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CLASSICAL ANOMALIES AND LOCAL VERSION
OF THE ATIYAH - SINGER THEOREM

N. K. Nielsen
NORDITA, Copenhagen

H. Römer ^{*)} and B. Schroer ^{**)}
CERN - Geneva

A B S T R A C T

We give a split-point derivation of the $m=0$ axial anomaly in an external Euclidean Riemann metric. We discuss two new anomalous currents which lead to the Euler-density and the Hirzebruch L density.

^{*)} On leave of absence from the Physikalisches Institut der Universität Bonn.

^{**)} On leave of absence from the Institut für Theoretische Physik der Freien Universität Berlin.

Usually physicists use the terminology of anomalous currents for certain operator currents (composite fields) which although being formally (i.e., naively canonically) conserved, nevertheless possess a true non-vanishing divergence as a result of renormalization properties. The famous example is the axial current in QED ¹⁾.

In the study of the Euclidean functional integral in gauge theories with fermions it is, however, mathematically useful to consider an affiliated "classical current", which also fulfils an anomalous divergence relation. The main ingredient in the construction of this classical gauge invariant current is the short-distance singular classical Green's function of the Euclidean Dirac operator

$$\gamma^\mu (\partial_\mu - ie A_\mu) + m$$

A particularly interesting situation arises for $m=0$, Dirac operator $\not{D} = \gamma_\mu (\partial_\mu - ie A_\mu)$.

In this case the following four steps lead to a local version ³⁾⁻⁵⁾ of the Atiyah-Singer index theorem ^{6),7)} for gauge theories.

- 1) Compactify $\mathbb{R}^4$ by (conformal) stereographic projection into S^4 using as a metric the natural metric on S^4 . The Dirac operator on this manifold leads to the following discrete eigenvalue problem ³⁾ :

$$\not{D} \psi_i = \lambda_i \frac{2}{1+x^2} \psi_i \quad (1a)$$

with normalization

$$\int \frac{2d^4x}{1+x^2} \psi_i^\dagger \psi_i = 1 \quad (1b)$$

- 2) Define the spinor Green's function

$$G(x,y;A) = \sum' \psi_i(x) \frac{1}{\lambda_i} \psi_i^\dagger(y) \quad , \quad (2)$$

where the prime denotes omission of zero eigenvalues of $\not{D}$.

- 3) Define a gauge invariant axial current by "geometric regularization" (point split technique and parallel transport) :

$$j_{\mu 5}(x) = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \sum_{\epsilon, -\epsilon} \text{tr} \gamma_{\mu} \gamma_5 G(x+\epsilon, x-\epsilon; A) I(x+\epsilon, x-\epsilon) . \quad (3)$$

Here

$$I(x', x) = P e^{ie \int_x^{x'} A_{\mu} d\xi_{\mu}} \quad (4)$$

is the operator of parallel transport from x to x' along the geodesic which connects the two points, P denotes the path ordering, and the trace is to be taken over Dirac - as well as "colour" indices.

The $\epsilon \leftrightarrow -\epsilon$ averaging serves to get rid of the odd terms $\epsilon_{\nu}/\epsilon^2$ and the limit is shown to be directional independent. In (4) the zero modes may even be included since the trace with $\gamma_{\mu} \gamma_5$ projects them out anyhow.

- 4) Compute the divergence of the current and obtain the modified anomaly relation

$$\partial_{\mu} j_{\mu 5} = \frac{e^2}{8\pi^2} \text{tr} F_{\mu\nu} \hat{F}_{\mu\nu} + 2 \frac{2}{1+x^2} \sum_i \psi_{0i}^{\dagger} \gamma_5 \psi_{0i} . \quad (5)$$

($\{\psi_{0i}\}$ is an orthonormal complete set of zero modes.)

