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TWO-DIMENSIONAL YANG-MILLS THEORY IN THE LEADING $1/N$ EXPANSION

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ABSTRACT

Recent controversies about the gauge invariance of the two-dimensional $SU(N)$ Yang-Mills theory in the 't Hooft limit of large N are resolved. The fermion (quark) propagator is found explicitly, and is qualitatively different from those in the previous literature.

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Lately, there has been a great deal of activity to study the $N \rightarrow \infty$ limit of the $SU(N)$ Yang-Mills field in two dimensions. This problem was just proposed and analyzed by 't Hooft¹⁾ in the light cone gauge. Recent work of Frishman et al.²⁾ and others³⁾ in more general gauges yields, however, different answers. It is the purpose of this note to expose the underlying reason for this contradiction, and to give a revised solution for this problem.

Since the revised bound-state equation is quite complicated and may not be easy to solve even numerically, only the fermion (quark) propagator is studied here. The present answer is different from those given previously, and contains the novel feature of a branch cut.

Since the Yang-Mills field⁴⁾ is massless, care must be exercised with the infra-red cut-off procedure. Einhorn⁶⁾ and Frishman (unpublished) have shown that 't Hooft's cut-off is equivalent to interpreting all integrals as principal values. Chernin procedure⁶⁾ can then be used to trace the lack of gauge invariance to the non-commutative nature of principal-value integrals, as given by the Poincaré-Bertrand formula⁷⁾

$$\int \frac{dx'}{x-x'} \int \frac{dx''}{x'-x''} f(x', x'') - \int dx'' \int dx' \frac{f(x', x'')}{(x-x')(x'-x'')} = -\pi^2 f(x, x), \quad (1)$$

where every integration is understood in the principal-value sense. Since the order of integration cannot be specified in a Feynman integral, Eq. (1) causes chaos. The introduction of $i\epsilon$ is of no help.

This difficulty is resolved by remembering the intimate relation between relativistic quantum field theory and Euclidean quantum field theory, as shown by Schwinger, Symanzik, Nelson, Osterwalder and Schrader⁸⁾. That Wick rotation is always possible restricts the choice of integration procedures. With light cone variables⁹⁾, the factor $(k_1-k_0)^{-2}$ in the Yang-Mills propagator becomes $(k_1+ik_2)^{-2}$ after Wick rotation, where $k_2 = ik_0$. Thus Wick rotation, in particular, excludes the principal-value prescription of Einhorn⁶⁾ and Frishman. Therefore, the correct prescription is to use $(k_1+ik_2)^{-2}$ together with symmetric integration.

Such a prescription is natural mathematically as well as physically. Generalization of the familiar one-dimensional principal-value integral to spaces of higher dimensions was first studied by Calderón and Zygmund¹⁰⁾ and has since been treated extensively in the mathematics literature. The present kernel $(k_1+ik_2)^{-2}$

is an especially simple special case of such generalizations, and has the additional remarkable property that the Poincaré-Bertrand right-hand side is zero, i.e.,

$$\int_{-\infty}^{\infty} dx' dy' \int_{-\infty}^{\infty} dx'' dy'' \frac{f(x', y'; x'', y'')}{[(x+iy)-(x'+iy')]^2 [(x'+iy')-(x''+iy'')]^2} - \int_{-\infty}^{\infty} dx'' dy'' \int_{-\infty}^{\infty} dx' dy' \frac{f(x', y'; x'', y'')}{[(x+iy)-(x'+iy')]^2 [(x'+iy')-(x''+iy'')]^2} = 0. \quad (2)$$

Equation (2) together with a very similar version in the general axial gauge²⁾ is the cornerstone of the present approach.

Consider the limit $N \rightarrow \infty$ with $g^2 = g_0^2 N$ fixed¹⁾, where g_0 is the Yang-Mills coupling constant. In the light cone gauge and the present prescription, the dressed fermion propagator is

$$S(\vec{p}) = -i \frac{p_1 + i p_2}{p_1^2 + p_2^2 + m^2 - (p_1 + i p_2) \Sigma(\vec{p})}, \quad (3)$$

where m is the fermion mass, and $\Sigma(\vec{p})$ satisfies the Wick-rotated version of Eq. (10) of 't Hooft¹⁾

$$\Sigma(\vec{p}) = -\frac{g^2}{\pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 \frac{1}{(k_1 + i k_2)^2} \frac{(k_1 + p_1) + i(k_2 + p_2)}{(k_1 + p_1)^2 + (k_2 + p_2)^2 + m^2 - [(k_1 + p_1) + i(k_2 + p_2)] \Sigma(\vec{k} + \vec{p})}. \quad (4)$$

The solution of this integral equation (4) is

$$\Sigma(\vec{p}) = \frac{g^2}{2\pi} (p_1 + i p_2)^{-1} \left\{ (p^2 + m^2) - [(p^2 + m^2)^2 - 4 g^2 p^2 / \pi]^{1/2} \right\}, \quad (5)$$

and hence the dressed fermion propagator is

$$S(\vec{p}) = -i\pi [2g^2(p_1 - ip_2)]^{-1} \{ (p^2 + m^2) - [(p^2 + m^2)^2 - 4g^2 p^2/\pi]^{1/2} \}. \quad (6)$$

It is instructive to expand (5)

$$\Sigma(\vec{p}) = \frac{g^2}{\pi} \frac{p_1 - ip_2}{p^2 + m^2} \left[1 + \frac{g^2}{\pi} \frac{p^2}{(p^2 + m^2)^2} + \dots \right]. \quad (7)$$

This is qualitatively different from the 't Hooft solution¹⁾, which is a function of $p_1 + ip_2$ only and is proportional to g^2 .

Solutions (5) and (6) show the novel feature of a branch cut, not present in any of the previous solutions of this problem. However, it is exceedingly bothersome that, for small g , the branch points of the dressed fermion propagator are not real. This point deserves a great deal of attention and further study.

The 't Hooft solution has been applied to strong interaction^{1),5),11)}. This work needs to be repeated. For this purpose, the bound state equation must be solved. The present version is similar to Eq. (15) of 't Hooft¹⁾

$$\begin{aligned} \Psi(\vec{p}, \vec{r}) = & - (4g^2)^{-1} [(p_1 - r_1) - i(p_2 - r_2)]^{-1} (p_1 - ip_2)^{-1} \\ & \{ (p-r)^2 + m_2^2 - [(p^2 - 2pr + r^2 + m_2^2)^2 - 4g^2(p-r)^2/\pi]^{1/2} \} \\ & \{ p^2 + m_1^2 - [(p^2 + m_1^2)^2 - 4g^2 p^2/\pi]^{1/2} \} \\ & \int_{-\infty}^{\infty} dk_1 dk_2 (k_1 + ik_2)^{-2} \Psi(\vec{p} + \vec{k}, \vec{r}). \end{aligned} \quad (8)$$

This is a two-dimensional integral equation.

The novel feature of a branch cut in (5) and (6) has previously been found to be present in the two-dimensional Ising field theory¹²⁾, where confinement is due to spontaneous symmetry breaking and phase transition. It is therefore tempting to speculate that branch cut is a general feature of confined propagators.

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