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PROPERTIES OF $O(4) \times O(2)$ SYMMETRIC SOLUTIONS
OF THE YANG-MILLS FIELD EQUATIONS

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1. A large amount of interest has been recently devoted to classical solutions of $SU(2)$ Yang-Mills field equations both in the Euclidean and in the Minkowski domain ¹⁾. In particular a non-trivial solution has been proposed ²⁾, which is everywhere regular in Minkowski space and leads to finite action and energy. Its original form reads

$$A_\mu = -i \sigma_{\mu\nu} \left(\frac{y_{+\nu}}{y_+^2} + \frac{y_{-\nu}}{y_-^2} \right) \quad (1)$$

where

$$y_+ = x + a, \quad y_- = x - a \quad (2)$$

and for the definition of $\sigma_{\mu\nu}$ and all other symbols we shall follow the paper by Jackiw and Rebbi ³⁾.

It has been noticed that the $SU(2)$ components, which are obtained in Minkowski space by projection from the solution (2), i.e.

$$A_\mu^\alpha = i T_2 (A_\mu \sigma^\alpha) \quad , \quad \alpha = 1, 2, 3 \quad (3)$$

are complex, although the Lagrangian density and the energy momentum tensor are real. For this reason the following version of the solution (1) has been derived ⁴⁾, which differs from the original one by a complex gauge transformation and leads to real A_μ^α :

$$A_0 = i \frac{x_0}{2\tau^2} \vec{\sigma} \cdot \vec{x}$$

$$A_i = \frac{-i}{2\tau^2} \left\{ \epsilon_{ijk} x_j \sigma_k + \frac{1+x^2}{2} \sigma_i + x_i \vec{\sigma} \cdot \vec{x} \right\}_{(4)}$$

$$\tau^2 = (1+t_+^2)(1+t_-^2) \quad , \quad t_\pm = x_0 \pm |\vec{x}|$$

This result follows quite beautifully and naturally by exploiting the invariance properties of the Yang-Mills equations under the $O(4) \times O(2)$ group and formulating the theory on a hypertoroidal surface.

As viewed from the Euclidean point of view the solution (1) exhibits very interesting "topological" properties. If we introduce the so-called topological charge of the gauge field configuration

$$q = \int_{S_\infty} dS_\mu I_\mu \quad (5)$$

$$I_\mu = -\frac{1}{8\pi^2} \epsilon_{\mu\alpha\beta\gamma} T_2 \left\{ A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma \right\}$$

where the surface S_∞ is taken at infinity, then the asymptotic behaviour of Eq. (1), $A_\mu \sim x^{-1}$ leads to a non-vanishing value of $q=1$. As shown in Ref. 1) it turns out that the topological density is concentrated in the points of singularity $x=+a$ $x=-a$: this results from the explicit expression of the quantity

$$\begin{aligned} \partial_\mu I_\mu = D(x) &= -\frac{1}{32\pi^2} T_2 \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} = \\ &= \frac{1}{2} [\delta(x+a) + \delta(x-a)] \end{aligned} \quad (6)$$

Thus any surface integral which is computed on a small sphere around one of the two singular points leads to a value $q=\frac{1}{2}$. Owing to their existence as separated lumps of fractional topological charge, these configurations have been called "merons" 5).

From the Minkowski point of view the situation is somewhat baffling. On one side the solution (1) is regular and well-behaved, on the other the absence of any singularity leads to a density $D(x)$ vanishing everywhere and therefore to the (apparent) absence of any interesting property concerning the surface integral of I_μ .

The aim of this letter is to discuss the "topological" properties of the real Minkowski solution, Eq. (4), and to show that:

- a) in the Euclidean domain the topological properties of solutions like (4) and (1) are quite different,
- b) in the Minkowski domain it is possible to obtain, in the case of the solution (4), a meaningful non-vanishing value of the surface integral of I_μ , i.e., of the "topological" charge q .