The last relation is the density version of the Atiyah-Singer theorem ⁶⁾. If we had kept $m \neq 0$ ⁴⁾ we also would have obtained after integration (m drops out) the global gauge Atiyah-Singer relation on a closed manifold. However, the density form is more flexible and allows us to discuss possible boundary effects on manifolds ⁸⁾.

There are many other classical problems in which vector currents with anomalies arise. A generalization of the Dirac equation to an arbitrary four-dimensional manifold with a Riemann metric leads to

$$\partial_{\mu} j_{\mu 5} = -\frac{1}{384\pi^2} \epsilon^{\alpha\beta\mu\nu} R_{\sigma\mu\nu} R_{\alpha\beta\sigma} + \frac{e^2}{8\pi^2} \text{tr} F_{\mu\nu} \hat{F}_{\mu\nu} + 2 \sum_i \psi_{0i}^{\dagger} \gamma_5 \psi_{0i} \quad (6)$$

Here, for simplicity, we have absorbed all metric factors (above, we have explicitly exhibited the conformal factors) in the x dependent γ_μ matrices, the spinor connection and the normalization of states.

Before we discuss in some detail the derivation of this generalization we briefly point out two entirely different anomalous currents, whose anomalies give the Euler characteristic ⁹⁾ and the signature of the base manifold. These currents are formed with a Lorentz $(0(4)-)$ reducible multiplet of scalar, vector, and antisymmetric tensor fields. The Dirac field ψ is replaced by a column of antisymmetric tensors

$$\phi = \begin{pmatrix} f \\ f_\mu \\ f_{\mu\nu} \\ f_{\mu\nu\varrho} \\ f_{\mu\nu\varrho\sigma} \end{pmatrix} \quad (7)$$

For both cases the Γ_μ matrices are of the Duffin-Kemmer type ¹⁰⁾

$$(\Gamma_\mu)_{JK} = \varepsilon(\mu, K) \delta^{J, \{\mu\} \cup K} \delta^{|K|, |J|+1} - (J \leftrightarrow K) \quad , \quad (8)$$

where the antisymmetric tensors are uniquely labelled by subsets $J, K, \dots$ of the set $\{1, 2, 3, 4\}$, $|K|$ denotes the number of elements of the set K , and

$$\varepsilon(\mu, K) = (-1)^{|\{j | j \in K, j < \mu\}|}$$

The analogues of γ_5 are

$$(\Gamma_5^E)_{JK} = (-1)^{|K|} \delta^{JK} \quad (9)$$

or

$$(\Gamma_5^H)_{JK} = i^{|K|(|K|+1)} \delta^{J, \{1, 2, 3, 4\} - K} \quad (10)$$

where E stands for the Euler current and H for the Hirzebruch current.

With the Γ matrices one can form (by the point splitting method and using the Γ algebra) covariant currents $j^{\mathbb{E},\mathbb{H}}_{\mu}$, which fulfil the relation, on a closed manifold (formally $j^{\mathbb{E}}_{\mu} = \phi^{\dagger} \Gamma_{\mu} \Gamma^{\mathbb{E}}_5 \phi$)

$$\partial_{\mu} j^{\mathbb{E}}_{\mu} = -\frac{1}{64\pi^2} R_{\alpha\beta\mu\nu} R_{\gamma\delta\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \varepsilon^{\mu\nu\rho\sigma} \frac{1}{\sqrt{g}} + 2 \sum_i \phi_{\alpha i}^{\dagger} \Gamma^{\mathbb{E}}_5 \phi_{\alpha i}, \quad (11)$$

whereas the Hirzebruch signature current obeys the divergence relation (formally $j^{\mathbb{H}}_{\mu} = \phi^{\dagger} \Gamma_{\mu} \Gamma^{\mathbb{H}}_5 \phi$)

$$\partial_{\mu} j^{\mathbb{H}}_{\mu} = \frac{1}{48\pi^2} \varepsilon^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu} R_{\rho\sigma\alpha\beta} + 2 \sum_i \phi_{\alpha i}^{\dagger} \Gamma^{\mathbb{H}}_5 \phi_{\alpha i}. \quad (12)$$

Integration of these density relations over the closed manifolds yields the Gauss-Bonnet theorem and Hirzebruch's signature theorem ⁷⁾.

