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A SIMPLE REDUCTION PROCEDURE FOR THE THREE- AND N BODY
SCATTERING EQUATIONS

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ABSTRACT

A generalized form of the two-body Kowalski-Noyes method is shown to provide a both simple and powerful unitary reduction of the three- and N body scattering equations. Employing generalized half-off-shell functions that satisfy off-shell but real and non-singular integral equations, the reduction directly leads to on-shell integral equations for the scattering amplitudes. Physically, it is a simple example of how the scattering problem can be split into an internal and an external part.

It was first shown by Kowalski¹⁾ and Noyes¹⁾ that the off-shell two-body t matrix can be written as a sum of a separable term and a real remainder, and that these terms can be obtained from real, non-singular integral equations. Noyes also pointed out that this representation might be useful in the study (and solution) of the three-body Faddeev equations. This idea has subsequently been exploited by several authors²⁾, but a straightforward three- (and N) body generalization of the Kowalski-Noyes (KN) method itself has up to now not been carried out. It is, therefore, the aim of this note to show how this can be done, and to point out some useful features of such an approach.

First, let us briefly recall the two-body results. From the Lippmann-Schwinger equation for the half-off-shell t matrix (e.g. for s waves; $\tilde{k}^2 = k^2/2\mu$)

$$t(p, k; \tilde{k}^2) = v(p, k) - \int_0^\infty v(p, q) \frac{q^2 dq}{q^2 - \tilde{k}^2 - i0} t(q, k; \tilde{k}^2) \quad (1)$$

one subtracts the same equation, with $p = k$ and multiplied by $v(p, k)/v(k, k)$, to get

$$f(p, k) = \frac{v(p, k)}{v(k, k)} - \int_0^\infty \left[v(p, q) - \frac{v(p, k)v(k, q)}{v(k, k)} \right] \frac{q^2 dq}{q^2 - \tilde{k}^2 - i0} f(q, k) \quad (2)$$

in terms of the real half-off-shell function $f(p, k) = t(p, k; \tilde{k}^2)/t(k, k; \tilde{k}^2)$. In contrast to (1), this is a real, non-singular integral equation. Taking $p = k$ in (1) one furthermore finds that f is related to the on-shell t through

$$t(k, k; \tilde{k}^2) = v(k, k) / \left[1 + \int_0^\infty v(k, q) \frac{q^2 dq}{q^2 - \tilde{k}^2 - i0} f(q, k) \right] \quad (3)$$

Equations (2) and (3) are frequently used instead of (1) for practical evaluation of $t(k, k; \tilde{k}^2)$. For the special case of a separable potential, (2) reduces to $f(p, k) = v(p, k)/v(k, k)$, and the only non-trivial part of the two-body problem is the evaluation of the integral in (3).

It can also be shown that $t(k, p'; \tilde{k}^2) = t(k, k; \tilde{k}^2)f(p', k)$ with the same half-off-shell function as above, and that the completely off-shell t matrix has the representation

$$t(p, p'; k^2) = f(p, k) t(k, k; k^2) f(p', k) + r(p, p'; k^2) \quad (4)$$

In (4), r is a real function that satisfies the integral equation (2) with the driving term replaced by $v(p, p') - v(p, k)v(k, p')/v(k, k)$. The function r vanishes on- and half-off-shell, and is identically zero for a separable potential.

In the three-body case the analogue of Eq. (1) is the half-off-shell Faddeev equation [for simplicity of presentation we will only consider s waves $J = \ell = \lambda = 0$, where $\vec{J} = \vec{\ell} + \vec{\lambda}$, i.e., only s wave interactions. As initial state we choose three free particles in channel 1 with relative momenta q_1^0 (for a pair) and p_1^0 (for the third particle relative to the pair). In the half-off-shell scattering amplitude $T^\beta(q_\beta, p_\beta; q_1^0, p_1^0; E)$ the initial state is on-the-energy-shell $\tilde{q}_1^0{}^2 + \tilde{p}_1^0{}^2 = E$]. Suppressing initial state labels in T^β we have $(\bar{\delta}_{\beta\gamma} = 1 - \delta_{\beta\gamma})$

$$T^\beta(q_\beta, p_\beta) = \frac{1}{p_1^2} \delta(p_1 - p_1^0) t_1(q_1, q_1; \tilde{q}_1^0{}^2) \delta p_1 + \int_0^\infty t_\beta(q_\beta, q, \tilde{q}_\beta^2) \tilde{q}^2 d\tilde{q} D_{\beta 1}(q_\beta, p_\beta; q_1^0, p_1^0) \bar{\delta}_{\beta 1} \quad (5)$$

$$- \sum_{\lambda \neq \beta} \int_0^\infty t_\beta(q_\beta, q, \tilde{q}_\beta^2) \tilde{q}^2 d\tilde{q} D_{\beta \lambda}(q_\beta, p_\beta; q'_\lambda, p'_\lambda) \frac{p_\lambda^2 d p'_\lambda}{\tilde{q}_\lambda^2 + \tilde{p}_\lambda^2 - E - i0} T^\lambda(q'_\lambda, p'_\lambda)$$

where $\tilde{q}_\beta^2 = E - \tilde{p}_\beta^2$, and where $D_{\beta\gamma}$ is a standard channel recoupling coefficient. Note that $D_{\beta\gamma}(q_\beta, p_\beta; q'_\gamma, p'_\gamma)$ is diagonal in the over-all energy, i.e., proportional to $\delta(\tilde{q}_\beta^2 + \tilde{p}_\beta^2 - \tilde{q}'_\gamma{}^2 - \tilde{p}'_\gamma{}^2)$, so that the t matrices in both the driving terms are half-off-shell.