2. Let us start our discussion for the Euclidean case by noticing that the asymptotic behaviour of solutions (1) and (4) are quite different. For solution (1) $A_\mu \sim x^{-1}$, producing a non-vanishing value of q , while for solution (4) $A_\mu \sim x^{-2}$ which implies the vanishing of the same surface integral. As a consequence the result (6) for $D(x)$ is certainly not valid for this form of the solution.

This fact leads one to suspect that in the case of the solution (4) the two singular points $a, -a$ are source of opposite topological charges, whose cancellation is responsible for a vanishing integral on the sphere at infinity. We shall show that this is indeed the case.

In order to achieve the goal it is fruitful to go through a detailed derivation of Eqs. (1) and (4), which will also allow a clear understanding of the differences and interrelations between the two solutions. Since the emphasis will be on "topological" charges, we shall use the $SU(2)$ formalism, which naturally distinguishes between meron (m) and antimeron ($\bar{m}$).

We define the "spin" four-vectors

$$S_\mu = (\epsilon \vec{\sigma}, 1) \quad , \quad \bar{S}_\mu = (-\epsilon \vec{\sigma}, 1) \quad (7)$$

$$(\mu = 1, 2, 3, 4)$$

from which one can build the "spin" matrices

$$\sigma_{\mu\nu} = \frac{1}{4i} (S_\mu \bar{S}_\nu - S_\nu \bar{S}_\mu)$$

$$\bar{\sigma}_{\mu\nu} = \frac{1}{4i} (\bar{S}_\mu S_\nu - \bar{S}_\nu S_\mu) \quad (8)$$

$$\sigma_{\epsilon\delta} = \bar{\sigma}_{\epsilon\delta} = \frac{1}{2} \epsilon_{\epsilon\delta\kappa} \sigma_\kappa \quad , \quad \sigma_{\epsilon 4} = -\bar{\sigma}_{\epsilon 4} = \frac{1}{2} \sigma_\epsilon$$

Following Ref. 3) it is useful to introduce the quantities

$$g(x) = \frac{(\bar{S} \cdot x)}{|x|} = \frac{x_4 - \epsilon \vec{\sigma} \cdot \vec{x}}{|x|} \quad ; \quad g^{-1} = \frac{(S \cdot x)}{|x|} \quad (9)$$

*) The Euclidean form of (4) is obtained with the familiar recipe

$$A_0 \rightarrow \epsilon A_0 = A_4 \quad , \quad A_\epsilon \rightarrow -A_\epsilon$$

Then the single meron solution can be written in the form

$$A_{\mu}^{(m)} = -i \frac{\sigma_{\mu\nu} x_{\nu}}{x^2} = \frac{1}{2} g^{-1} \partial_{\mu} g \quad (10)$$

and

$$\int_{S_{\infty}} I_{\mu} dS_{\mu} = \frac{1}{2}$$

Similarly for the single antimeron solution we find

$$A_{\mu}^{(\bar{m})} = -i \frac{\bar{\sigma}_{\mu\nu} x_{\nu}}{x^2} = \frac{1}{2} g \partial_{\mu} g^{-1} \quad (11)$$

and

$$\int_{S_{\infty}} I_{\mu} dS_{\mu} = -\frac{1}{2}$$

Using the above expressions one can easily check that the meron and the antimeron configurations are simply related by the (singular) gauge transformation built through the function $g(x)$ itself ^{*)}, namely

$$A_{\mu}^{(\bar{m})} = g A_{\mu}^{(m)} g^{-1} + g \partial_{\mu} g^{-1} \quad (12)$$

and

$$A_{\mu}^{(m)} = g^{-1} A_{\mu}^{(\bar{m})} g + g^{-1} \partial_{\mu} g \quad (13)$$

The single meron and antimeron solutions given in Eqs. (10) and (11) are singular both at the origin and at infinity.