For general dimensions, the anomalies are :

- (a) $2 \cdot \hat{A}(x, 0, \eta) = 2 \text{ch } \eta \hat{A}$ for the Dirac current in the presence of a gauge field, where $\text{ch } \eta$ is the Chern character of the gauge bundle and $\hat{A}$ is the Hirzebruch $\hat{A}$ genus density, which can be expressed in terms of the curvature tensor (via Pontryagin classes of the manifold) and the gauge field strength (via the Chern classes of the gauge bundle) ;
- (b) $2E$ for the Euler current, where E is the Euler class (Pfaffian of the curvature form), which yields the Euler characteristic when integrated over the compact manifold ;
- (c) $2L$ for the Hirzebruch current, where L is the L polynomial in the Pontryagin classes of the manifold (whose dimension has to be a multiple of four) and gives Hirzebruch's L genus after integration.

The following remark may serve as an example, how intricate a relationship there is between Euclidean and Minkowskian field theory. It is known that a manifold with non-vanishing Euler class does not admit a Minkowskian structure with the Lorentz group as structure group. So, the topological nature has to change under "Wick rotation". Still, the topological properties of the "Wick rotated" theory are vital for the Minkowskian version.

In the following we shall perform the explicit construction for the case of the axial anomaly in curved compact Euclidean space-time. We start out from the field equation in the vierbein formulation ¹¹⁾

$$i \gamma_\mu(x) \nabla_\mu \psi = 0, \quad (13a)$$

$$\nabla_\mu \psi = (\partial_\mu + A_\mu) \psi, \quad (13b)$$

$$A_\mu = \frac{1}{2} A_\mu^{\alpha\beta} C_{[\alpha,\beta]}, \quad (13c)$$

$$A_\mu^{\alpha\beta} = -h^{\alpha\nu}_{,\mu} h^{\beta\mu}_{,\nu} - \Gamma_{\mu\sigma\nu} h^{\alpha\sigma} h^{\beta\nu}, \quad (13d)$$

$$C_{[\alpha,\beta]} = \frac{1}{2} [\gamma_\alpha, \gamma_\beta], \quad (13e)$$

$$A_{\mu,\nu} - A_{\nu,\mu} + [A_\mu, A_\nu] = R_{\mu\nu} = \frac{1}{2} C_{[\alpha,\beta]} R_{\alpha\beta\mu\nu}. \quad (13f)$$

The "naïve" Euclidean current

$$j_{\mu 5}^{(0)}(x) = i g^{1/2} \bar{\psi} \gamma_\mu(x) \gamma_5 \psi = i g^{1/2} \text{tr} \gamma_\mu \gamma_5 G \quad (14a)$$

is formally conserved

$$\partial_\mu j_{\mu 5} = 0 \quad (14b)$$

In properly point split form, however, it reads ¹²⁾

$$j_{\mu 5}(x', x) = \frac{i}{4} \sum_\epsilon g^{1/2} \text{tr} \gamma_\mu(x) \gamma_5 G(x, x') I(x', x), \quad (15)$$

where $I(x', x)$ has been introduced above, and $G(x, x')$ is the Euclidean Green's function with zero modes extracted.

The leading singularity of G is

$$G(x, x') \approx \frac{-i}{2\pi^2} \frac{\delta_\mu \sigma_\mu}{(\sigma_\rho \sigma_\rho)^2},$$

where σ_μ is the tangent at x to the geodesic from x to x' normalized such that $\sigma_\rho \sigma_\rho$ is the square of the geodesic distance between x and x' .

