( \overline{D}_\nu \psi^a + \frac{1}{2} (\omega_{ab}(e, b) + \kappa_{ab}(e, \psi)) \sigma^{ab} + \frac{1}{2} b_\nu + \frac{1}{2} A^\nu \gamma_5 \right) \psi^a
$$

(6)

For these multiplets, the multiplication rule is the same as in global supersymmetry, yielding a multiplet which transforms as in (5) with weight $\lambda_1 + \lambda_2$. This constitutes the tensor calculus for conformal supergravity. We now turn to the problem of constructing actions. If after one or more multiplications one has obtained a composite multiplet with weight $\lambda = 3$, then an invariant density (up to a four-divergence) is given by

$$
\mathcal{L} = e^{\frac{1}{2}} \overline{\psi} \gamma \chi + e^{\frac{1}{2}} \overline{\psi} \gamma^\nu (A - i \gamma_5 B) \psi^\nu
$$

(7)

as one may verify from (3) and (4) using $\lambda = 3$.

We observe that the terms depending on the dilaton field $b_\mu$ in the transformation rules for both the gauge fields and the fields of the multiplet form an $S$ supersymmetry transformation with field-dependent parameter $-\beta^e$. 


\( \delta_Q^c(\epsilon)(b_{\mu} \neq 0) = \delta_Q^c(\epsilon)(b_{\mu} = 0) + \delta_S^c(-\frac{1}{2} \epsilon) \)

(8)

while all b-dependent terms in the action cancel due to its invariance under proper conformal boosts.

We now give four examples. The first concerns the Maxwell-Weyl system (which coincides with the Maxwell-Einstein system 6). The local vector multiplet \( W_\alpha \) with weight \( \lambda = \frac{3}{2} \) is the same as the previously given global vector multiplet but with completely covariant derivatives

\[
\overline{D}_\mu \lambda = \partial_\mu \lambda + \frac{i}{2} \left( \omega_{\mu ab}(\epsilon, b) + k_{\mu ab}(\epsilon, b) \right) \sigma^{ab} \lambda
+ \frac{i}{2} \partial_\mu \psi \lambda - \frac{i}{2} \bar{F}_{ab} \sigma^{ab} \psi - \frac{i}{2} D \psi \lambda
\]

and

\[
\bar{F}_{ab} = \partial_a \psi_b + \frac{i}{2} \bar{\psi}_a \psi_b - (a \leftrightarrow b)
\]

(it is \( \bar{F}_{ab} \) which transforms simply and not \( \bar{F}^{\mu \nu} \)). Taking the square \( W_\alpha \otimes W_\alpha^{\text{cov}} \) one obtains a scalar multiplet with weight \( \lambda = 3 \) and applying to this multiplet the rule in Eq. (7) yields the following Lagrangian

\[
\mathcal{L} = -\frac{e}{8} T_{\alpha} \left( \chi_{\beta \alpha} \chi_{\beta'} \chi_{\alpha'} \Sigma^{\beta \beta'} \right) - \frac{e}{8} T_{\alpha} \bar{\Sigma}^{\alpha}
+ \frac{e}{4} \left( \bar{\psi}_{\alpha} \chi_{\beta} \right) \chi_{\beta} \chi_{\alpha} + \frac{e}{8} \bar{\psi}_{\alpha} \chi_{\beta} \chi_{\alpha} = \left( \bar{\psi}_{\alpha} \chi_{\beta} \chi_{\alpha} \right)
\]

(9)

where \( \chi_{\beta \alpha} = (\sigma \cdot \Sigma^{\mu} + i \gamma_{5} D)_{\beta \alpha} \). One may verify that this reproduces the action obtained in Refs. 9, 10). The above results remain valid for the Yang-Mills-Weyl system provided one introduces the usual Yang-Mills covariant derivatives and tensors.