Conformal invariance allows to displace those singularities to the points

*) Similar relations hold between instanton and anti-instanton solutions [Ref. 3)].

$$\begin{aligned} y_+ &= x + a \rightarrow 0 \\ y_- &= x - a \rightarrow 0 \end{aligned} \quad (14)$$

where [as in Ref. 2)] $a_\mu = (0, 0, 0, 1)$.

We thus perform

- a) a translation $x_\mu \rightarrow x_\mu - 2a_\mu$,
- b) an inversion $x_\mu \rightarrow x_\mu / x^2$
- c) a translation $x_\mu \rightarrow x_\mu + a_\mu$.

We shall thus be led to the new solutions

$$A_\mu^{(m \bar{m})} = \frac{1}{2} f^{-1} \partial_\mu f \quad (15a)$$

$$A_\mu^{(\bar{m} m)} = \frac{1}{2} f \partial_\mu f^{-1} \quad (15b)$$

where

$$f = \frac{(S \cdot z)}{|z|} \quad (16)$$

$$z_\mu = \frac{a_\mu}{2a^2} - \frac{(x+a)_\mu}{(x+a)^2}$$

The solution $A_\mu^{(m \bar{m})}$ given in Eq. (15) coincides with the one given in Eq. (4). The identification is made easier by rewriting the first of Eqs. (15) in the form

$$A_\mu^{(m \bar{m})} = \frac{-2i}{y_+^2 y_-^2} \left\{ 2a^2 \sigma_{\mu\nu} y_{+\nu} - y_+^2 \sigma_{\mu\nu} a_\nu + 2 y_{+\mu} \sigma_{\lambda\nu} y_{+\lambda} a_\nu \right\}$$

It is now easy to show that (as exhibited by the notation) $A_\mu^{(m \bar{m})}$ corresponds to a meron located at $y_+ = 0$ and an antimeron at $y_- = 0$ (conversely $A_\mu^{(\bar{m} m)}$ will have a meron at $y_- = 0$ and an antimeron at $y_+ = 0$).

The operators f and f^{-1} can be rewritten in the convenient form ^{*})

$$f(x) = \frac{(\bar{s} y_+)}{|Y_+|} \frac{(s y_-)}{|Y_-|} = \frac{(s y_-)}{|Y_-|} \frac{(\bar{s} y_+)}{|Y_+|} \quad (17)$$

$$f^{-1}(x) = \frac{(s y_+)}{|Y_+|} \frac{(\bar{s} y_-)}{|Y_-|} = \frac{(\bar{s} y_-)}{|Y_-|} \frac{(s y_+)}{|Y_+|}$$

Equations (15) and (17) show that the two solutions are interchanged if one performs "meron conjugation" $s_\mu \rightarrow \bar{s}_\mu$ and also if one performs the $y_+ \rightleftharpoons y_-$ exchange. One can also immediately check that

$$\text{at } y_+ \rightarrow 0 \quad A_\mu^{(m \bar{m})} \rightarrow \frac{1}{2} (s y_+) \partial_\mu (\bar{s} y_+) ; \quad A_\mu^{(\bar{m} m)} \rightarrow \frac{1}{2} (\bar{s} y_+) \partial_\mu (s y_+) \quad (18)$$

$$\text{at } y_- \rightarrow 0 \quad A_\mu^{(m \bar{m})} \rightarrow \frac{1}{2} (\bar{s} y_-) \partial_\mu (s y_-) ; \quad A_\mu^{(\bar{m} m)} \rightarrow \frac{1}{2} (s y_-) \partial_\mu (\bar{s} y_-) \quad (18)$$

Equations (18) show that our previous identification of the meron and anti-meron content of the solutions (15) is indeed correct. As a consequence we have

$$D(x) = \pm \frac{1}{2} \left\{ \delta(x+a) - \delta(x-a) \right\} \quad (19)$$

the signs +, - apply to the solutions (15a) and (15b), respectively. From Eq. (19) one gets

$$\int_{S_\infty} I_\mu dS_\mu = 0 \quad (20)$$

in agreement with the fast decrease at infinity of $A_\mu^{(m \bar{m})}(x)$.

