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A MODIFIED FUNCTIONAL MEASURE FOR THE DETERMINATION
OF ENERGY LEVELS IN GAUGE THEORIES

K. Dietz
Physikalisches Institut der Universität Bonn

and

H. Römer
CERN - Geneva

A B S T R A C T

It is shown that in the functional formalism the computation of energy levels in a weak external gauge field calls for a modification of the Faddeev-Popov functional measure. Physical implications are discussed.

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In this letter we should like to discuss further a problem raised in a previous publication ¹⁾. The source of the difficulties described there lies in the fact - typical for gauge theories - that even for "time-independent" configurations minima of the gauge-invariant energy functional need not be solutions of the equations of motion.

This phenomenon is clearly exhibited for the case of stationary electric configurations

$$\vec{E} = 0 \quad , \quad \vec{B} = 0 \quad (1)$$

interacting with a scalar matter field, where it results in a reversal of sign of the diamagnetic term as compared to the coupled equations of motion. In the following, to cope with this difficulty, we shall try to give a procedure which consists in modifying the functional measure which defines the partition function and we shall discuss which are the physical ingredients suggesting such a modification.

For definiteness we consider scalar, massive QED and parametrize condition Eq.(1) by the following four-potential

$$A_\mu = (\Phi(\vec{x}), \vec{0}) + \partial_\mu \chi \quad (2)$$

As a physical, gauge-invariant definition of time independence we demand the current $j_\mu(\vec{x}, t)$ to be independent of time. Even more, condition Eq.(1) together with Maxwell's equations implies

$$\vec{j} = 0 \quad (3)$$

hence, with a polar decomposition of the matter field $\psi(\vec{x}, t)$

$$\psi(\vec{x}, t) = \varphi e^{ie\theta} \quad (4)$$

we have

$$\vec{j} = -e^2 \varphi^2 (\vec{\partial} \theta - \vec{\partial} \chi) = 0$$

and

$$\theta = \chi + f(t) .$$

Furthermore, taking $\varphi = \varphi(\vec{x})$ for consistency,

$$\frac{d}{dt} j_0 = 0 \quad (5)$$

means that

$$\psi = \varphi(\vec{x}) e^{ie\chi(\vec{x},t) + iew.t} \quad (6)$$

Inserting this time-independent electric configuration Eqs.(2) and (6) into the gauge-invariant energy functional

$$H = \int d^3x T_{00}^{Bel}(\psi, A_\mu) , \quad (7)$$

where $T_{00}^{Bel} = -\mathcal{L}_E$ is the 00-component of the Belinfante tensor and equals the Euclidean action density, we have

$$H = \frac{1}{2\alpha} \int d^3x \{ (\nabla \varphi)^2 + (\nabla \Phi)^2 + m^2 \varphi^2 + \varphi^2 (\Phi - \omega)^2 \} . \quad (8)$$

Stationary points of $H[\varphi, \Phi]$ are given by

$$(-\Delta + \varphi^2)(\Phi - \omega) = 0 \quad (9)$$

and

$$(-\Delta + m^2 + (\Phi - \omega)^2) \varphi = 0, \quad (10)$$

equations which differ by the sign of the diamagnetic term $(\Phi - \omega)^2 \varphi$ from the equations of motion. Maxwell's equations remain unchanged.

Analogously, for non-Abelian gauge fields with a compact gauge group coupled to a scalar matter field ψ , configurations which are gauge equivalent to

$$A_\mu = (\Phi(\vec{x}), \vec{0}) \quad , \quad \psi = \varphi(\vec{x}) e^{i\omega t} \quad (\varphi \text{ real}) \quad (11)$$

fulfil our electric time independence conditions Eq.(1) and lead to a time independent current. The energy functional then has the form Eq.(8).

Generating functionals of our Euclidean field theories are usually obtained from the measure

$$D P = e^{S_E} \cdot \Delta \cdot \delta(f[A_\mu, \psi]) D A_\mu D \psi D \psi^* \quad (12)$$

$$S_E = - \int d^4x T_{00}^{Bel} \quad ,$$

where $f[A_\mu, \psi] = 0$ is some gauge-defining condition and Δ the corresponding Faddeev-Popov determinant.

It is clear that the generating functional

$$Z[j, J] = \int D P e^{\int d^4x (\psi j^* + \psi^* j + A_\mu J^\mu)} \quad , \quad (13)$$

expanded around the trivial stationary point $A_\mu = 0, \psi = 0$, generates the Wick-rotated Feynman amplitudes of the theory.

Likewise, this measure should yield the energy levels of our system (as poles of the Laplace transform) via the Feynman-Kac formula since it is the physical gauge invariant energy density T_{00}^{Bel} which occurs in the exponent :

$$Z(\beta) = \int D\mathcal{P}_\beta, \quad (14)$$

where the index β indicates that we are to integrate over periodic configurations with period β without Wick-rotation.