In analogy with the two-body case we introduce the representation

$$T^\beta(q_\beta, p_\beta) = \sum_\lambda \int_0^\infty F_{\beta\lambda}(q_\beta, p_\beta; p'_\lambda) p_\lambda^2 d p'_\lambda T^\lambda(q'_\lambda, p'_\lambda) \quad (6)$$

for a half-off-shell scattering amplitude component in terms of the on-shell amplitude components (κ is always understood to be related to p through the three-body on-shell condition $\tilde{\kappa}^2 = E - \tilde{p}^2$). Taking $q_\beta = \kappa_\beta$ in (6), multiplying by $f_\beta(q_\beta, \kappa_\beta)$ and subtracting the result from (6) we obtain an equation for the half-off-shell function $F_{\beta\lambda}$ which is analogous to (2):

$$F_{\beta\lambda}(q_\beta, p_\beta; p'_\lambda) = \delta_{\beta\lambda} \frac{1}{p_\beta^2} \delta(p_\beta - p'_\lambda) f_\beta(q_\beta, \kappa_\beta) - \sum_{\gamma \neq \beta} \int_0^\infty r_\gamma(q_\gamma, q; \tilde{\kappa}_\gamma^2) q^2 dq D_{\beta\gamma}(q, p_\beta; q''_\gamma, p'_\lambda) \frac{q_\gamma^2 dq_\gamma'' p_\gamma^2 dp_\gamma''}{q_\gamma^2 + \tilde{p}_\gamma^2 - E - i0} F_{\gamma\lambda}(q''_\gamma, p''_\gamma; p'_\lambda) \quad (7)$$

The δ function in $D_{\beta\gamma}$ allows us to write the propagator in (7) as

$$(\tilde{p}_\beta^2 + \tilde{q}^2 - E - i0)^{-1} = (\tilde{q}^2 - \tilde{\kappa}_\beta^2 - i0)^{-1}.$$

Recalling that r_β in (7) vanishes when $q = \kappa_\beta$ (half-on-shell), we conclude that Eq. (7) is a real, non-singular, two-variable integral equation for $F_{\beta\lambda}$. In particular, $F_{\beta\lambda}$ is itself a real function.

In order to obtain the on-shell T^β from $F_{\beta\gamma}$, i.e., the analogue of Eq. (3), we simply take $q_\beta = \kappa_\beta$ in (5) and use (6) to get

$$T^\beta(\kappa_\beta, p_\beta) = T^\beta_0(\kappa_\beta, p_\beta) - \sum_\lambda \int_0^\infty [K_{\beta\lambda}(p_\beta, p'_\lambda) + R_{\beta\lambda}(p_\beta, p'_\lambda)] p_\lambda^2 dp'_\lambda T^\lambda(\kappa'_\lambda, p'_\lambda) \quad (8)$$

where we have split the kernel into an on-shell piece $K_{\beta\lambda}$ [corresponding to the on-shell part of $F_{\beta\lambda}$, $\delta_{\beta\lambda}(1/p_\beta^2)\delta(p_\beta - p'_\lambda)$], and an off-shell piece $R_{\beta\lambda}$. The three-body analogue of (3) is therefore not an algebraic expression but a one-variable integral equation for the on-shell scattering amplitude in terms of the half-off-shell function $F_{\beta\lambda}$. In the particularly simple case of a separable two-body interaction, the function r in (4) and (7) is absent, and $F_{\beta\lambda}$ is trivially given by the driving term in (7). Therefore, the solution of (8) is in this case the only non-trivial part of the three-body problem.

This observation can be given an interesting interpretation: the three-body scattering amplitudes that are obtained using the separable approximation are essentially unitarized [via Eq. (8)] Born approximations to Eq. (7)^{*}). In such a perspective the well-known, but not equally well understood, success of the separable approximation is less astonishing.

