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CYLINDRICAL CONFIGURATION OF CLASSICAL YANG-MILLS FIELDS
AND ITS SYMMETRICAL PROPERTIES

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ABSTRACT

An explicit construction of classical Yang-Mills equations in Euclidean four-space is given on the basis of single external constant vector ansatz for $SU(2)$ gauge potentials. The presence of a symmetry group leads to an essential simplification of the equations for gauge invariant combinations of structure functions (entering in field ansatz) and shows the remarkable group properties of the corresponding field structures.

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In the recent few months some advance has been achieved in the investigation of the Yang-Mills self-duality equations in R_4 for an $SU(2)$ gauge group ¹⁾. This progress affects the general considerations and theorems as well as explicit constructions of self-dual solutions of corresponding equations in the framework of appropriate ansätze. At the same time it seems important to learn not only the self-dual (anti-self-dual) case, but also the general Yang-Mills equations and to discover more deeply the group structure of previously known solutions and results.

In this letter our purpose is to investigate some symmetrical aspects of $SU(2)$ gauge theory in R_4 (including an explicit construction of corresponding equations) on the basis of the following ansatz for Yang-Mills field ²⁾ :

$$A_i = h_1 T_{ij} x_j + h_2 T_{ij} b_j + (h_3 x_i + h_4 b_i)(x_\kappa T_{\kappa\epsilon} b_\epsilon) \quad (1)$$

where b is an arbitrary constant four-vector, structure functions h_i depending on two variables : $s_1 = x_i \cdot b_i$ and $s_2 = [x_i^2 - (x_i \cdot b_i)^2]^{\frac{1}{2}}$; $T_{ij} = \sigma_a (\epsilon_{abc} \delta_{bi} \delta_{cj} + \delta_{ia} \delta_{j4} - \delta_{i4} \delta_{ja})$, $1 \leq a, b, c \leq 3$, σ_a are generators of $SU(2)$ gauge group ; b_i^2 is put to be equal to unity for simplicity. Notice that if $b_i = \delta_{i4}$ Eq.(1) reduces to the ansatz of Witten ³⁾, who discovered in this symmetric case the solutions of the self-dual equations with an arbitrary number n of instantons.

The following notation will be used ^{*)} :

$$\bar{U} = (u_1, u_2, u_3, u_4) = \{-s_2^2 h_3 - s_1 h_1 - h_2, s_2(s_1 h_3 + h_4 - h_1), s_2(s_1 h_1 + h_2), 1 + s_2^2 h_1\},$$

$$\psi \equiv \arctg \frac{u_3}{u_4}, \quad R_\mu = \frac{1}{2} \{ s_2^2 \bar{\nabla} \bar{U} + (-1)^\mu (u_{2+\mu}^2 - 1) \}, \quad \mu, \nu = 1, 2;$$

$$\Delta \equiv \partial^2 / \partial s_\mu^2, \quad \bar{\nabla} \equiv (\partial / \partial s_1, \partial / \partial s_2).$$

*) Throughout this letter we shall not consider functions u_i (and their derivatives) containing singularities like δ functions.

We choose the same gauge constraint as in Refs. 2), 3) :

$$\varepsilon_{\mu\nu} \frac{\partial u_\mu}{\partial s_\nu} = 0, \quad \varepsilon_{\mu\nu} \equiv (-1)^\mu \delta_{\mu+\nu,3}.$$

(This choice ensures the finiteness of the structure functions h_i in the self-dual case.)

Let us briefly describe the symmetry groups of the problem. First of all there is the $O(2,1)$ subgroup of the conformal group, whose algebra consists of dilatations $\hat{D}(\equiv -L_2)$, translations $\hat{P}(\equiv L_1 + L_3)$ and special conformal transformations $\hat{K}(\equiv -L_1 + L_3)$ along the direction b with generators (which act on the column vector $v \equiv \{u_3, u_4, u_2, s_1 u_2 - s_2 u_1, 1\}$)

$$\hat{D}_{\alpha\beta} = \delta_{\alpha\beta} (\bar{S} \bar{\nabla}) + \delta_{\alpha 3} \delta_{\beta 3}, \quad \hat{P}_{\alpha\beta} = \delta_{\alpha\beta} \partial / \partial s_1 - \delta_{\alpha 4} \delta_{\beta 3},$$

$$\hat{K}_{\alpha\beta} = \delta_{\alpha\beta} [s_1 (\bar{S} \bar{\nabla}) - s_2 \varepsilon_{\mu\nu} s_\mu \partial / \partial s_\nu] + 2 \delta_{\alpha 3} \delta_{\beta 4} - \bar{S}^2 \delta_{\alpha 4} \delta_{\beta 3} + 2 s_1 \delta_{\alpha 4} \delta_{\beta 1}$$

$[L_a]$ are generators of $O(2,1)$ group]. The second part of symmetry group is a subgroup of a gauge group with generator

$$\hat{\Omega}_{\alpha\beta} = \omega (\delta_{\alpha 1} \delta_{\beta 2} - \delta_{\alpha 2} \delta_{\beta 1}) + \frac{\partial \omega}{\partial s_1} \delta_{\alpha 3} \delta_{\beta 5} + \bar{S} (\bar{\nabla} \omega) \delta_{\alpha 4} \delta_{\beta 5}, \quad \omega \equiv s_2 \varphi_g,$$

where the gauge phase φ_g is an arbitrary function of s_1 and s_2 . The quantities R_μ are gauge invariants. Notice that in the self-dual or anti-self-dual cases there is only one gauge invariant because then $R_2 = 0$ or $R_1 = 0$.

