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AN INEQUALITY ON THE SLOPE OF THE DIFFRACTION PEAK
IN ELASTIC COLLISIONS OF PARTICLES OF ARBITRARY SPINS

G. Auberson, A. Martin and G. Mennessier

CERN -- Geneva

A B S T R A C T

We obtain inequalities linking the slope of the diffraction peak, the total cross-section and the elastic cross-section of the form

$$\left. \frac{d}{dt} \log \frac{d\sigma^A}{d\Omega} \right|_{t=0} > R \frac{1}{18\pi} \frac{(\sigma_{\text{tot}})^2}{\sigma_{\text{el}}},$$

for elastic collisions of particles of arbitrary spins ($d\sigma^A/d\Omega$ is the "absorptive" differential cross-section).

In the limit of infinite energy we get $R > 0.9965$, to be compared to $R=1$ for the scalar case. Results for finite energies and finite slopes are also obtained, such as $R > 0.989$ as soon as the logarithmic derivative with respect to $\cos\theta$ of the absorptive differential cross-section is greater than 33.

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1. - INTRODUCTION

Several years ago, an inequality on the slope of the diffraction peak in elastic collisions of spinless particles was obtained ¹⁾

$$\left. \frac{d}{dt} \log A(s, t) \right|_{t=0} \geq \frac{1}{36\pi} \left(\frac{\sigma_{\text{tot}}^2}{\sigma_{\text{el}}} - \frac{4\pi}{k^2} \right) \quad (1)$$

$A(s, t)$ designates the absorptive part of the scattering amplitude, s the square of the centre-of-mass energy, t the squared four-momentum transfer, k the centre-of-mass momentum.

To compare this inequality with experiment, we have to neglect real part effects and spin effects, since there is no direct measurement of spin 0 - spin 0 scattering ($\pi\pi$ is obtained by extrapolation). When real parts and spins are neglected, one finds that (1) is not far from being saturated by experiment over a very large range of energies, from 5 to 2000 GeV lab. energy. To be on the safe side, let us say that the ratio L.H.S./R.H.S. does not exceed 1.2.

Concerning real part effects, we can mention the rigorous result that, if the odd signature amplitudes are negligible at high energies, and if total cross-sections decrease less fast than $s^{-\frac{1}{2}}$, the widths of the diffraction peaks defined by A and by the modulus $|F|$ of the amplitude coincide in the high energy limit ²⁾, so that asymptotically it is legitimate to neglect real parts.

As regards spin effects, the only case completely treated is that of pion-nucleon scattering, i.e., spin 0 - spin $\frac{1}{2}$ with parity conservation, where one gets essentially the same inequality as in the scalar case ^{3), 4)}, if one replaces A^2 by the absorptive unpolarized differential cross-section (see below). To generalize these results to arbitrary spins, one has to generalize in some way the positivity properties. Indeed, inequality (1) rests essentially on the positivity of the partial wave expansion coefficients of A .

2. - POSITIVITY FOR ARBITRARY SPINS ; THE NOTION OF ABSORPTIVE CROSS-SECTION

In the general spin case, the elastic differential cross-section can be expressed in terms of a set of amplitudes F_i

$$\frac{d\sigma}{d\Omega} = N(s) \sum_i |F_i|^2 \quad (2)$$

The set of F_i 's is of course not unique. Acceptable sets (so that there are no cross terms) are, for instance, helicity or transversity amplitudes. Each amplitude F_i has an absorptive part, which, by a convenient choice of phases can be taken to be $\text{Im } F_i$. The absorptive differential cross-section is defined to be :

$$\frac{d\sigma^A}{d\Omega} = N(s) \sum_i (\text{Im } F_i)^2 \quad (3)$$

The absorptive differential cross-section is therefore obtained by neglecting all real part contributions in the differential cross-section. What matters is that this is an invariant notion, independent of the choice of the basis for the amplitudes.