Next consider the cube of the canonical scalar multiplet \( \Sigma(\lambda = 1) \). It has weight \( \lambda = 3 \) and the action in Eq. (7) now reproduces all internal coupling terms of the scalar-Weyl system itself. Defining the superconformal kinetic multiplet with weight \( \lambda = 2 \) by \( T(\Sigma) = \left[ \Sigma, \gamma_{5}, \Sigma^{\alpha}, \Sigma^{\alpha} \right] \) and taking the product \( \Sigma \otimes T(\Sigma) \) so that one obtains a scalar multiplet with weight \( \lambda = 3 \), application of Eq. (7) indeed reproduces the action of Ref. 10).
Note that the last term in $\delta^O_\mu \chi$ which is the $S$ connection, gives rise to the improved supersymmetry current while the complete $\chi$ field equation is $\gamma^\mu \delta^O_\mu \chi = 0$ with $F = G = 0$. With $\delta^O_F$ in Eq. (3), this explains why $F$ and $G$ only appear in the action as $\frac{1}{2}(F^2 + G^2)$.

The completely covariant D'Alembertians $\Box^0 A$ and $\Box^0 B$ in $\mathbb{T}(\Sigma)$ follow either from $\delta^O_\mu (\delta^O_\chi)$ using Eq. (3), or by evaluating $\delta^O_\mu A = e^{-1} D^\nu_\mu (\sigma^\mu \nu \delta^O_\chi)$ with $\delta^O A$ given in Eq. (5) and by adding all $S$, $A$, $K$, $D$ and $Q$ connections. One thus finds

$$
\delta^O_\mu A = e^{-1} D^\nu_\mu \left( q^{-1} e^{D^c_\mu} A \right) + \left( \frac{1}{2} \overline{F} \cdot \gamma \chi + \overline{F} \cdot \phi A \right) \\
+ \left( -\frac{1}{3} A \cdot A - \frac{1}{9} A A^2 - \frac{i}{4} \overline{F} \chi \gamma^5 A \cdot \psi + \frac{i}{12} \overline{F} \chi \gamma^5 A \gamma^5 \chi + \frac{1}{2} b A B \right) \\
+ \left( \frac{1}{6} R^b A + \frac{1}{3} \overline{F} \gamma^5 \gamma^\mu \phi \chi \right) + \left( b \gamma^\mu D^c_\mu \right) \\
+ \left( -\frac{1}{2} \overline{F} \gamma^5 \gamma^\mu D^c_\mu \chi - \frac{i}{2} \overline{F} \gamma^5 \chi \gamma^\mu \phi \chi \right)
$$

(10)

Although $A$ is $K$ inert, $\delta^O_\mu A$ is not and in the $K$ connection the symbol $R^\alpha_\mu$ denotes the scalar curvature with $b$ and $\psi$ torsion. The reader may verify that the separate connections are not invariant under the symmetries to which they are not related, but that the sum $\delta^O A$ transforms indeed without derivatives on any gauge parameter of the superconformal group. All $b$ dependent terms in $\delta^O A$ and $\delta^O_\chi$ cancel separately.

The last example is the superconformal gauge action itself $^4)$. Starting from a scalar multiplet with curvatures $^5) \ R_{\alpha \beta \gamma} = (R_{\alpha \beta \gamma}(Q), \ldots)$ the square of $\delta^O_\mu \chi$ has weight $\lambda = 3$ and starts with the components $(R^2_{\mu \nu}(Q), R_{\mu \nu}(Q) \gamma^5 H^\mu \nu(Q), \ldots)$. Thus one can re-obtain in this way the superconformal gauge action by using Eq. (7). The constraints $R_{\mu \nu}(Q) + \frac{1}{2} \gamma^5 H^\mu \nu(Q) = - \frac{1}{2} R^\mu \nu(Q) = 0$ are equivalent to the requirement that $R_{\alpha \beta \gamma}(Q)$ has three symmetric undotted indices, while $R^\mu \nu(Q) = 0$ specifies the torsion in the connection.

We now consider Poincaré supergravity using superconformal notions. We will put consistently $\kappa = 1$. The basic multiplet is again the scalar multiplet with arbitrary weight $\lambda$. The transformation rules are now $^2), ^3) \delta^P_{\chi}(\epsilon) = \delta^c_{\chi}(\epsilon) + \delta_S(p = -\frac{1}{3} \frac{\langle S \rangle}{\epsilon} + \frac{1}{3} \gamma_5 P \epsilon + \frac{i}{3} \gamma_\nu \gamma_5 \epsilon + \frac{i}{6} \gamma_\nu \gamma_5 \epsilon)$

