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THE CRITICAL EXPONENTS OF REGGEON FIELD THEORY

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A B S T R A C T

By using the method of Lipatov to determine the behaviour of perturbation theory at large order, we derive a convergent numerical procedure for the calculation of the critical exponents of Reggeon field theory, based on the loop expansion and the ϵ expansion. We find the results $\gamma = -0.26 \pm 0.02$, $z = 1.13 \pm 0.01$, and $\lambda = 0.49 \pm 0.01$.

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Recently, Lipatov ¹⁾ has developed a simple method for determining the behaviour of perturbation theory at large order in quantum field theory. The technique has been extended and applied to the calculation of critical exponents by Brézin et al. ²⁾.

The critical exponents of Reggeon field theory ³⁾ determine the asymptotic behaviour of elastic scattering when $\alpha(0)=1$ according to the following laws :

$$\sigma_{\text{tot}} \propto (\ln s)^{-\gamma} [1 + O((\ln s)^{-\lambda})] \quad (1)$$

$$\sigma_{\text{el}} \propto (\ln s)^{-2\gamma-z} [1 + O((\ln s)^{-\lambda})] \quad (2)$$

Previous calculations of these exponents, based on the loop expansion, the ϵ expansion, or other methods ⁴⁾, have been unreliable because of their lack of convergence or of any estimate of the error involved. We shall show that the addition of the knowledge of the large order behaviour of the perturbation expansion enables one to convert the loop expansion and the ϵ expansion into convergent procedures. The current knowledge of the loop expansion ^{5),6)} (out to three loops), and the ϵ expansion ^{7),8)} [out to $O(\epsilon^2)$], then gives the exponents to an accuracy of better than 10%.

The Green functions of Reggeon field theory are defined by the generating functional

$$Z[J, \bar{J}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(-A[\psi, \bar{\psi}, J, \bar{J}]) \quad (3)$$

where $A[\psi, \bar{\psi}, J, \bar{J}] =$

$$\int d^4y d^D b \left\{ \frac{1}{2} \bar{\psi} \overleftrightarrow{\partial}_y \psi + \alpha_0 \nabla \bar{\psi} \cdot \nabla \psi + \Delta \bar{\psi} \psi \right. \\ \left. + \frac{1}{2} i \gamma_0 \bar{\psi} \psi (\bar{\psi} + \psi) + J \psi + \bar{J} \bar{\psi} + \text{counterterms} \right\} \quad (4)$$

In the loop expansion ⁵⁾, the renormalization group functions are defined in the theory with $\Delta > 0$. All quantities of interest will be expanded in the dimensionless coupling $g_0 = r_0 / (\alpha'_0 \Delta)^{\frac{1}{2}}$ in $D = 2$. Thus we set $\alpha'_0 = \Delta = 1$ and $r_0 = g_0$ for convenience. The method of Lipatov ¹⁾ consists of using the Cauchy formula for the coefficient of g_0^n in the functional integral, and estimating this using the method of steepest descent as $n \rightarrow \infty$. The calculation for Reggeon field theory follows closely that of Brézin et al. ²⁾ for ϕ^{2N} theory, and will be given in detail elsewhere. The results are as follows ; for a Green function $G^{(r)}$ with r external lines,

$$G^{(r)} = \begin{cases} \dots + \sum_{n \text{ large}} C^{(r)} \Gamma(n + \frac{r}{2} + \frac{p}{2}) (-g_0^2 / \bar{A})^n & (r \text{ even}) \\ \dots + g_0 \sum_{n \text{ large}} C^{(r)} \Gamma(n + \frac{r}{2} + \frac{p}{2} + \frac{1}{2}) (-g_0^2 / \bar{A})^n & (r \text{ odd}) \end{cases} \quad (5)$$

where $C^{(r)}$ depends on the external momenta, etc., but not on n . $\bar{A}$ is the action of a solution of the equations of motion with an unphysical value of the coupling constant (in this case, a real triple-Pomeron coupling)

$$\bar{A}[\phi, \bar{\phi}] = \int dy d^p b \left[\frac{1}{2} \bar{\phi} \Delta_y \phi + \nabla \bar{\phi} \cdot \nabla \phi + \bar{\phi} \phi - \frac{1}{2} \bar{\phi} \phi (\bar{\phi} + \phi) \right] \quad (6)$$