The absorptive differential cross-section possesses a positivity property which generalizes the positivity properties of the scalar amplitude and which was obtained in two successive contributions by Cornille and Martin ⁵⁾ and Mahoux ⁶⁾, and which in the final form of Mahoux is :

$$\frac{d\sigma^A}{d\Omega} = \sum_{n=0}^{\infty} (2n+1) c_n(s) P_n(\cos\theta) \quad , \quad c_n(s) \geq 0 \quad (4)$$

$P_n(\cos\theta)$ is a Legendre polynomial whose argument is $\cos\theta = 1 + t/(2k^2) = \cos\theta$ of the centre-of-mass scattering angle. The expansion in (4) converges inside an ellipse. That was expected. What is non-trivial is the positivity of the c_n 's.

It is clear that this property holds in the scalar case for

$$A(s,t) = \bar{A}(\cos\theta, s) = \sum_{\ell} (2\ell+1) \text{Im } f_{\ell}(s) P_{\ell}(\cos\theta) \quad , \quad \text{Im } f_{\ell}(s) \geq 0 \quad (5)$$

Then one notices that the square of an expansion with positive coefficients is also an expansion with positive coefficients.

To get an inequality of the type (1), we have to relate the absorptive differential cross-section to the elastic and total cross-sections. First of all, it is trivial to obtain

$$\sigma_{el} \geq \int d\Omega \frac{d\sigma^A}{d\Omega} \quad (6)$$

The total cross-section is linked to the forward absorptive cross-section by the following chain of arguments :

- in the helicity basis, for instance, we have

$$\frac{d\sigma^A}{d\Omega}(\cos\theta=1, s) = M(s) \sum_{\lambda_1, \lambda_2} |\langle \lambda_1, \lambda_2 | \text{Im} T | \lambda_1, \lambda_2 \rangle|^2 \quad (7)$$

where the only non-vanishing terms are such that $\lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$; if we retain only the truly diagonal terms

$$\frac{d\sigma^A}{d\Omega}(\cos\theta=1, s) \geq M(s) \sum_{\lambda_1, \lambda_2} |\langle \lambda_1, \lambda_2 | \text{Im} T | \lambda_1, \lambda_2 \rangle|^2 \quad (8)$$

- for each separate term the optical theorem applies and we get

$$\frac{d\sigma^A}{d\Omega}(\cos\theta=1, s) \geq \frac{k^2}{(4\pi)^2} \frac{\sum_{\lambda_1, \lambda_2} (\sigma_{\lambda_1, \lambda_2}^{\text{tot}})^2}{(2s_1+1)(2s_2+1)} \quad (9)$$

where $\sigma_{\lambda_1, \lambda_2}^{\text{tot}}$ represents the total cross for initial helicities λ_1, λ_2 (s_1 and s_2 are the spins of the particles) ;

- applying now the Schwarz inequality to the right-hand side of (9), we get

$$\frac{d\sigma^A}{d\Omega}(\cos\theta=1, s) \geq \frac{k^2}{(4\pi)^2} \sigma_{\text{tot}}^2 \quad (10)$$

where σ_{tot} is the true unpolarized cross-section, i.e., the cross-section defined by the attenuation of an unpolarized beam through an unpolarized target.

Notice that to make (10) saturated, the non-diagonal absorptive amplitudes such that $\lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$ but $\lambda_1 \neq \lambda_1'$ have to be negligible (such an amplitude is the double flip in p-p scattering). Also the Schwarz inequality must be saturated, which means that the various polarized total cross-sections $\sigma_{\lambda_1 \lambda_2}^{\text{tot}}$ should be equal to one another.

3. - MATHEMATICAL FORMULATION OF THE PROBLEM AND ITS SOLUTION

For simplicity, we shall write

$$\frac{d\sigma^A}{d\Omega}(\cos\theta, s) = Q(x) \quad , \quad x = \cos\theta \quad (11)$$

We shall try to find inequalities which relate

$$Q(x)^{-1} \left. \frac{d}{dx} Q(x) \right|_{x=1}, \frac{1}{2} \int_{-1}^{+1} Q(x) dx \quad \text{and} \quad Q(1) \quad (12)$$

By using (6) and (10) these inequalities will indeed relate the slope of the diffraction peak, the elastic and the total cross-sections.

