Ref.TH.2327-CERN

EUCLIDEAN SUPERSYMMETRY AND THE MANY-INSTANTON PROBLEM

B. Zumino

CERN -- Geneva

ABSTRACT

There is no Hermitean supersymmetry in Euclidean four-space. The simplest supersymmetry has complex four-component spinorial parameters. We give its algebraic structure and the automorphisms of the algebra, as well as a representation in terms of fields and an invariant Lagrangian. The results are relevant to the counting and the construction of the solutions of the many-instanton problem.

Ref.TH.2327-CERN
31 May 1977

Recently there has been considerable interest [1-16] in the study of the Yang-Mills field coupled with a massless, spin one-half field in Euclidean space-time. When the Yang-Mills field has non-vanishing winding number there are non-trivial solutions for the spinor. It has been observed [14-16] that, if the spinor field is in the adjoint representation of the internal symmetry group, the equations of motion possess partial supersymmetry properties analogous to those of the corresponding equations of the supersymmetric Yang-Mills theories in Minkowski space [17-19]. We will show here that, if the theory is extended by the inclusion of a scalar and pseudoscalar field, there exists a complete supersymmetry, with transformations which form a closed algebra in the usual sense. These transformations can be used to obtain some information on the solutions of the equations of motion. In particular, some of the results of Ref. [15] emerge as a special case.

In Euclidean four-space there are no real four by four gamma matrices ^{*)} and therefore no Majorana spinors. As a consequence, the generators of the simplest supersymmetry are complex four-component spinors. Since the spinor generators are complex, the supermultiplets are larger than they would be for a Hermitean supersymmetry. We consider a supermultiplet of fields consisting of a vector V_μ , a complex spinor ψ , a real scalar A and a real pseudoscalar B . All these fields are assumed to belong to the adjoint representation of some internal gauge group of which V_μ is the gauge potential. The internal symmetry indices will not be explicitly indicated and we shall use the cross product defined by the structure constants of the group. For instance, $D_\mu A = \partial_\mu A + gV_\mu \times A$ is the covariant derivative of the field A and $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + gV_\mu \times V_\nu$ is the field strength. In terms of the anticommuting (Grassmannian), complex, spinor parameter α (a singlet under the internal symmetry) the infinitesimal supersymmetry transformation is given by

$$\begin{aligned}\delta A &= i\alpha^* \psi - i\psi^* \alpha \\ \delta B &= i\alpha^* \gamma_5 \psi - i\psi^* \gamma_5 \alpha \\ \delta V_\mu &= i\alpha^* \gamma_\mu \psi - i\psi^* \gamma_\mu \alpha \\ \delta \psi &= -F_{\mu\nu} \Sigma_{\mu\nu} \alpha + (D_\mu A + \gamma_5 D_\mu B) \gamma_\mu \alpha - A \times B \gamma_5 \alpha \\ \delta \psi^* &= \alpha^* F_{\mu\nu} \Sigma_{\mu\nu} + \alpha^* \gamma_\mu (D_\mu A + \gamma_5 D_\mu B) - \alpha^* \gamma_5 A \times B\end{aligned}\tag{1}$$

) Our Euclidean gamma matrices are Hermitean and satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$. We take γ_1, γ_2 and γ_3 real and symmetric, γ_4 imaginary and antisymmetric. Then $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ is Hermitean, imaginary and antisymmetric. The matrix $C = \gamma_1 \gamma_2 \gamma_3$ is real, antisymmetric and satisfies $C\gamma_\mu = \gamma_\mu^ C = \gamma_\mu^T C$, $C^2 = -1$. We also use $\Sigma_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu]$, $\epsilon_{1234} = 1$.

The commutator of two transformations (1) is a translation accompanied by a field dependent gauge transformation. For instance,

$$[\delta_2, \delta_1] A = \xi_\mu \partial_\mu A + g \Lambda \times A \quad (2)$$

where

$$\xi_\mu = 2i \alpha_1^* \gamma_\mu \alpha_2 - 2i \alpha_2^* \gamma_\mu \alpha_1 \quad (3)$$

and

$$\begin{aligned} \Lambda = & \xi_\mu V_\mu - (2i \alpha_1^* \alpha_2 - 2i \alpha_2^* \alpha_1) A \\ & + (2i \alpha_1^* \gamma_5 \alpha_2 - 2i \alpha_2^* \gamma_5 \alpha_1) B . \end{aligned} \quad (4)$$

