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THEORY OF THE FROISSARON EXCHANGE

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ABSTRACT

The results of the multi-Pomeron exchange theory considered earlier ⁶⁾⁻⁸⁾, are shown to depend crucially on the threshold behaviour of the Pomeron contribution at $t = 4\mu_\pi^2$, under the condition $\alpha(0) - 1 = \Delta > \alpha'(0)4\mu_\pi^2$. The t channel partial wave $f(t)$ of the multi-Pomeron exchange is calculated. In the limit $\Delta > 1$, where $\xi = \ln \Delta / 4\mu^2$, it corresponds to the scattering on the black disk of the expanding radius $b_0 \approx a \cdot \xi$ where $a = \Delta / 2\mu_\pi - \alpha' 2\mu_\pi$. Due to the threshold singularity influence, it does not violate the t channel unitarity condition. At a sufficiently small value of the Froissaron coupling constant $g_{00} \sim \Delta^3 / a^2$, the theory is shown to be simultaneously s unitary.

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1. INTRODUCTION

The French group ¹⁾ has remarked first that the Pomeron theory with $\alpha_P(Q) = 1 + \Delta > 1$ leads to a good description ²⁾ of high energy, low $p_{\perp}$ experimental data. At the same parameters, the theory describes simultaneously the rising cross-sections, the $d\sigma/dt$ behaviour, and a number of features of particle production processes ^{3),4)}; in particular, the mean multiplicities and the distribution of particles over the multiplicities and rapidities.

The rescattering corrections (Fig. 1a) are very important and the exchange by a large number of Pomerons, $n \sim e^{\xi \Delta}$, is essential at very high energy, where $\xi \Delta > 1$, $\xi = \ln s/m^2$. The resulting scattering amplitude, $M_0(\xi, k^2)$, $k^2 = -t$, which we shall call the "Froissaron", yields a Froissart-type rise of $\sigma^{\text{tot}} \sim \xi^2$ at $\xi \rightarrow \infty$. For the diffraction dissociation total cross-section, the same approach gives $\sigma^{\text{diffr}} \sim \xi$, at $\xi \rightarrow \infty$.

At such extremely high energies, $\xi \rightarrow \infty$, the contribution to $M(\xi, k^2)$ of the so-called enhanced graphs ⁵⁾ is also important. They contain chains of the multi-Pomeron exchange blocks of the type shown in Figs. 1b, 1c, 1d connected to each other along the t channel by multi-Pomeron interaction vertices $g_{m,n}$. The main idea of handling them ⁶⁾⁻⁸⁾ was to sum up first all the Pomeron parallel line contributions in each block, i.e., to eikonalize it, forming the Froissarons [or "superpropagators" in the terminology of Ref. 6].

Then a new type of graphs appears, made up of these Froissaron lines and of the vertices g_{00} of the interaction of Froissarons. They are shown in the right-hand side of Fig. 1. Their contribution is determined by the usual Gribov rules ⁵⁾, since the zero-order Froissaron contribution can be shown ⁶⁾ to be factorizable, at least at very high energy. The multi-Froissaron interaction vertices are the same ⁶⁾ g_{00} for any number of adjoint Froissaron lines.

We shall call exact Froissaron the scattering amplitude $M(\xi, k^2)$ given by the sum of all graphs of Fig. 1. In the impact parameter representation it is given by the profile function

$$F(\xi, b) = 1 - e^{2i\delta(\xi, b)} = \int \frac{d^2k}{2\pi} M(\xi, k) e^{i\vec{k}\vec{b}}, \quad (1)$$

where $\delta(\xi, b)$ is the scattering phase. The function $F(\xi, b)$ happens to be almost real at high energy, where $M(\xi, k^2)$ and $\delta(\xi, b)$ are almost imaginary.

The analysis of the Froissaron features made previously in our papers ^{7),8)} has shown that, in general, s unitarity can be preserved in the Froissaron theory at a sufficiently small value of the vertex g_{00} .