We try to solve the following problem :

- knowing that

$$Q(x) \geq 0 \quad \text{for} \quad -1 \leq x \leq 1 \quad (13)$$

(positivity of the differential cross-section)

$$Q(x) = \sum_n (2n+1) c_n P_n(x) \quad , \quad c_n \geq 0 \quad (14)$$

(positivity of the expansion coefficients)

- we want to find, for given p,

$$R(p) = \inf \left[\frac{9}{4} \frac{Q'(1)}{Q(1)} \frac{\frac{1}{2} \int_{-1}^{+1} Q(x) dx}{Q(1)} \right] \quad (15)$$

where

$$p = \alpha'(1) / \alpha(1) \quad (16)$$

A lower limit for $R(p)$ has already been obtained by Cornille and Martin ⁵⁾

$$R(p) \geq 9/16 \quad \text{for} \quad p \geq 1/2 \quad (17)$$

To try to solve the problem, it is useful to return to the scalar case ¹⁾ where $\alpha(x) = A(x)^2$ and $A(x)$ has itself positive expansion coefficients. Then the solution is known. In particular, for

$$p = N(N+2)/3, \quad N \text{ integer} \quad (18)$$

the minimizing A is

$$A_m(x) = (1+x) [P_N''(x) + P_{N+1}''(x)] \quad (19)$$

and

$$R = N(N+2)/(N+1)^2 = 1 - 1/(N+1)^2 \quad (20)$$

For $p \rightarrow \infty$, which corresponds to infinite energy, $R \rightarrow 1$ and this corresponds to neglecting $4\pi/k^2$ in (1).

To treat the non-scalar case, we build a functional

$$q_N^1(f) \equiv \frac{1}{2} \int_{-1}^1 f(x) dx - \lambda_N f(1) + \mu_N f'(1) - \nu_N f(-1) - \sum_{i=1}^{N-1} w_{iN} f(x_{iN}) \quad (21)$$

To fix the $2N+1$ weights w_{in} and abscissas x_{in} 's, we demand, in analogy to the Gauss integration method that :

$$q_N^1(f) = 0 \quad \text{for any } f = \text{polynomial of degree } \leq 2N \quad (22)$$

The solution is such that the x_{in} 's are the zeros of A_m (19), all located between -1 and +1. The weights w_{in} are positive and, explicitly,

$$\lambda_N = \frac{8}{3(N+1)^2}, \quad \mu_N = \frac{4}{N(N+2)(N+1)^2}, \quad \nu_N = \frac{1}{N(N+2)}, \quad \lambda_N + \nu_N + \sum_{i=1}^{N-1} w_{iN} = 1 \quad (23)$$

Now suppose that for a given N , we have

$$q_N^1(p_e) \geq 0 \quad \forall p \quad (24)$$

Then $q_N^1(Q) \geq 0$ and since $Q(x) \geq 0$ for $-1 \leq x \leq +1$, we get

$$\frac{1}{2} \int_{-1}^{+1} Q(x) dx \geq \lambda_N Q(1) - \mu_N Q'(1) \quad (25)$$

so that

$$R(p) \geq \frac{9}{4} p (\lambda_N - \mu_N p) \quad (26)$$

Maximizing with respect to p

$$R(p) = N(N+2)/(N+1)^2 \quad \text{for} \quad p = N(N+2)/3 \quad (27)$$

This is exactly the scalar answer. It can be shown that it is sufficient to test condition (24) for

$$2N < p < N(N+2)/\sqrt{2}$$

One finds numerically that condition (24) is satisfied for $N \leq 9$. So, in particular,

$$R = .99 \quad \text{for} \quad p = 33. \quad (28)$$

Then the question is : what do we do for larger values of the slope ? (for lower values, we have the discrete points $N=1,2,\dots,8$ and an adaptation for intermediate values can be made, but this is not physically very interesting).