The integer p is the number of arbitrary constants appearing in such a solution. In the physical case $D = 2$, we have $p = 3$, corresponding to translational invariance in rapidity and impact parameter. For this case, we have been unable to find a solution of finite action analytically, nor prove that it must exist. However, it is intuitively clear that such solutions should exist, and, since only the action is required, one can use variational techniques to estimate it. The error should be small even if the trial functions are rather crude. We use trial functions of the form

$$\begin{aligned} \phi(y, \vec{b}) &= f(y) u(\vec{b}) \\ \bar{\phi}(y, \vec{b}) &= \bar{f}(y) u(\vec{b}) \end{aligned} \quad (7)$$

The functional $\bar{A}[\phi, \bar{\phi}]$ then reduces to the form

$$\bar{A}[\phi, \bar{\phi}] = (\int u^2 d^D b) \int dy \left\{ \frac{1}{2} \bar{f} \tilde{\Delta}_y f + \tilde{\Delta}_0 \bar{f} f + \frac{1}{2} i \tilde{r}_0 \bar{f} f (\bar{f} + f) \right\} \quad (8)$$

where

$$\tilde{\Delta}_0 = \int [(\nabla u)^2 + u^2] d^D b / \int u^2 d^D b \quad (9)$$

$$-i \tilde{r}_0 = \int u^3 d^D b / \int u^2 d^D b \quad (10)$$

Apart from the factor $\int u^2 d^D b$, Eq. (8) has the form of the action for the $D=0$ problem, for which the solutions of the classical equations have been classified⁹⁾. The only solutions with finite action at arbitrarily high rapidity have the action $-2\tilde{\Delta}_0^2/r_0^2$ ($\tilde{\Delta}_0 > 0$). The problem is thus reduced to finding the lowest extremum of the functional

$$\bar{A}[u] = 2 \left\{ \frac{\int [(\nabla u)^2 + u^2] d^D b}{\int u^3 d^D b} \right\}^2 (\int u^2 d^D b) \quad (11)$$

Variational estimates with combinations of rational and exponential functions yield $\bar{A}/4\pi \approx 4.47$ in $D=2$.

The renormalization group functions $\beta(g)$, $\gamma(g)$, $z(g)$ are defined in terms of derivatives with respect to Δ of the two and three-point functions evaluated at zero external energy and momentum. This differentiation introduces an extra factor of n . One must then substitute the renormalized coupling g for the bare coupling g_0 . This does not change the large n dependence. Finally, we have, in $D=2$

$$\beta(g) = -\frac{1}{2}g + \dots + g \sum_{n \text{ large}} C \Gamma(n + \frac{1}{2}) (-g^2/\bar{A})^n \quad (12)$$

$$\gamma(g) = -\frac{1}{8}(g^2/4\pi) + \dots + \sum_{n \text{ large}} C' \Gamma(n + \frac{1}{2}) (-g^2/\bar{A})^n \quad (13)$$

$z(g)$ has a similar character to $\gamma(g)$, while $\lambda(g) = \beta'(g)$ gains an extra factor of n on $\beta(g)$.

These results imply the divergence of the perturbation expansions, but indicate that they are Borel summable. The Borel transform of $\beta(g)$, defined by dividing the coefficient of g^{2n+1} by $n!$, has radius of convergence $\bar{A}$, with a leading singularity of the form $[t + \bar{A}]^{-4.5}$. If instead we define a modified Borel transform by

$$\beta(g) = g \sum_{n=0}^{\infty} \beta_n g^{2n} \quad (14)$$

$$\tilde{\beta}(t) = \sum_{n=0}^{\infty} \frac{\beta_n}{\Gamma(n + 3/2)} g^{2n} \quad (15)$$

then $\tilde{\beta}(t)$ will have only a weak singularity at $t = -\bar{A}$. $\beta(g)$ can be recovered from $\tilde{\beta}(t)$ by

$$\beta(g) = g \int_0^{\infty} e^{-t} t^{1/2} \tilde{\beta}(tg^2) dt \quad (16)$$

We note that (16) involves a continuation of $\tilde{\beta}(t)$ outside the circle of convergence, where the perturbation expansion in t diverges. This difficulty can be overcome by mapping the t plane, cut from $-\infty$ to $-\bar{A}$, into the unit circle $|w| < 1$, for example by the transformation

$$w = \frac{(t + \bar{A})^{1/2} - \bar{A}^{1/2}}{(t + \bar{A})^{1/2} + \bar{A}^{1/2}} \quad (17)$$

which also maps $t=0$ into $w=0$. If $\tilde{\beta}(t)$ is then analytic in the cut plane [which assumes that all extrema of (6) have real action $\geq \bar{A}$] then, as a function of w , it is analytic in $|w| < 1$, and therefore has a convergent perturbation expansion in w for all t in the interval $(0, \infty)$.