The situation, however, is less clear if one expands around a non-trivial stationary point of the exponent in DP. Take, for example, the time-independent electric configurations of scalar QED introduced above. Starting with a Coulomb-ansatz for $\Phi(\vec{x})$, we observe that

$$\Phi_0 = \frac{Q}{r} + E, \quad \varphi = 0 \quad (15)$$

is a stationary point of the Euclidean action with an additional point source term (without a stabilizing external source, non-trivial stationary points with finite action do not exist for massive scalar QED). Fluctuations around this configuration are expected to yield the energy levels to lowest order in $\hbar$ of hydrogen-like atoms and are given ¹⁾ by the spectrum of the differential operator $[-\Delta + m^2 + (\Phi_0 - \omega)^2]$. However, due to the repulsive sign of the diamagnetic term already discussed above, the relativistic fine structure comes out with an inverted sign as compared to the result of a Klein-Gordon calculation. It is the latter energy spectrum which is apparently confirmed by experiment ²⁾.

It is noteworthy that because of linearity such a problem does not occur for spin $\frac{1}{2}$ fields moving in a gauge field.

Notwithstanding the physically unacceptable option of working with imaginary Q there is one possibility to avoid this result. One can restrict the set of admissible functions in the functional integrals by imposing a constraint equation

$$\nabla \vec{E} = j_0 . \quad (16)$$

This gauge invariant constraint changes the functional measure into

$$\hat{D}P = DP \delta(\nabla \vec{E} - j_0) . \quad (17)$$

For our electric time independent configurations it is easy to see that this constraint leads to the desired change of sign. Introducing the constraint with a Lagrange parameter $\gamma(\vec{x})$ we get a modified Euclidean action (from now on we write Φ instead of $\Phi - w$)

$$\begin{aligned} \hat{S}_E = -\frac{\beta}{2\alpha} \int d^3x & ((\nabla \Phi)^2 + (\nabla \varphi)^2 + m^2 \varphi^2 + \Phi^2 \varphi^2 \\ & + 2\gamma(x) (-\Delta \Phi + \varphi^2 \Phi)) . \end{aligned} \quad (18)$$

Therefore, from

$$\frac{\delta \hat{S}_E}{\delta \Phi} = -\frac{\beta}{\alpha} (-\Delta \Phi + \varphi^2 \Phi - \Delta \gamma + \gamma \varphi^2) = 0$$

and

$$\frac{\delta \hat{S}_E}{\delta \varphi} = -\frac{\beta}{\alpha} ((-\Delta + m^2 + \Phi^2) \varphi + 2\gamma \Phi \varphi) = 0 ,$$

we get

$$\Phi = -\gamma$$

and, hence,

$$(-\Delta + m^2 - \Phi^2) \varphi = 0 . \quad (19)$$

This now has the attractive diamagnetic term.

The same recipe also works for general configurations. Variation of the generalized Euclidean action incorporating the constraint (17)

$$\hat{S}_E = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} |D_\mu \psi|^2 - \frac{m^2}{2} |\psi|^2 - \gamma(x) (\partial_\mu F_{0\mu} + j_0) \right\},$$

and choosing the gauge condition $f[A_{\mu,\psi}] = 0$ such that

$$\nabla \cdot (\vec{A} - \vec{\partial} \theta) = 0$$

we, of course, reproduce Maxwell's equations but change the sign of the term $e^2 A^2 \varphi$ in the equation for the matter field.

The following remarks seem to be appropriate :

- The subsidiary condition is of purely kinematical origin without involving time development. In a canonical formalism it had to be imposed ³⁾ to restrain phase space such as to arrive at the correct number of independent canonical degrees of freedom.
- On the constrained phase space the canonical and the Belinfante energy density coincide up to surface terms.
- It is immediately seen that Feynman rules derived from the modified generating functional

$$\hat{Z}[j, J] = \int \hat{D}\rho \, e^{\int d^4x (j\psi^* + j^*\psi + J_\mu A_\mu)} \quad (20)$$

give amplitudes with Wick-rotated momenta even for a perturbation expansion around a non-trivial stationary point.

- For known monopole ⁴⁾ and dyon ⁵⁾ solutions of non-Abelian gauge theories the diamagnetic term happens not to contribute so that they extremalize both the energy functional Eq.(7) and the action. For these solutions time independence is gauge invariantly defined for the matter fields by

$$D_0 \psi = 0 \quad (21)$$

For scalar QED this would exclude all non-trivial electric configurations. Accordingly, we had to generalize the notion of time independence by imposing it on observable, gauge invariant quantities like currents.

- The sign difference between Eqs.(10) and (19) has very important physical consequences. It is particularly interesting to note that for the repulsive sign in Eq.(10) which arises without subsidiary condition there is no ⁶⁾ instability ⁷⁾ at $Q = \frac{1}{2}$. This is reminiscent to the observation that for large external point sources ^{6),8)} charge screening may occur. In particular Mandula ⁸⁾ showed that for massless QED beyond the critical value $Q = \frac{1}{2}$ complete asymptotic extinction occurs even for all orders of perturbation theory ⁹⁾. This screening will be incomplete ^{6),8)} in the massive case. It is, therefore, tempting to speculate that the unconstrained measure Eq.(12) is relevant for the description of a system with an external source beyond the catastrophic value $Q = \frac{1}{2}$ and describes the motion in the residual potential after collapse. In such an interpretation the subsidiary condition Eq.(16) demanding local charge conservation prevents a "phase transition" to a collapsed state. The question which of the measures is to be used pertains to deciding which is the relevant physical framework to be incorporated from the outset by distinguishing appropriate collective modes around which quantum fluctuations should be most effectively computed.

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