However, the problem of the t unitarity of the theory has remained up to now unresolved. The difficulty was due to the lack of an accurate analysis and is as follows. At $\xi \rightarrow \infty$ the zero order Froissaron profile $F_0(\xi, b)$, corresponding to the multi-Pomeron exchange in Fig. 1a, appears to be close to unity in the wide interval $0 \leq b \leq b_0$ of the values of b , where

$$b_0 = a\xi \left(1 + O\left(\frac{1}{\xi} \ln \xi\right)\right) \quad (2)$$

and a is some constant length. At $b > b_0$ the profile $F_0(\xi, b)$, shown in Fig. 2, falls off rapidly within the interval $\Delta b \simeq \beta^{-1}$ which can be shown to be energy independent. For this reason one takes, in the zero order approximation, $F_0(\xi, b)$ at $\xi \rightarrow \infty$ in the form of the Θ function

$$F_0(\xi, b) \simeq C_0 \Theta(b_0 - b) \quad (3)$$

where $C_0 = 1$ (dotted line in Fig. 2). It corresponds to the scattering on the black sphere with sharp edges, and yields the well-known diffraction scattering result $M_0 \rightarrow \mathcal{M}_{b_0}(\xi, k)$, where

$$\mathcal{M}_b(\xi, k) = \frac{i b^2 J_1(kb)}{2kb} \quad (4)$$

Indeed, in general

$$M(\xi, k) = \int_0^\infty \frac{i}{2} F(\xi, b) J_0(kb) b db = \int_0^\infty \frac{i b J_1(kb)}{2k} \left(-\frac{\partial F(\xi, b)}{\partial b}\right) db \quad (5)$$

and $(-\partial F/\partial b) \rightarrow \delta(b_0 - b)$ for the profile (3). If one now neglects the $\ln \xi = \ln \ln s$ term in (2), putting $b \approx b_0 \approx a\xi$ in (4), then the t channel partial wave

$$f(\omega, k) = \int_0^\infty M(\xi, k) e^{-\omega \xi} d\xi, \quad \omega = j-1, \quad (6)$$

of the amplitude $M \rightarrow \mathcal{M}_{b_0}(\xi, k)$ will have the form of the so-called Schwarz-type singularity

$$f(\omega, k) \rightarrow f_0(\omega, k) \simeq \frac{a^2/2}{(\omega^2 - a^2 t)^{3/2}} \quad (7)$$

where $t = -k^2$.

This result is quite misleading because it strongly violates, at $4\mu_\pi^2 < t < 16\mu_\pi^2$ and $\omega \rightarrow a\sqrt{t}$, the t channel unitarity condition

$$\delta f = \frac{1}{2i} [f(\omega, t_+) - f(\omega, t_-)] = A \cdot (t - \kappa^2)^{\omega+3/2} f(\omega, t_+) f(\omega, t_-), \quad (8)$$

where $\kappa^2 = 4\mu_\pi^2$, $\kappa = 2\mu_\pi$. The violation is due not only to the fact that the right-hand side of this equation tends to infinity as $\omega \rightarrow a\sqrt{t}$. Moreover, the discontinuity of $f(\omega, t)$ over the right-hand cut in the t plane was neglected from the very beginning - both in the Pomeron and in the Froissaron contribution. Thus, for the amplitude (7) the left-hand side of the unitarity equation is zero and the right-hand side at $\omega \rightarrow a\sqrt{t}$ tends to infinity.

It is shown below that these difficulties can be easily avoided if :
i) terms of the order of $\ln s$ will be retained in (2), and ii) the threshold singularity of the type $\sim (t - \kappa^2)^\gamma$ will be taken into account in the Pomeron contribution. They change significantly its profile, leading - in particular - to a larger value of the constant a in Eq. (2). This expands at $\xi \rightarrow \infty$ by a large amount the radius of the Froissaron profile in Fig. 2.

The partial t channel amplitude (6) of the "eikonalized" Pomeron, Fig. 1a, is calculated accurately⁹⁾ below at different values of the variables ω and k . It is shown to have a "soft" singularity at $\omega \rightarrow \pm i\kappa a$ of the type $(\omega \pm i\kappa a)^\gamma$, where at $\sqrt{t} > \kappa$ the power $(-\gamma)$ is positive - due to the influence of the threshold singularity of a Pomeron of type $\sim (t - \kappa^2)^\gamma$ at $\gamma > 1$. This preserves the t channel unitarity, at least in the form of Eq. (8).