Similarly

$$[\delta_2, \delta_1] V_\lambda = \xi_\mu \partial_\mu V_\lambda - D_\lambda \Lambda \quad (5)$$

and so on. In evaluating the commutator on the spinor field one makes use of its Grassmannian nature. It is easy to verify that the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu A)^2 - \frac{1}{2} (D_\mu B)^2 - \frac{i}{2} (\psi^* \gamma_\mu D_\mu \psi - D_\mu \psi^* \gamma_\mu \psi) \\ & + ig \psi^* \cdot \left[(A - \gamma_5 B) \times \psi \right] + \frac{g^2}{2} (A \times B)^2 \end{aligned} \quad (6)$$

changes by a divergence under (1). It is Hermitean. Observe, however, the sign of the kinetic term of the field A . To understand its origin we observe that the supersymmetry algebra admits as automorphisms the non-compact chiral transformations given infinitesimally by

$$\delta'A = 2B, \delta'B = 2A, \delta'\psi = \gamma_5 \psi, \delta'\psi^* = \psi^* \gamma_5, \delta'V_\mu = 0 \quad (7)$$

(we have dropped the infinitesimal parameter). This is also an invariance of the Lagrangian. There are three more automorphisms of the algebra which leave also the Lagrangian invariant. They are, on ψ ,

$$\begin{aligned} \delta_1 \psi &= i \psi, & \delta_1 \psi^* &= -i \psi^* \\ \delta_2 \psi &= C \psi^*, & \delta_2 \psi^* &= C \psi \\ \delta_3 \psi &= -i C \psi^*, & \delta_3 \psi^* &= i C \psi \end{aligned} \quad (8)$$

while the other fields do not change, i.e., $\delta_1 A = 0$ etc. While δ_1, δ_2 and δ_3 commute with the chiral transformation δ' , they form an $SU(2)$ structure, $[\delta_1, \delta_2] = \delta_3$, etc., cyclically. All the above can be verified directly on the field supermultiplet. We are in the presence of a non-compact $U(2)$ with a compact $SU(2)$ subgroup. In Euclidean four-space the Lorentz transformations become compact rotations, while the chiral transformations, which are compact in Minkowski space, become non-compact. This fact has been noticed before [20,21]. The conserved axial vector current is

$$J_\mu^{(5)} = A D_\mu B - B D_\mu A - \frac{i}{2} \psi^* \gamma_5 \gamma_\mu \psi \quad (9)$$

The conserved spinor current of the supersymmetry (1) is

$$\begin{aligned} J_\mu &= F_{\lambda\nu} \cdot \Sigma_{\lambda\nu} \gamma_\mu \psi + \gamma_\lambda D_\lambda (A + \gamma_5 B) \cdot \gamma_\mu \psi - g (A \times B) \cdot \gamma_5 \gamma_\mu \psi \\ &+ \frac{2}{3} (\gamma_\mu \gamma_\lambda \partial_\lambda - \partial_\mu) [(A - \gamma_5 B) \cdot \psi] \quad (10) \end{aligned}$$

The last term, which is automatically conserved, has been added so that the spinor current satisfies also

$$\gamma_\mu J_\mu = 0 . \quad (11)$$

The supersymmetry transformations δ can be split into the sum of δ_+ and δ_- by separating the parameter α into a part of positive chirality and one with negative chirality

$$\gamma_5 \alpha_+ = \alpha_+ , \quad \gamma_5 \alpha_- = -\alpha_- . \quad (12)$$

Two transformations of the type δ_+ or two of the type δ_- commute, while the commutator of a δ_+ with a δ_- gives a translation (up to field dependent gauge transformations). We mention also that, since the Lagrangian is invariant under conformal transformations one can enlarge the supersymmetry algebra by introducing conformal supersymmetry transformations. This is related to the existence of the spinor current (10) which satisfies (11).

We do not write out the obvious equations following from (6) but observe that they take a very simple form if one imposes on the solutions the restrictions, invariant under supersymmetry transformations,

$$\begin{aligned} \bar{F}_{\mu\nu} &= \hat{\bar{F}}_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} \bar{F}_{\lambda\rho} \\ \psi &= \psi_- = \frac{1}{2} (1 - \gamma_5) \psi \\ A &= -B \end{aligned} \quad (13)$$

or, alternatively,

$$\bar{F}_{\mu\nu} = -\hat{\bar{F}}_{\mu\nu} , \quad \psi = \psi_+ , \quad A = B . \quad (14)$$

If we take (13), for instance, the equations reduce to

$$\gamma_\mu D_\mu \psi_- = 0 \quad (15)$$

$$D_\mu^2 A + ig \psi_-^* \times \psi_- = 0 \quad (16)$$

$$\bar{F}_{\mu\nu} = \hat{\bar{F}}_{\mu\nu} \quad (17)$$

and the supersymmetry transformations to

$$\begin{aligned}\delta_+ A &= 0 \\ \delta_+ \psi_- &= \gamma_\mu D_\mu (2A \alpha_+) \\ \delta_+ V_\mu &= i \alpha_+^* \gamma_\mu \psi_- - i \psi_-^* \gamma_\mu \alpha_+\end{aligned}\tag{18}$$

and

$$\begin{aligned}\delta_- A &= i \alpha_-^* \psi_- - i \psi_-^* \alpha_- \\ \delta_- \psi_- &= -F_{\mu\nu} \Sigma_{\mu\nu} \alpha_- \\ \delta_- V_\mu &= 0\end{aligned}\tag{19}$$