It is now useful to clarify the precise relation between the original solution (1) and the one given by Eqs. (15) and (17). This can be easily achieved by performing an appropriate singular gauge transformation which changes the antimeron at $x=-a$ into a meron (at the same point) thus

^{*}) In deriving Eqs. (17) we have used the fact that when a is oriented along the four axis $(sa)/(a) = (\bar{s}a)/(a) = 1$. For a general vector a_μ we would have had e.g.

$$f(x) = \frac{(\bar{s} y_+)}{|Y_+|} \frac{(s y_-)}{|Y_-|} \frac{(\bar{s} a)}{|a|}$$

leading from an $\bar{m}$ - m amplitude to an m - m one. A glance to the above formulae shows that the required transformation should be of the form. We first transform Eqs. (15b) and (17) into the convenient form

$$A_{\mu}^{(\bar{m}m)} = \frac{1}{2} f \partial_{\mu} f^{-1} = i \frac{(\bar{5} y_+)}{|y_+|} \sigma_{\mu\nu} \left(\frac{y_{+\nu}}{y_+^2} - \frac{y_{-\nu}}{y_-^2} \right) \frac{(\bar{5} y_+)}{|y_+|} \quad (21)$$

We now perform the gauge transformation

$$A_{\mu}^{(\bar{m}m)} \rightarrow g^{-1}(y_+) A_{\mu}^{(\bar{m}m)} g(y_+) + g^{-1}(y_+) \partial_{\mu} g(y_+) \quad (23)$$

$$g(y_+) = \frac{(\bar{5} y_+)}{y_+}$$

where, as in the single meron case, $g(y_+)$ is singular at the antimeron position. One then immediately obtains the form (1)

$$A_{\mu}^{(mm)} = -i \sigma_{\mu\nu} \left(\frac{y_{+\nu}}{y_+^2} + \frac{y_{-\nu}}{y_-^2} \right)$$

which is indeed the dimeron configuration obtained in Ref. 2).

3. The following, simple remark, still in the Euclidean domain, will provide the required insight into the properties of solutions (1)-(4) in Minkowski space-time. The key argument for this final development is the value of the surface integral

$$\int_S d\sigma_{\mu} I_{\mu} = \int_{x_4=0} I_4 d^3\vec{x} \quad (24)$$

to be evaluated on the $x_4=0$ hyperplane, which is invariant under the Wick rotation $x_4 \rightarrow it$.

Let us now consider the integral of the "topological" current density on the closed surface, which encloses the $x=-a$ singular point, and consists of the $x_4=0$ hyperplane and of the lower hemisphere at infinity.

For both the (mm) solution (23) and the $(m\bar{m})$ solution (15a) the overall integral, which represents the topological charge at the point $y_+ = 0$ has the value $\frac{1}{2}$, however,

- a) for the (mm) solution (23) the integral on the hemisphere does already have the value $\frac{1}{2}$ so that

$$\int_{x_4=0} I_4 d^3\vec{x} = 0 \quad (25a)$$

- b) for the $(m\bar{m})$ solution (15a) the integral on the hemisphere at infinity gives a vanishing contribution (due to the fast decreasing asymptotic behaviour and therefore

$$\int_{x_4=0} I_4 d^3\vec{x} = \frac{1}{2} \quad (25b)$$

- c) in the case of the $(\bar{m}m)$ solution (15b) one gets of course:

$$\int_{x_4=0} I_4 d^3\vec{x} = -\frac{1}{2} \quad (25c)$$

4. We are finally ready to discuss the main features of the $O(4) \times O(2)$ solutions in Minkowski space time. We shall concentrate our attention on the $A_\mu^{(m\bar{m})}$ solution of Eqs. (4) and (15a) which [together with its "conjugate" (15b) solution] has the property of being real. The generating operator $f(\vec{x}, t)$ has now the form