The last part of the paper is devoted to the analysis of the exact equation for the sum of contributions of all enhanced graphs in Fig. 1, obtained earlier^{7),8)}. The method of its successive iteration is developed with zero order Froissarons, Fig. 1a, taken at zero order approximation. It is shown to be convergent (provided the s channel unitarity of the theory) for a sufficiently small value of the vertex $|g_{00}|$ of the Froissaron interaction and, at the same condition, $\gamma > 1$ which is required by the t channel unitarity.

A number of papers, concerning the Pomeron theory at $\alpha(0) > 1$ should be noted. Some of them ⁶⁾⁻⁸⁾ were mentioned above. There were earlier attempts made by Cheng, Wu and co-workers ¹⁰⁾⁻¹³⁾, to obtain a description of the experimental data, using the unmovable Pomeron pole, fixed at $\alpha = 1 + \Delta = \text{const}$, with a complicated form of the residue. The rescattering terms were there very essential. These works were followed by a number of Russian papers ¹⁴⁾⁻¹⁶⁾. Rescattering terms were considered there in a different variant of the "eikonalization". However, no attempts were made to investigate t and s channel unitarity problems, in particular in connection with the enhanced graphs' contribution.

Quite apart of them, there are a number of researches ¹⁷⁾⁻²³⁾ made in the frame of the Pomeron field theory at $\alpha(0) > 1$. Some kind of phase transition was found in the theory when the bare value of $\alpha = \alpha_0(0)$ rises and passes the critical value $\alpha_0(0) = \alpha_c > 1$, which is close to unity. However, these papers are mostly concerned with the properties of the Pomeron Green's function and only three-Pomeron interaction was taken into account there. As it will be clear from the following, at $\alpha(0) > 1$ (in contrast to the situation when $\alpha(0) = 1$) there is no simple connection between the scattering amplitude and the Pomeron Green's function and all multi-Pomeron interactions are essential.

2. THRESHOLD SINGULARITIES AND THE POMERON PROFILE

The Pomeron profile

$$F_P = U(\xi, b) = \int_0^\infty \frac{2}{i} M_P(\xi, k) J_0(kb) k dk \approx \int_{-\infty}^\infty \frac{2}{i} M_P(\xi, k) J_0^{(1)}(kb) k dk \quad (9)$$

is determined by the Pomeron contribution of Fig. 3 :

$$\frac{1}{i} M_P \approx g^2 \exp [(\alpha(-k^2) - 1) (\xi - \frac{i\pi}{2})]. \quad (10)$$

Here $\xi = \ln s / 2m_\pi^2$; we have put $g_A = g_B = g = \text{const}$ and

$$\alpha(t) = 1 + \Delta + \alpha' t + \varepsilon \cdot (\alpha^2 - t)^\delta, \quad t = -k^2, \quad (11)$$

is the Pomeron trajectory. The last term represents its singular part, ϵ is some small coefficient, $\kappa^2 = 4u_\pi^2$, $\kappa^2 - t = e^{-i\pi}(t - \kappa^2)$ and γ is some non-integer number. Due to the rescattering, or to the P^n contribution, the result of Gribov and Pomeranchuk⁹⁾, concerning the value of γ ($\gamma = \alpha(\kappa^2) + \frac{1}{2}$), is not valid (see below). However, it will be seen that their restriction $\epsilon_1 = \epsilon \sin \pi\gamma > 1$ still remains at $\alpha(0) > 1$. The expression in the right-hand part of (9) was obtained by the substitution: $J_0(kb) = \frac{1}{2}(H_0^{(1)}(kb) + H_0^{(1)}(-k_+b))$, where $k_+ = k + i\tau$, $\tau > 0$, $\tau \rightarrow 0$; it allows one to shift the integration contour on the upper part of the complex k plane.

In what follows, the most important will be the large values of the impact parameter b of the order of ξ : $b = \lambda_0 \xi$. Here only large values of kb will be essential in (9) and the asymptotic form

$$J_0^{(1)}(kb) \approx \frac{2}{\sqrt{2\pi i k b}} e^{ikb}$$

can be used there.

