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RELATIVE MAGNITUDES OF THE J/ψ AND ψ'
WAVE FUNCTIONS AT THE ORIGIN

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ABSTRACT

We prove that in the approximation of a non-relativistic quark- antiquark model the 2S wave function at the origin is smaller (larger) than the 1S wave function if the binding potential is a concave (convex) function of the inter-particle distance.

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1. INTRODUCTION AND OUTLINE

In non-relativistic charmonium models of the J/ψ , interpreted as a 1S state and ψ' interpreted as a 2S state, a crucial quantity is the wave function at the origin. In particular, the leptonic width is controlled, via the Weisskopf-Van Royen formula ¹⁾, by this quantity :

$$\Gamma_e = 3 \alpha^2 \frac{16\pi}{3} \times \frac{4}{9} |\psi(0)|^2 \quad (1)$$

to fit the experimental width the magnitude of $\psi_{2S}(0)(\psi')$ must be 1.5 smaller than the magnitude of $\psi_{1S}(0)(J/\psi)$.

The question we want to investigate is if this ratio is sensitive to the details of the potential chosen to build the quark-antiquark pair. Previously ^{2),3)}, it has been shown that the order of levels of charmonium is relatively insensitive to details. In particular any potential of the form

$$-\int_{-1}^0 \sigma(\alpha) r^\alpha d\alpha + \int_0^2 \rho(\alpha) r^\alpha d\alpha, \sigma > 0, \rho > 0, \quad (2)$$

gives the "canonical" ordering

$$1S, 2P, 2S, 3D$$

which is consistent with the most favoured interpretation of experiment, including the last discovered resonance at Spear at 3.77 GeV ^{4),5)}.

We shall see that the situation is not exactly the same for the magnitude of the wave functions at the origin. We prove two theorems.

I) If the total binding potential $V(r)$ is convex, i.e.,

$$\frac{d^2 V}{dr^2} > 0 \quad (3)$$

$$|\psi_{2S}(0)| > |\psi_{1S}(0)|$$

this could be the case of a pure harmonic oscillator potential.

II) If $V(r)$ is concave,

$$\frac{d^2 V}{dr^2} < 0$$

$$|\psi_{1s}(0)| > |\psi_{2s}(0)| \quad (4)$$

this is the case of a pure "Coulomb" attractive potential, but also of Coulomb + linear potential which is frequently used in charmonium models.

2. PROOF OF THEOREM I

We call $u = r\psi_{1s}$ the reduced wave function of the ground state, $v = r\psi_{2s}$ the reduced wave function of the first radial excitation :

$$\begin{aligned} -u'' + V(r)u &= E_1 u \\ -v'' + V(r)v &= E_2 v \end{aligned} \quad (5)$$

$$E_2 > E_1$$

u is nodeless and can be taken positive. v has one node, and we take $v > 0$ for $r < r_0$, $v < 0$ for $r > r_0$. Like in Ref. 2) we prove that $u-v$ vanishes at most once in $r < r_0$ and that $u+v$ vanishes at most once in $r > r_0$.

Indeed

$$uv' - vu' = (E_1 - E_2) \int_0^r uv dr < 0 \quad (6)$$

for $r < r_0$. Hence if $u-v=0$, $(u'-v') > 0$, so that $u-v$ cannot vanish more than once in $r < r_0$.

Similarly

$$uv' - vu' = (E_2 - E_1) \int_r^\infty uv \, dr < 0 \quad \text{for } r > r_0 \quad (7)$$

and at $u+v=0$ we have

$$u' + v' < 0.$$

We conclude that $u^2 - v^2$ vanishes at most once in $0 < r < r_0$ and at most once in $r_0 < r < \infty$.

Let us now make one further remark: if $(u'(0))^2 > (v'(0))^2$, $u^2 - v^2$ does not vanish in $0 < r < r_0$, because $u^2 - v^2$ is positive at $r = r_0$, and must vanish once in $r_0 < r < \infty$ if u^2 and v^2 are normalized. Indeed

$$\int_0^\infty (u^2 - v^2) \, dr = 0. \quad (8)$$

So for $r \rightarrow \infty$, $u^2 - v^2 < 0$.