(11)
where $\delta_Q^C$ was given in Eq. (3) and $\delta_S$ by

$$
\delta_S A = \delta_S B = 0; \quad \delta_S \chi = \lambda (A + i \gamma_5 B) \gamma \gamma_5
$$
$$
\delta_S F = (1 - \lambda) \gamma \gamma_5 \gamma \gamma_5 \gamma \gamma_5; \quad \delta_S G = -i (1 - \lambda) \gamma \gamma_5 \gamma \gamma_5 \gamma \gamma_5
$$
(12)

For completeness, we also record the transformation rules of the gauge fields of Poincaré supergravity

$$
\delta^P_Q \psi^\mu = \varepsilon \gamma^\mu \psi^\mu,
$$
$$
\delta^P_Q \psi = \frac{1}{2} D^\rho \varepsilon + \frac{1}{3} \varepsilon \gamma^\rho (A - i \gamma_5 B) \varepsilon, \varepsilon (A - \frac{1}{3} \gamma_5 B) \gamma_5 \varepsilon
$$
$$
\delta^P_Q \bar{\psi} = \frac{1}{2} \left( \varepsilon \gamma_5 \gamma_5, \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5 \right) \varepsilon (A - \frac{1}{3} \gamma_5 B) \gamma_5 \varepsilon
$$
$$
\delta^P_Q \psi = \frac{1}{2} \left( \varepsilon \gamma_5 \gamma_5, \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5 \right) \varepsilon (A - \frac{1}{3} \gamma_5 B) \gamma_5 \varepsilon
$$
$$
\delta^P_Q \psi = \frac{1}{2} \left( \varepsilon \gamma_5 \gamma_5, \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5 \right) \varepsilon (A - \frac{1}{3} \gamma_5 B) \gamma_5 \varepsilon
$$
(13)

where $D^\rho \varepsilon$ and the spin $\frac{3}{2}$ field equation $R^\mu$ contain only the $Q$ connection ($\psi$ torsion). From Eq. (13) one deduces Eq. (11) except that the $S$ transformation with parameter $\delta_S$ is added in order that the transformation $\delta^P_Q(\varepsilon)$ be $\varepsilon$ independent, as Poincaré rules should be. Multiplying two scalar multiplets with weights $\lambda_1$ and $\lambda_2$ leads to a third scalar multiplet which transforms again according to Eqs. (11), (12) with $\lambda = \lambda_1 + \lambda_2$. This is the tensor calculus for Poincaré supergravity. The solution of the density problem is now as follows. Up to a four-divergence a super-Poincaré invariant density is given by

$$
\mathcal{L} = e F + \frac{e}{2} \bar{\psi} \gamma \chi + \frac{e}{2} \bar{\psi} \gamma \gamma_5 A (A - i \gamma_5 B) \gamma \gamma_5, \gamma \gamma_5, \gamma \gamma_5, \gamma \gamma_5 \gamma \gamma_5 + e \frac{3 - \lambda}{3} (SA + PB)
$$
(14)

For the special case of $\lambda = 3$, the action is superconformal invariant, and hence separately invariant under $\delta_Q^C$ and $\delta_Q^P$. 
It is preferable to formulate the tensor calculus and construction of actions for Poincaré supergravity using super-Poincaré notions only. In particular, a $\lambda$ independent formulation will enable one to consider also scalar multiplets for which no definite weight can be defined, such as the sum of two multiplets with different weights. In fact, one expects such a formulation to be possible since (Weyl) weights play no rôle in the super-Poincaré group. Indeed this is possible. From Eqs. (11) and (12) one may deduce that

$$
\delta^P_A = \varepsilon \chi \quad ; \quad \delta^P_B = -i \bar{\varepsilon} \chi_5 \chi
$$

$$
\delta^P\chi = \bar{\chi} \bar{\chi} (A - i \chi_5 B) \varepsilon + (F' + i G' \chi_5) \varepsilon
$$

$$
\delta^P F' = \bar{\varepsilon} \bar{\chi} \bar{\chi} + \bar{\chi}_5 \chi \quad ; \quad \delta^P G' = i \varepsilon \chi_5 \bar{\chi}_5 \chi - i \bar{\chi}_5 \chi_5 \chi
$$

$$
F' = F - \frac{1}{3} (SA + PB) \quad ; \quad G' = G - \frac{1}{3} (SB - PA)
$$