We can try to replace the interval $[-1,+1]$ by $[1-2\alpha,1]$, in the definition of q_N , and we get by rescaling

$$q_N^\alpha(f) \equiv \frac{1}{2\alpha} \int_{1-2\alpha}^1 f(x) dx - \lambda_N f(1) + \alpha \mu_N f'(1) - \nu_N f(1-2\alpha) - \sum_{i=1}^{N-1} w_{i,N} f(1-\alpha x_{i,N}) \quad (29)$$

Assuming $q_N^\alpha(P_\ell) \geq 0 \quad \forall \ell$, and using the fact that the elastic cross-section is certainly greater than the contribution from angles $1-2\alpha \leq \cos\theta \leq 1$, one gets an inequality for $R(p)$ of the type (26), with p replaced by αp , and the value of $R(p)$ given by (27) but for $p = N(N+2)/(3\alpha)$.

Naturally, by construction $q_N^\alpha(f)$ vanishes if f is a polynomial of degree $\leq 2N$. But the positivity tests $q_N^\alpha(P_\ell) \geq 0$ do not coincide with the previous ones. Again we have only to carry out a finite number of tests, $2N < \ell < (N(N+2))/\sqrt{2} \times \sqrt{2-\alpha}/\sqrt{\alpha}$. This range of ℓ can be further narrowed using the expression of the error of the generalized Gauss integration formula

$$q_N^\alpha(f) = c_N^\alpha f^{(2N+1)}(\xi) \quad , \quad 1-2\alpha \leq \xi \leq 1 \quad (30)$$

where c_N^α is known to be positive. By combining Eq. (30) with some inequalities on the derivatives of the Legendre polynomials ⁷⁾ which guarantee that $P_\ell^{(2N+1)}(z)$ is positive in the interval $(1-2\alpha,1)$, one can reduce the range in which positivity tests are necessary to ℓ such that

$$j_{2N+1,1} < \eta = (2\ell+1)\sqrt{\alpha} < \sqrt{2} (N+1)^2 \sqrt{2-\alpha} \quad (31)$$

($j_{2N+1,1}$ is the first zero of the Bessel function of order $2N+1$). Inequality (31) is important because, in the limit $\alpha \rightarrow 0$, it shows that though the range in ℓ goes to infinity, the range in η is finite and the asymptotic representation of Legendre polynomials by a Bessel function can be used. The conclusion of the numerical study is the following :

$$q_N^2(P_2) \geq 0 \quad \forall p \quad \text{for } N=9 \quad \text{and} \quad 0 < \alpha \leq .42 \quad (32)$$

This means that

$$R(p) \geq .99 \quad \text{for } p \geq .33 / .42 \approx 78.6 \quad (33)$$

However, some further progress can be made by considering positive linear combinations

$$q(p) = \sum_j \beta_j q_{N_j}^{\alpha_j}(p)$$

for which positivity tests $q(P_2) \geq 0$ may work, and using the corresponding generalized inequalities of the type (26). The present conclusions, obtained with two terms $N_1 = N_2$, are the following :

$$- R \geq .989 \quad \text{for } 33. \leq p \leq 38.7 \quad (34)$$

$$- R \geq .99 \quad \text{for } p \geq 38.7 \quad (35)$$

$$- R \geq .9965 \quad \text{for } p \rightarrow \infty \quad (36)$$

(obtained using $N=19$, α_1 and $\alpha_2 \rightarrow 0$ with $\alpha_1/\alpha_2 = 0.94$).

4. - DISCUSSION

From a practical point of view, inequalities (34), (35) and (36) are sufficient. Indeed, from (34), we get, for nucleon-nucleon kinematics :

$$\left\{ \begin{array}{l} \frac{d}{dt} \log \frac{d\sigma^A}{d\Omega} \Big|_{t=0} \geq .989 \frac{1}{18\pi} \frac{\sigma_{tot}^2}{\sigma_{el}} \\ \text{for } E_{kinetic, lab} \geq 56 \text{ GeV} \quad \text{and} \quad \frac{d}{dt} \log \frac{d\sigma^A}{d\Omega} \Big|_{t=0} \geq 7 \text{ GeV}^{-2} \end{array} \right. \quad (37)$$

The discrepancy of less than 1.1% with the scalar case is negligible with respect to the experimental uncertainty on σ_{el} .