The quantity of interest for us is

$$|\psi_{1s}(0)|^2 - |\psi_{2s}(0)|^2 = (u'(0))^2 - (v'(0))^2 \quad (9)$$

Multiplying equations (5) to the left by $u'(r)$ and $v'(r)$, respectively, we get the relation ⁶⁾

$$(u'(0))^2 - (v'(0))^2 = \int_0^\infty (u^2 - v^2) \frac{dV}{dr} \, dr \quad (10)$$

We are now ready to prove Theorem I. Assume $d^2V/dr^2 > 0$. Assume further $|u'(0)|^2 - |v'(0)|^2 > 0$. This leads to a contradiction. Indeed it has been noticed that if $|u'(0)|^2 > |v'(0)|^2$, $u^2 - v^2$ vanishes once and only once, at $r = r_1$. Then

$$\int (u^2 - v^2) \left[\frac{dV}{dr}(r) - \frac{dV}{dr}(r_1) \right] \, dr$$

is strictly negative, in contradiction with (10). Therefore if $d^2V/dr^2 > 0$, $(u'(0))^2 < (v'(0))^2$.

3. PROOF OF THEOREM II

Here let us sharpen slightly a remark that we made in the previous Section. If $|u'(0)|^2 = |v'(0)|^2$, $u^2 - v^2$ is strictly positive for $0 < r \leq r_0$ where r_0 is the first zero of u . This is because the effective potential $V - E_2$ is more attractive than $V - E_1$, and follows from standard Sturm-Liouville theory. Then the normalization condition (8) implies that $u^2 - v^2$ is negative for $r \rightarrow \infty$.

First we shall prove that if

$$\frac{d^2V}{dr^2} < 0$$

it is impossible to have $|u'(0)|^2 = |v'(0)|^2$. Indeed then $u^2 - v^2$ vanishes once at $r = r_1$ and

$$\int_0^\infty (u^2 - v^2) \frac{dV}{dr} dr = \int_0^\infty (u^2 - v^2) \left[\frac{dV}{dr}(r) - \frac{dV}{dr}(r_1) \right] dr > 0$$

Equation (10) would therefore indicate that $(u'(0))^2 > (v'(0))^2$. However, if we assume $(u'(0))^2 < (v'(0))^2$ we do not get a really obvious contradiction from Eq. (10) because $u^2 - v^2$ may vanish twice, and we have to use a more subtle argument.

Let us remark that there exist concave potentials for which $(u'(0))^2 - (v'(0))^2 > 0$. For instance, an attractive Coulomb potential admits as $\ell=0$ normalized reduced wave functions

$$\begin{aligned} u &= 2r \exp -r \\ v &= \frac{1}{\sqrt{2}} \left(r - \frac{r^2}{2} \right) \exp -\frac{r}{2} \end{aligned} \quad (11)$$

Let us call V_0 a concave potential such that $|u'(0)|^2 > |v'(0)|^2$. V_0 can be precisely an attractive Coulomb potential.

We solve the Schrödinger equation for a family of potentials

$$V_{\alpha} = (1-\alpha)V_0 + \alpha V, \quad 0 \leq \alpha \leq 1$$

The energy levels and the wave functions are continuous functions of the parameter α . The levels never cross. The potentials V_{α} are all concave if V_0 and V are concave.

If it happens that u and v , corresponding to the potential ($\alpha=1$) are such that $|u'(0)|^2 < |v'(0)|^2$ there will be a value of α such that

$$|u'_{\alpha}(0)|^2 = |v'_{\alpha}(0)|^2.$$

However, this is impossible as has been remarked before. So we conclude that if

$$\frac{d^2V}{dr^2} < 0, \quad |u'(0)|^2 > |v'(0)|^2.$$

4. CONCLUDING REMARKS

The two theorems we have obtained explain why experimental data favour a Coulomb-like + linear potential, since the leptonic width of J/ψ is larger than the leptonic width of ψ' . Even if they look convenient because of their simplicity harmonic oscillator wave functions should only be used with extreme caution.

Of course our theorems do not tell us anything on a mixed situation in which d^2V/dr^2 has not a constant sign like, for instance, in

$$V = -\frac{\alpha_s}{r} + Cr^2$$

Clearly α_s must be sufficiently large then to make agreement with experiment possible. In the relativistic case, for which intensive calculations have been made ^{5),7)} one has first to carry a non-relativistic reduction before applying our results.

R E F E R E N C E S

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