(15)

and that the multiplication rule in (1) is consistent with these transformations. The derivatives $\overline{D}^P_\mu$ are now covariantized with respect to the super-Poincaré group. Hence

$$
\overline{D}^P_\mu A = \partial_\mu A - \frac{1}{2} \overline{\psi}_\mu X
$$

$$
\overline{D}^P_\mu X = \partial_\mu X + \frac{1}{2} (\sigma_{ab} (e) + \xi_{ab} (e, \psi)) \chi_5 \chi
$$

$$
- \frac{1}{2} \overline{D}^P (A - i \chi_5 B) \psi_\mu - \frac{1}{2} (F' + i G' \chi_5) \psi_\mu
$$

$$
\overline{\xi} = - \frac{1}{3} \varepsilon (S - i \chi_5 P + \frac{i}{2} \chi_5 \chi_5)
$$

(16)

The basic multiplet $(A, B, X, F', G')$ is again multiplied according to the flat-space rule in Eq. (1), but now both the constituent as well as the composite scalar multiplets transform according to Eq. (15). In terms of the multiplet $(A, B, X, F', G')$ an invariant density up to a four-divergence is given by

$$
\mathcal{L} = e F' + \frac{e}{2} F \chi_5 \chi + \frac{e}{2} \bar{\chi}_5 \bar{\chi} \sigma^{\mu \nu} (A - i \chi_5 B) \psi_\nu + e (SA + PB)
$$

(17)
This realizes our goal of obtaining a $\lambda$ independent action and transformation rules. Thus one can now obtain arbitrarily many new actions by simply multiplying scalar multiplets, possibly with internal indices, and then using our general rule in Eq. (17). In particular all super-conformal models have actions defined by (17) and (15). Note that only for $\lambda = 3$ the action in Eq. (17) is $S$ and $P$ independent.

We now give some examples. Defining the $n^{th}$ power of the canonical scalar multiplet $\Sigma$ by $\Sigma^n$ with weight $\lambda = n$, we consider the general scalar multiplet

$$ f(\Sigma) = \sum_{n,m} a_{nm} \Sigma^n \otimes T(\Sigma^m) $$

where $T(\Sigma^0) = (1,0,0,0,0)$. Special cases are:

(i) only $a_{00} \neq 0$; this is the $\lambda = 0$ multiplet $(0,0,0,0,0)$ which yields the de Sitter terms of pure supergravity $E$;

(ii) only $a_{30}$ and $a_{41}$ non-zero; this multiplet $\alpha \Sigma^3 + \beta \Sigma \otimes T(\Sigma)$ with weight $\lambda = 3$ reproduces the scalar Weyl system with an additional self-coupling $10$.

For arbitrary coefficients $a_{nm}$, Eq. (18) leads to a Poincaré locally supersymmetric interaction of the scalar multiplet. For all these cases the action follows from Eq. (17) when applied to the multiplet $f(\Sigma)$.

Summarizing, we have obtained a tensor calculus for scalar multiplets, possibly with internal indices [For example $SU(n)$ or spinorial indices]. We have also shown how to construct supergravity actions out of these multiplets. Our results were obtained by extending flat-space global supersymmetry results to curved space-time. Since these global supersymmetry results can also be obtained from a superspace approach $7,11$), it is conceivable that our tensor calculus and construction of densities can be re-obtained from a superspace formulation of supergravity $12$).

We have not yet considered the actions which are not based on scalar multiplets such as the Poincaré supergravity action itself which is constructed out of two vector multiplets $5$). It would be very interesting
to extend our analysis to cover also these cases \textsuperscript{\textasteriskcentered}). Our methods based on completely covariant derivatives and the complete auxiliary field structure seem sufficiently powerful for this purpose.

\textsuperscript{\textasteriskcentered) Because in global supersymmetry the $D$ component of a vector multiplet can always be related to the $F$ component of a scalar multiplet, it is possible that our results can be adapted to all relevant cases.}
REFERENCES


2) S. Ferrara and P. van Nieuwenhuizen - CERN Preprint TH. 2453 (1978), to be published in Phys.Letters B.

3) K. Stelle and P.C. West - Imperial College Preprint ICTP 77-78/6 (1978).


12) J. Wess and B. Zumino - Phys.Letters 66B, 361 (1977);
    R. Grimm, J. Wess and B. Zumino - Phys.Letters 73B, 415 (1978);
    J. Wess and B. Zumino - CERN Preprint TH. 2453 (1978);