Now, for taking the divergence, $j_{\mu 5}(x, x')$ has to be differentiated both at x and x' , and the differentiation at x' is moved to x by a geodesic parallel transport operator $g_\mu^{\nu'}$. One finds, using the properties of G :

$$\begin{aligned} \partial_\mu j_{\mu 5} = & g^{1/2} \epsilon_{\gamma 5} G_{; \nu'}(x, x') \{ g_\mu^{\nu'} I(x', x) \gamma^{\nu'}(x) - \gamma^{\nu'}(x') I(x', x) \} \\ & + g^{1/2} \epsilon_{\gamma 5} G(x, x') \{ g_\mu^{\nu'} I_{; \nu'}(x', x) + I_{; \mu}(x', x) \} \\ & + 2 \sum_i \psi_{0i}' \gamma_5 \psi_{0i}. \end{aligned} \quad (16)$$

Here, primed indices always refer to the argument x' .

In the coincidence limit $x \rightarrow x'$, the quantities in the curly brackets vanish, but there are compensating singularities in the Green's function. The expansion of the brackets in terms of σ_μ is obtained by successive covariant differentiation of identities like

$$\sigma_\mu I_{; \mu} = 0 \quad (17)$$

Details will be given in a forthcoming publication.

The result is that the first bracket is of order σ^5 and does not contribute to $\partial_\mu j_{\mu 5}$, whereas the second bracket just compensates the leading singularity of $G(x, x')$.

Insertion into Eq. (16) yields

$$\begin{aligned}
 \partial_\mu j_{\mu 5} &= i g^{1/2} \epsilon_{\mu\nu\lambda} \gamma_\mu \gamma_5 \left(\frac{-i}{2\pi^2} \right) \frac{\gamma_\lambda \sigma_\lambda}{(\sigma_3 \sigma_3)^2} \frac{1}{24} C_{[\alpha, \beta]} R_{\delta\tau\mu\nu} R_{\alpha\beta\delta\sigma} \sigma^\tau \sigma^\nu \sigma^\sigma \\
 &+ 2 \sum_i \psi_{0i}^\dagger \gamma_5 \psi_{0i} \\
 &= - \frac{1}{384\pi^2} \epsilon^{\mu\nu\alpha\beta} R_{\delta\mu\nu} R_{\alpha\beta\delta\sigma} + 2 \sum_i \psi_{0i}^\dagger \gamma_5 \psi_{0i} , \tag{18}
 \end{aligned}$$

which is the gravitational anomaly of the axial current ^{(13), (14)}.

In deriving the last line of Eq. (18) one has to employ the identity

$$\begin{aligned}
 &\epsilon^{\mu\lambda\alpha\beta} R_{\delta\tau\mu\nu} R_{\alpha\beta\delta\sigma} \\
 &= \frac{-1}{16} \epsilon^{\mu\sigma\alpha\beta} R_{\delta\mu\nu} R_{\alpha\beta\delta\sigma} \left(\frac{1}{3} g^{\lambda\tau} g^{\nu\sigma} - \frac{1}{3} g^{\lambda\sigma} g^{\nu\tau} + g^{\lambda\nu} g^{\tau\sigma} \right) \\
 &\quad + \text{antisymmetric terms} .
 \end{aligned}$$

To summarize, we derived the gravitational contribution to the axial current anomaly in the framework of classical field theory, without even mentioning QFT one-loop graphs. The Atiyah-Singer theorem gives a topological explanation for the emergence of anomalous divergences by relating this familiar physical feature to a "skewness" of the zero mode structure of certain elliptic operators, which has a topological meaning. The Atiyah-Singer theorem also provides a clue how to formulate other topological quantities as anomalous divergences of appropriate currents.

On the other hand, it is to be remembered that one is always arguing with the Euclidean counterpart of the underlying classical field theory, whose relationship to the Minkowskian QFT is quite intricate, although it exhibits salient topological features of it.

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