From the work of Dash and Harrington ⁵⁾ and Harrington ⁶⁾, we know the renormalization group functions through three loops. We can therefore calculate the perturbation expansion of $\tilde{\beta}$, etc., in w through $O(w^3)$. For example, from Ref. 6)

$$\beta(g)/g = -\frac{1}{2} + 0.344h - 0.165h^2 + 0.124h^3 + O(h^4) \quad (18)$$

where $h = g^2/4\pi$.

The perturbation expansion of $\tilde{\beta}(t)$ in w is as follows :

$$\tilde{\beta}(t) = \frac{1}{\Gamma(\frac{13}{2})} \left\{ -\frac{1}{2} + 0.946w + 0.811w^2 + 0.224w^3 + O(w^4) \right\} \quad (19)$$

Although (19) is apparently no more convergent than (18), we know that in fact the higher order terms in (19) must go to a constant (since the radius of convergence is unity), while in (18) they will start to diverge rapidly [around $O(h^6)$].

We then reconstruct $\beta(g)$ via (16) and look for a zero $g = g_1$ at which $\beta'(g_1) > 0$. The same method is then applied to the other renormalization group functions to evaluate them at $g = g_1$. This yields the exponents γ , z and λ . The results obtained by truncating the series at the two and three loop levels are shown in Table 1. The results at the single loop level are not sensible. The closeness of the two and three loop results indicates the rapid convergence of our procedure.

We conclude with several remarks.

- 1) One can test for the sensitivity of the results to the exact value of $\bar{A}$. We find that a 10% error in $\bar{A}$ results in a similar error in g_1 , but only a small variation ($\sim 1\%$) in the exponents.
- 2) One can further modify the Borel transform (15) by dividing by $\Gamma(n+b')$ instead of $\Gamma(n+13/2)$. The results are insensitive to such a change if $|b' - 13/2| \lesssim 1$.
- 3) The small change between the two and three loop results then indicates (since the procedure is convergent) that the error at the three loop level is less than 10%, at least for γ and z .
- 4) These results agree with the Padé-Borel analysis by Bronzan et al.¹⁰⁾ of the three loop expansion, but improve the error involved. We have applied the same techniques to the zero of $\beta(g)$ in $D=0$, where it is known¹⁰⁾ that $g_1 = 2$. At the three loop level we find $g_1 \approx 2.07$, which is an improvement of a factor of more than two in the error over the Padé-Borel method¹⁰⁾, which finds $g_1 \approx 1.84$ at three loops.

- 5) We have also determined the behaviour of the ϵ expansion of the exponents at large orders. The details of this calculation will be given elsewhere. We find

$$\gamma = -\frac{\epsilon}{12} + \dots + \sum_{n \text{ large}} C'' \Gamma(n+6) (-\epsilon/R)^n \quad (20)$$

where $R \approx 6.91$. The exponents z and λ have a similar character. The techniques described above can then be applied to calculate at $\epsilon = 2$, using the $O(\epsilon^2)$ results of Bronzan and Dash ⁷⁾. These results are shown in Table 2. For γ and z they do not converge, nor agree with the loop expansion. This is because (20) predicts that the ϵ expansion be an alternating series for large n , but, to $O(\epsilon^2)$, γ and z do not alternate. However, the series for λ does alternate, and diverges rapidly, and the method is very successful, giving a better result than the loop expansion.

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	Number of loops	
	2	3
$g_1^2/4\pi$	3.15 ± 0.3	3.00 ± 0.3
$-\gamma$	0.24	0.26
z	1.12	1.13
λ	0.68 ± 0.01	0.58 ± 0.01

TABLE 1 : Exponents calculated in the loop expansion, to two decimal places. Errors corresponding to a 10% uncertainty in $\bar{A}$ are shown, where significant.

	$O(\epsilon)$	$O(\epsilon^2)$
$-\gamma$	0.08 ± 0.01	0.18 ± 0.01
z	1.04	1.09 ± 0.01
λ	0.49 ± 0.01	0.49 ± 0.01

TABLE 2 : Exponents calculated in the ϵ expansion, to two decimal places. Errors corresponding to a 10% uncertainty in R are shown, where significant.

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