$$f(\vec{x}, t) = \frac{(1 - \epsilon t_+ \vec{\sigma} \vec{u})}{\sqrt{1 + t_+^2}} \frac{(1 + \epsilon t_- \vec{\sigma} \vec{u})}{\sqrt{1 + t_-^2}} \quad (26)$$

where

$$\vec{u} = \frac{\vec{x}}{|\vec{x}|}$$

As pointed out in Ref. 4) the real solution (15), (26) leads to the same (finite and convergent) values of $L(x)$ and of $\theta_{\mu\nu}$ obtained in Ref. 2) for the solution (1). We recall for example that

$$L = 12 \left\{ \frac{1}{(1+t_+^2)(1+t_-^2)} \right\}^2 \quad (27)$$

Again the axial density

$$D = \frac{\partial I_\mu}{\partial x_\mu} = 0 \quad (28)$$

vanishes everywhere.

As already noticed in the Euclidean treatment, the fundamental new feature of the $(m-\bar{m})$ solution is its asymptotic behaviour. In the Minkowski case, because of the hyperbolic metric, such a behaviour is not the same for all directions and requires a detailed discussion.

- a) For $x_\mu \rightarrow \infty$ along a space-like direction, A_μ decreases like $1/|\vec{x}|^2$.
- b) A similar $1/t^2$ behaviour is found when $x_\mu \rightarrow \infty$ along a time-like direction.
- c) The situation is completely different when $x_\mu \rightarrow \infty$ along a light-like direction. For $t_+ \rightarrow \infty$ in correspondence of finite values of t_- , A_μ tends to the finite, non-vanishing, expression:

$$A_\mu = \left(\frac{1 - it_- \vec{\sigma} \vec{u}}{\sqrt{1+t_-^2}} \right) \frac{\partial}{\partial x_\mu} \left(\frac{1 + it_- \vec{\sigma} \vec{u}}{\sqrt{1+t_-^2}} \right) \quad (29)$$

We wish now to define a Minkowski "axial charge" and to discuss its properties.

Following the usual prescription for constructing "charges" in quantum field theory we define:

$$q = \int I_\mu dS_\mu \quad (30)$$

where S_μ can represent any space-like surface.

The vanishing of D together with the fast decrease of A_μ for space-like directions ensures that the integral (30) is the same for all space-like surfaces. The integral can thus be computed for the $t=0$ hyperplane leading to

$$q = \int I_\mu dS_\mu = \begin{cases} +\frac{1}{2} & \leftarrow \text{for the solution (16a)} \\ -\frac{1}{2} & \leftarrow \text{for the solution (16b)} \end{cases} \quad (31)$$

It is amusing to notice that the result (31), valid for all space-like surfaces, holds also when the surface S coincides with the $t \rightarrow -\infty$, $t \rightarrow +\infty$ hyperplanes which correspond to the "incoming" and "outgoing" charges. The fact that, even with the fast convergence of A_μ , those asymptotic charges do not vanish, is related to the peculiar asymptotic behaviour of A_μ near the light cone. Indeed the whole contribution to the $t \rightarrow -\infty$ or $t \rightarrow +\infty$ charges comes from the immediate surrounding of the light cone.

This paper confirms that the $m-\bar{m}$ solution of Ref. 4) are the natural solutions of Yang-Mills equations in Minkowski space-time. Beyond the well-known properties of leading finite and convergent values for L and $\theta_{\mu\nu}$ they lead to a meaningful quantized values for the axial charge.

Equations (31) show that they fully deserve the name of "merons" also in the Minkowski framework. In Ref. 5) the original dimeron solution (1) has been used in the Euclidean framework and the semi-integer topological charge was shown to be relevant in connection with the problem of confinement, however, the presence of the two singular points was leading to the handicap of a logarithmically divergent action.

It is possible that the use of the real solution (4) in its own Minkowski framework might still retain some of the advantages of the Euclidean treatment of Ref. 5) without the appearance of any infinite quantities.

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