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"FRACTIONAL WINDING NUMBERS" AND SURFACE EFFECTS

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A B S T R A C T

We discuss the modification in the axial anomaly relation due to boundary effects. We relate the surface effects to the generalized index theorem of Atiyah, Patodi and Singer. As an illustration we solve Hawking's problem of the Euclidean Dirac operator in a Riemann metric of Taub-NUT type.

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Although our main motivation for understanding boundary effects in the axial anomaly relation is the problem of colour confinement, we illustrate boundary modifications of the axial anomaly relation in the following with the help of a "natural" example from general relativity.

Recently Hawking ¹⁾ proposed to discuss the Taub-NUT space as a special Euclidean Riemannian background metric in order to understand the axial anomaly relation ^{2),3)} in general relativity. To be more specific, Hawking abstracts a classical Euclidean subproblem from the yet ill-defined formulation of "quantum geometry" in terms of Euclidean Feynman-Kac integrals.

Using the split-point technique for the classical Euclidean $m=0$ Dirac current in a Riemann metric one finds ⁴⁾

$$\partial_\mu j_{\mu 5} = -2 \left\{ \frac{1}{24} p_1 + \sum_i \psi_{0i}^\dagger \gamma_5 \psi_{0i} \right\} \quad (1)$$

$$p_1 = \frac{1}{32\pi^2} \epsilon^{\alpha\beta\mu\nu} R_{\rho\sigma\mu\nu} R_{\alpha\beta\rho\sigma} \quad (2)$$

If one were to ignore boundary terms, one would find by integration the ordinary Atiyah-Singer relation ^{5),4)}

$$\gamma \equiv n_+ - n_- = -\frac{1}{24} \int p_1, \quad (3)$$

which because of $\int p_1 = 2$ for the Euclidean Taub-NUT metric ¹⁾ would lead to the puzzle of where the compensating fractional number comes from. This difficulty is easily resolved if one remembers that Hawking's problem should be considered as a limit of boundary value problems where the surface goes to infinity. In fact any Dirac problem on an initially non-compact space, in particular also the problem in an instanton field ⁶⁾, should be approached in this way. Whether the limiting situation is a closed compactified manifold or whether there remain persistent boundary effects should emerge in a natural fashion. For Hawking's problem we find a persistent boundary effect.

In general the component of the axial current in the direction orthogonal to the boundary leads to three effects ⁷⁾: surface zero modes, say h linear independent ones, a global contribution of all other surface modes η and an effect resulting from the geodesic curvature K near the boundary.

Hence the integration of the axial anomaly relation will lead precisely to the general index theorem with global boundary terms of Atiyah, Patodi and Singer ⁸⁾ : from

$$\int_X \partial_\mu j_{\mu 5} dx = -\frac{1}{12} \int_X p_1 - 2 \sum_i \int_X \psi_{0i}^\dagger \gamma_5 \psi_{0i} dx \quad (4)$$

we obtain

$$\frac{\eta + h}{2} + K = -\frac{1}{12} \int p_1 - 2 \text{index } D \quad (5)$$

D is the Dirac operator with "spectral boundary conditions" ⁸⁾ on the boundary $Y = \partial X$ which roughly speaking corresponds to well-behaved Dirac wave functions (section through the spin-bundle) if Y is moved to infinity. The resulting Green's function obtained by incorporating these boundary conditions serves to construct an axial current in the usual way ⁴⁾, which then in addition to having the anomalous divergence and the zero mode term leads to the boundary contribution on the left-hand side of (4).

h = number of zero surface modes,

$\eta = \lim_{s \rightarrow 0} \sum_{\lambda \neq 0} \text{sign } \lambda |\lambda|^{-s} = \text{A.P.S. invariant } ^{8)}$,

K = contribution from geodesic curvature.

Here the summation over λ is the summation over non-zero surface modes.