The differential equations (15) to (17) are covariant under (18) and (19). While our results require in general that all spinors be treated as Grassmannian variables, some partial results do not require Fierz rearrangements and are valid also if one treats the spinors as c numbers. This applies, for instance, to the covariance of the equations (15) to (17) under the δ_- transformation (19). Therefore, if one starts from a solution for which $A = \psi_- = 0$, but the Yang-Mills field has non-vanishing winding number, (19) generates two independent solutions of the Dirac equation in that external field and two more are generated by the corresponding conformal supersymmetry transformation $(\alpha \rightarrow \gamma_\mu x^\mu \alpha)$. For δ_+ the situation is different and the covariance of the Dirac equation (15) is not true for c number spinors. However, the other two equations have zero δ_+ variation if ψ_- satisfies the Dirac equation^{*)}. Therefore, to each solution of the Dirac equation in a given many-instanton configuration, $\delta_+ V_\mu$ in (18) gives a fluctuation which satisfies the correct perturbation equations and which depends on α_+ (two complex components, four real parameters). In reality only two of these four

*) For (17), this is a consequence of the identity

$$\gamma_\nu D_\mu - \gamma_\mu D_\nu = \frac{1}{2} \varepsilon_{\nu\mu\lambda\rho} \gamma_5 (\gamma_\lambda D_\rho - \gamma_\rho D_\lambda) + \frac{1}{2} (\gamma_\nu \gamma_\mu - \gamma_\mu \gamma_\nu) \gamma_\lambda D_\lambda$$

solutions are independent, since the other two can be obtained by applying to ψ_- the automorphism δ_2 of (8) and therefore they correspond to another solution of the Dirac equation. We recover in this way the result of Ref. [15], which states that the solutions with winding number n of the $SU(2)$ Yang-Mills equation, are twice as many ($8n$) as those ($4n$) for a Dirac field belonging to the adjoint representation in a Yang-Mills field of that winding number [10-13] (more precisely $8n-3$, subtracting the three parameters of a global gauge rotation).

The finite transformations corresponding to (18) can be used to construct certain explicit solutions of the Yang-Mills equation (17). Since for $SU(2)$ particular solutions of (17) with winding number n are known and the general solution (depending on $4n$ parameters) of (15) in one of those n instanton fields has also been given [16], one can exponentiate (18) to generate the finite corrections to the Yang-Mills potential. In this way one obtains an n instanton configuration depending on $8n$ parameters, which is an exact solution of (17), but only in virtue of the anticommuting Grassmannian nature of the parameters. Its form is correspondingly simpler, since the exponentiation of (18) requires only a finite number of terms. The relation of this type of solution to the pure c number solutions and the role it may play in the quantization problem are as yet unclear.

Acknowledgements

Discussions with J. Ellis, D. Olive and W. Nahm are gratefully acknowledged.

REFERENCES

- [1] A.A. Belavin, A.M. Polyakov, A.S. Schwarz and Yu.S. Tyupkin, Phys. Letters 59B (1975) 85.
- [2] A.M. Polyakov, Nordita preprints (1976 and 1977).
- [3] G. 't Hooft, Phys. Rev. Letters 37 (1976) 8; Phys. Rev. D14 (1976) 3432.
- [4] R. Jackiw and C. Rebbi, Phys. Rev. Letters 37 (1976) 172; Phys. Rev. D14 (1976) 517; Phys. Letters 67B (1977) 189.
- [5] R. Jackiw, C. Nohl and C. Rebbi, Phys. Rev. D15 (1977) 1642.
- [6] E. Witten, Phys. Rev. Letters 38 (1977) 121.
- [7] E. Corrigan and D.B. Fairlie, Phys. Letters 67B (1977) 69.
- [8] R. Ore, Phys. Rev. D, in press.
- [9] B. Grossmann, Phys. Letters 61A (1977) 86.
- [10] A.S. Schwarz, Phys. Letters 67B (1977) 172; Proc. Symp. in Alushta, USSR, April 1976, p. 224.
- [11] J. Kiskis, Los Alamos preprint (1976).
- [12] N.K. Nielsen and B. Schroer, CERN preprint TH.2317 (1977).
- [13] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Oxford Mathematical Institute preprint (1977).
- [14] S. Chadha, A. D'Adda, P. Di Vecchia and F. Nicodemi, Phys. Letters 67B (1977) 103.
- [15] L.S. Brown, R.D. Carlitz and C. Lee, University of Washington, Seattle, preprint (1977).
- [16] R. Jackiw and C. Rebbi, MIT preprint (1977).
- [17] S. Ferrara and B. Zumino, Nuclear Phys. B79 (1974) 413.
- [18] A. Salam and J. Strathdee, Phys. Letters 51B (1974) 353.
- [19] B. de Wit and D.Z. Freedman, Phys. Rev. D12 (1975) 2286.
- [20] J. Schwinger, Phys. Rev. 115 (1959) 721.
- [21] J.D. Stack, University of Illinois preprint (1977).