The Yang-Mills potentials, A_i , satisfy the field equations

$$\partial_i F_{ij} + [A_i, F_{ij}] = 0, \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j],$$

provided R_μ and ψ are the solutions of the following system of non-linear partial differential equations :

$$\Delta R_\mu = (-1)^\mu (\bar{\nabla} R_\mu)^2 / (R_2 - R_1 + 1) + 2 S_2^{-2} R_\mu (R_2 - R_1 + 1), \quad (2)$$

$$\Delta \psi = (R_2 - R_1 + 1)^{-2} \varepsilon_{\mu\nu} \partial R_1 / \partial s_\mu \cdot \partial R_2 / \partial s_\nu \quad (3)$$

The functions u_1 and u_2 can be recovered from the formulae

$$u_\mu = \varepsilon_{\mu\nu} \frac{\partial \psi}{\partial s_\nu} + \frac{1}{2} (R_2 - R_1 + 1)^{-1} \frac{\partial (R_1 + R_2)}{\partial s_\mu} \quad (4)$$

The gauge invariant expressions for action and topological charge densities are constructed with R_μ which satisfy Eq.(2) :

$$\begin{aligned} Sp F_{ij} F_{ij} &= 4 S_2^{-2} \sum_\mu \left[\frac{1}{2} (R_\mu)^2 / (R_2 - R_1 + 1) + S_2^{-2} R_\mu^2 \right], \\ Sp F_{ij}^* F_{ij} &= 4 S_2^{-2} \sum_\mu \left[-\frac{1}{2} \Delta R_\mu + S_2^{-2} R_\mu \right], \quad F_{ij}^* \equiv \frac{1}{2} \epsilon_{ijkl} F_{kl}. \end{aligned} \quad (5)$$

Notice that the system (2) contains three particular cases :

$R_2 = 0$ ($F_{ij} = *F_{ij}$), $R_1 = 0$ ($F_{ij} = -*F_{ij}$) and $R_1 = -R_2$ ($Sp F_{ij}^* F_{ij} = 0$). The transition between self-duality and anti-self-duality corresponds to the substitution $R_1 \rightleftharpoons -R_2$, $\psi \rightarrow -\psi$, which conserves Eqs.(2) and (3).

Let us introduce angular variables for the pseudo-orthogonal $O(2,1)$ vector :

$$S_2^{-1} = ch \theta + sh \theta \cdot \sin \varphi, \quad S_1/S_2 = \cos \varphi \cdot sh \theta.$$

If R_μ does not depend on φ , $(\hat{P} + \hat{K})R_\mu = 0$, the system (2) reduces to the ordinary differential equations :

$$\frac{d}{dx} \left[(x^2 - 1) \frac{dR_\mu}{dx} \right] = (-1)^\mu (x^2 - 1) (R_2 - R_1 + 1)^{-1} \left(\frac{dR_\mu}{dx} \right)^2 + 2 R_\mu (R_2 - R_1 + 1), \quad (6)$$

where $x \equiv ch \theta$. In the above-mentioned cases ($R_2 = 0$, $R_1 = 0$ and $R_1 + R_2 = 0$) Eq.(6) becomes simpler :

$$sh^2 y (\ddot{f} - f^{-1} \dot{f}^2) = f(f^2 - 1) \quad \text{if} \quad R_1 = 0 \quad \text{or} \quad R_2 = 0 \quad (7_1)$$

$$sh^2 y \ddot{f} = f(f^2 - 1) \quad \text{if} \quad R_1 + R_2 = 0 \quad (7_2)$$

where we write $\dot{f}$ for df/dy ; $f \equiv (R_2 - R_1 + 1)^{\frac{1}{2}}$, $ch y \equiv x$.

The finite action solutions f_n of Eq.(7₁) corresponding to arbitrary number n of pseudo-particles coincide with those of Witten, if one puts all parameters a_i in Ref. 3) equal to unity. It is easy to show that the functions $g_n \equiv f_n^{-1}$ are the correctly normalized matrix elements $t_{00}^{4,n}(y)$ of class 1 Lorentz group representation

$$g_n(y) = t_{00}^{4,n}(y) = ch^2 y {}_2F_1 \left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{3}{2}; th^2 y \right) \quad (8)$$

satisfying the Laplace equation

$$\left[(x^2-1) \frac{d^2}{dx^2} - n(n+2) \right] g_n(x) = 0. \quad (9)$$

There are lowering and raising operators

$$L_{\pm}^{(n)} = \sqrt{x^2-1} \frac{d}{dx} \mp (n+1 \pm 1) \frac{x}{\sqrt{x^2-1}}$$

which connect functions (8) with different topological charge values,

or

$$\begin{aligned} L_{\mp}^{(n)} g_n &= \pm (n+1 \mp 1) g_{n \mp 1} \\ L_{-}^{(n)} t_{00}^{4,n} &= n t_{00}^{4,n-1}, \quad L_{+}^{(n)} t_{00}^{4,\bar{n}} = \bar{n} t_{00}^{4,\bar{n}+1} \end{aligned} \quad (10)$$

where $t_{00}^{4,\bar{n}} (\equiv t_{00}^{4,n})$ is matrix element of adjoint representation ;
 $\bar{n} \equiv -(n+2), \quad L_{+}^{(n)} \equiv L_{-}^{(\bar{n})}.$

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