We now take Hawking's Euclidean Taub-NUT space, defined by the metric ¹⁾ :

$$ds^2 = V^{-1} \left[d\tau \pm 4m \sin^2 \frac{\theta}{2} d\varphi \right]^2 + V \left[dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]$$

$$m > 0 ; V = 1 + \frac{2m}{R} ; \int p_1 = \pm 2 \quad (6)$$

The surfaces $R = \text{const.}$ have the topology of S^3 and are left invariant under the action of the $SO(3)$ subgroup of the isometry group of the metric which is generated by the Killing vectors :

$$\begin{aligned}
 K_1 &= -\sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \left[\cot \theta \frac{\partial}{\partial \varphi} \pm 2m \tan \frac{\theta}{2} \frac{\partial}{\partial \tau} \right] \\
 K_2 &= \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \left[\cot \theta \frac{\partial}{\partial \varphi} \pm 2m \tan \frac{\theta}{2} \frac{\partial}{\partial \tau} \right] \\
 K_3 &= \frac{\partial}{\partial \varphi} \mp 2m \frac{\partial}{\partial \tau}
 \end{aligned} \tag{7}$$

Moreover, this group is transitive on the surfaces $R = \text{const.}$ and thus provides their natural parametrization. The K_i form an orthonormal basis with respect to the natural bi-invariant metric. The metric, restricted to $R = \text{const.}$ reads

$$\begin{aligned}
 g &= \sum_i \omega^i \otimes \omega^i \quad \text{with} \\
 \omega^1 &= \frac{\mp 1}{\sqrt{V}} (d\tau \pm 4m \sin^2 \frac{\theta}{2} d\varphi) \\
 \omega^2 &= R\sqrt{V} d\theta \\
 \omega^3 &= R\sqrt{V} \sin \theta d\varphi
 \end{aligned} \tag{8}$$

The fields K_i are Killing vector fields, thus

$$K_i g(K_r, K_s) = g([K_i, K_r], K_s) + g(K_r, [K_i, K_s]) \quad , \tag{9}$$

which means that the metric $\tilde{g}$, defined by

$$\tilde{g}(h) = g(h) \circ \text{Ad } h^{-1} \tag{10}$$

is left-invariant.

g and $\tilde{g}$ have the same eigenvalues which can easily be found by inserting (7) into (8), for instance at the convenient values $\theta = \pi/2$, $\varphi = 0$. One finds

$$\mu_1(R) = \mu_2(R) = R^2 V \quad ; \quad \mu_3(R) = \frac{4m^2}{V} \quad (11)$$

The Dirac operators with respect to g and $\tilde{g}$ are bound to have the same eigenvalues. With this information it is now a merely algebraic problem to write down the Dirac operator in terms of μ_1, μ_2, μ_3 and to find its eigenvalues, and in particular the quantities $\eta_R(0)$ and h_R .

The eigenvalues are 9)

$$\begin{aligned} \frac{n}{\mu} + \frac{m}{2} & \quad (\text{multiplicity } 2p) \\ & \quad (n, q \text{ integer, } > 0) \\ \frac{m}{2} \pm \sqrt{4nq\mu^2 + (n-q)^2} / \mu & \quad (\text{multiplicity } n+q) \end{aligned} \quad (12)$$

Here

$$\mu = \mu(R) = \pm \frac{\mu_3^{1/2}(R)}{\mu_1^{1/2}(R)} \xrightarrow{R \rightarrow \infty} 0$$

This gives

$$\eta_R(0) = \pm(-1 + 2\mu^2(R) - \mu^4(R)) / 6 \xrightarrow{R \rightarrow \infty} \mp \frac{1}{6} \quad (13)$$

(for $|\mu| < 4$)

and

$$h_R \xrightarrow{R \rightarrow \infty} 0$$

Looking at Eq. (1) we see that for $R \rightarrow \infty$ where K vanishes because of asymptotic flatness

$$\text{index } D = \mp \frac{1}{12} \pm \frac{1}{12} = 0$$

The second term which comes from the boundary (it is the η term) just compensates the "fractional instanton" contribution $-1/24 \int p_1$ and thus solves Hawking's paradox. For finite R one can establish by a more lengthy computation that η and K are functions of m/R which again

together with the Pontryagin density contribution leads to a complete compensation. In order to get rid of any persistent boundary contribution one presumably has to glue together 12 Taub-NUT spaces in such a way that the boundary drops out ¹⁾.

REFERENCES AND FOOTNOTES

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