One can now proceed and derive a representation analogous to (4) for the fully-off-shell three-body scattering amplitude. This should not be done directly for T^β but for the more symmetric Faddeev components $M_{\beta\alpha}$ [which satisfy the equation $M_{\beta\alpha} = t_\beta \delta_{\beta\alpha} - t_\beta G_0 \sum_{\gamma \neq \beta} M_{\gamma\alpha}$ and the relation $T^\beta = \sum_\alpha M_{\beta\alpha}$]. Generalizing the two-body manipulations that lead to (4) one finds

$$\begin{aligned}
 M_{\beta\alpha}(q_\beta, p_\beta; q'_\alpha, p'_\alpha; E) = & \sum_{\gamma, \lambda} \int_0^\infty F_{\beta\gamma}(p_\beta, q_\beta; p''_\gamma) p''_\gamma{}^2 dp''_\gamma \times \\
 & \times M_{\gamma\lambda}(k''_\gamma, p''_\gamma; k'''_\lambda, p'''_\lambda; E) p'''_\lambda{}^2 dp'''_\lambda F_{\lambda\alpha}^+(p'''_\lambda; q'_\alpha, p'_\alpha) + \\
 & + M_{\beta\alpha}^r(q_\beta, p_\beta; q'_\alpha, p'_\alpha; E)
 \end{aligned} \tag{9}$$

where $F_{\beta\gamma}$ is the real half-off-shell function, $F_{\lambda\alpha}^+$ stands for its right-hand side counterpart, and where $M_{\gamma\lambda}(k''_\gamma, p''_\gamma; k'''_\lambda, p'''_\lambda; E)$ is on-shell. The remainder function $M_{\beta\alpha}^r$ satisfies an equation similar to (7) but with the driving term replaced by

$$\delta_{\beta\lambda} t_\beta(q_\beta, q'_\beta; k_\beta^2) \frac{1}{f_\beta^2} \delta(p_\beta - p'_\beta)$$

It is, therefore, a real function that vanishes on- and half-off-shell, and (9) is indeed a sum of a "separable" part and a real remainder.

³⁾*) This point of view is similar to that of Amado³⁾ who argues that the three-body equations in the separable approximation mainly serve to provide the scattering amplitudes with appropriate analyticity properties.

It should now be clear for anyone familiar with the four- and N body Faddeev-Yakubovski formalism^{4,5)} that the above results directly generalize to the N body case if only the N body equations are written down for properly chosen operators. In view of the large number of operators available, it is a slightly non-trivial result that the appropriate choice is the Yakubovski generalization of the three-body $M_{\beta\alpha}$ operator [in the four-body case, this is the $Y_{\beta\alpha}^{\sigma\tau}$ of Ref. 5); it satisfies the equations

$$\begin{aligned} Y_{\beta\alpha}^{\sigma\tau} &= \delta^{\sigma\tau} M_{\beta\alpha}^{\sigma} - \sum_{s,\gamma} K_{\beta\gamma}^{\sigma} G_0 \delta^{\sigma s} Y_{\gamma\alpha}^{s\tau} \\ &= \delta^{\sigma\tau} M_{\beta\alpha}^{\sigma} - \sum_{s,\gamma} Y_{\beta\gamma}^{\sigma s} \delta^{\sigma\tau} G_0 \tilde{K}_{\gamma\alpha}^{\tau} \end{aligned}$$

For this choice the N body versions of (5)-(9) can more or less be written down by inspection.

It should be mentioned that there are some anomalies connected with the simplest version of the two-body KN method⁶⁾ [e.g., divergence of $f(p,k)$ when $t(k,k;\tilde{k}^2) = 0$]. Such anomalies might need some attention also in the more general case considered here, but they are not expected to limit the applicability of the method more than they have done in the two-body case.

The reduction of the Faddeev equations worked out here is a special (and particularly simple) case of a more general reduction scheme⁷⁾ based on an analysis of the physically intuitive multiple scattering structure of the Faddeev equations. This fact provides some insight into the physical content of the equations obtained here [e.g., they represent a separation of the scattering problem into an internal and an external part, see Ref. 7)] and is taken as an excuse for leaving out such aspects of the reduction in this note. On the other hand, the present work extends the analysis of Ref. 7) by providing an explicit example of how the function $R_{\beta\lambda}$ in Eq. (8) is constructed. This point was not discussed in detail in the more general context of Ref. 7).

The splitting of the three-body problem into two parts is characteristic of the reduction procedure for the Faddeev equations developed in this note: the solution of a real, non-singular, two-variable integral equation [Eq. (7)], and the subsequent solution of an on-shell, one-variable integral equation [Eq. (8)]. The

last equation is similar in structure to the three-body equations in the separable approximation and mainly serves to unitarize the solution to the first equation. In particular, Eq. (8) can be combined with any approximate (but real) solution to the more complicated Eq. (7) to generate unitary scattering amplitudes [see Ref. 7)].

Let us conclude with the following remark. The natural separation of the unitarization into a comparatively simple equation is clearly a very powerful feature of the reduction presented here, but on the other hand, any reasonable unitary reduction scheme ought to have this feature. It might, therefore, be considered an equally important feature that the equation that here carries the details of the dynamics, i.e., Eq. (7), is completely free from propagator singularities, and therefore has a much simpler structure than the original Faddeev equations (it might be recalled that in the three-body K matrix formalism⁸⁾ which is a competing unitary reduction scheme, the corresponding equation contains awkward principle part type singularities). In fact, it is this feature that has made the original Kowalski-Noyes method a standard tool for solving the two-body scattering equations.

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