The fact that inequalities of the type (37) are not far from being saturated is indicative that spin effects are relatively small, i.e., that the double flip terms are not very large, and that the spin dependence of the total cross-section is weak. However, this is not a very sensitive test of spin effects. The thing on which the saturation gives more information is the partial wave distribution. Disregarding completely spin effects, one can see that, for instance, the distribution produced by a black disk or a disk with fixed opacity (i.e., $\text{Im } f_\ell = \text{const.}$ for $\ell \leq \ell_{\text{max}}$) gives a value of $R = 2.25$, completely incompatible with the data at presently accessible energies.

On the theoretical side, there remains the slightly irritating problem that the best possible answer to the problem is not known. Why do we not get $R \rightarrow 1$ for $p \rightarrow \infty$?

Here one can take two attitudes. It is possible that by taking into account the whole positivity content of the S matrix, one could get back to the scalar answer exactly, or at least asymptotically. This is what happens for pion-nucleon scattering.

On the other hand, if we use only the limited information contained in Eq. (4), we have been able to prove by building an explicit counter example that $R(\infty) < 0.99987$. The counter example is built as follows. We write $\mathcal{Q} = A^2$ and represent both A and $\mathcal{Q}$ as infinite "partial wave" series :

$$\begin{aligned} A(\cos \theta) &= \int_0^\infty d\ell \, \ell \, a(\ell) J_0(\ell \theta) \\ \mathcal{Q}(\cos \theta) &= \int_0^\infty dn \, n \, c(n) J_0(n \theta) \end{aligned} \quad (38)$$

The coefficients $a(\ell)$ and $c(n)$ are related by a "Bessel convolution" :

$$c(n) = \frac{2}{\pi} \iint_{[\ell] \geq 0} d\ell \, d\ell' \, \ell \, \ell' \, a(\ell) a(\ell') \left[(n^2 - (\ell - \ell')^2) ((\ell + \ell')^2 - n^2) \right]^{-1/2} \quad (39)$$

One then starts from the saturating partial wave distribution of the scalar case, $a_0(l) = \theta(L-l) [1 - (l/L)^2]$, which gives $R=1$, and try to perturb it by adding a small part $\Delta a(l)$, in such a way that R decreases and the condition $c(n) \geq 0$ be maintained. This can be shown to be achievable when $\Delta a(l)$ is sufficiently small and has a suitable negative bump above $l=L$ followed by a positive tail. We choose :

$$\Delta a(l) = \begin{cases} \mu [1 - (l/L)^2] + \varphi (1 - l/L) & , l < L \\ -\beta & , L < l < L_1 \\ \gamma (L/l)^m & , l > L_1 \end{cases} \quad (40)$$

Requiring that σ_{tot} and p undergo no change when $a_0(l) \rightarrow a_0(l) + \Delta a(l)$ allows us to express μ and φ as linear combinations of β and γ . Then $\Delta \sigma_{el}$ (and thus ΔR), as well as $\Delta c(n)$, are second order polynomials in β, γ . One finds that the region of the (β, γ) plane defined by :

$$\begin{cases} \Delta R < 0 \\ c_0(n) + \Delta c(n) \geq 0 \quad \forall n \geq 0 \end{cases} \quad (41)$$

is non-empty. A crude optimization with respect to the free parameters gives $m=12$, $L_1=1.07L$, $\beta=0.011$, $\gamma=0.0016$ and $\Delta R=-0.00013$.

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REFERENCES

- 1) S.W. McDowell and A. Martin - Phys.Rev. 135B, 960 (1964).
- 2) H. Cornille and A. Martin - Nuclear Phys. B101, 411 (1975).
- 3) K.H. Mütter - Nuclear Phys. B31, 589 (1971).
- 4) G. Mennessier, S.M. Roy and V. Singh - to be published.
- 5) H. Cornille and A. Martin - Nuclear Phys. B115, 163 (1976).
- 6) G. Mahoux - Phys.Letters 65B, 139 (1976).
- 7) A. Martin - CERN Preprint TH. 2199 (1976), to appear in
SIAM Journal on Mathematical Analysis.