Let us calculate the profile (9) in two regions $\lambda_0 < 2\alpha'\kappa$ and $\lambda_0 > 2\alpha'\kappa$. We will consider the case $\xi > 1/\Delta \gg 1$, and shall disregard, in (10), the small $-i\pi/2$ term.

i) The $\lambda_0 < 2\alpha'\kappa$ region

As will be seen in this case the singular term in the trajectory $\delta\alpha(t) = \epsilon(\kappa^2 - t)^\gamma$ is not important, and we can put $\epsilon = 0$. The calculation of the integral in the middle part of Eq. (9) is straightforward and gives the well-known result

$$v(\xi, b) = \frac{g^2}{\alpha'\xi} e^{\xi\Delta - b^2/4\alpha'\xi}, \quad b < 2\alpha'\xi. \quad (12)$$

It will be useful to rederive it by calculating the integral in the right-hand side of Eq. (9)

$$v(\xi, b) \approx 2g^2 e^{\xi\Delta} \int_{-\infty}^{\infty} e^{\varphi_b(\xi, k) + \delta\alpha(-k^2)\xi} \sqrt{\frac{k}{2\pi i b}} dk, \quad (13)$$

$$\varphi_b(\xi, \kappa) = i\kappa b - \alpha' \xi \kappa^2 = -\alpha' \xi \left(\kappa - \frac{i b}{2\alpha' \xi} \right)^2 - \frac{b^2}{4\alpha' \xi}, \quad (14)$$

using the saddle-point method. This approach allows one to consider easily also the case $\lambda_0 > 2\alpha' \kappa$. As it is seen at $\epsilon = 0$, the saddle-point

$$k = k_0 = \frac{i b}{2\alpha' \xi}$$

lies on the imaginary axis of the k plane of Fig. 4.

In general, at $\epsilon \neq 0$, two cuts have to be drawn in this plane along the imaginary axis, from $-i\omega$ to $-i\kappa$ and from $i\kappa$ to $i\omega$, corresponding to the threshold singularity at $t = \kappa^2 = 4u_\pi^2$ of M_P in the $t = -k^2$ plane. Note that the purely imaginary values $k = -i|k|$ correspond to values of t lying on the upper bound of the cut $t \rightarrow t + i\tau$, and $k = -i|k|$ corresponds to $t \rightarrow t - i\tau$. As at $\lambda_0 < 2\alpha' \kappa$ we have $|\lambda_0| < \kappa$, then the integration contour in (13) can be shifted in the upper part of the k plane to pass through k_0 (broken line in Fig. 4). In the integral over this contour, the singular term $\delta\alpha(-k^2)\xi$ in (13) is not essential. It can be taken at the saddle-point and then developed in the power series $t = -k^2$. This will lead (at small ϵ) only to a small change of the values of Δ and α' in (11)

$$\Delta \rightarrow \Delta_c = \Delta + \delta\alpha(0), \quad \alpha' \rightarrow \alpha'_c = \alpha' - \frac{\chi}{2\xi} \delta\alpha(0)$$

with $\delta\alpha(0) = \epsilon \kappa^{2\gamma}$, $|\delta\alpha(0)| < \Delta$. The shift of the contour $C \rightarrow C_1$ corresponds to a change of variable $k_1 = k - ib/2\alpha' \xi$ and gives

$$v(\xi, b) \simeq 2g^2 \sqrt{\frac{k_0}{2\pi i b}} e^{\xi \Delta_c} \int_{-\infty}^{\infty} e^{-\alpha_c \xi k_1^2} dk_1$$

where $\sqrt{k}$ was replaced by the value at the saddle-point. This leads exactly to the result (12) with the corrections (16) of first order in ϵ . They are not essential for the following, and we shall disregard them, assuming that $\delta\alpha(0) = \epsilon \kappa^{2\gamma} \ll \Delta$. The Pomeron profile $v(\xi, b)$, determined by (12), is very large in the small b region, at $\Delta > 1$. Only for $b \simeq a_0 \xi$, where $a_0 = 2\sqrt{\alpha'/\Delta}$, does it become of the order of unity. The length a_0 is smaller than $2\alpha' \kappa$ ($a_0 < 2\alpha' \kappa$) in the case $\Delta > \alpha' \kappa^2$ to which we shall restrict ourselves, in correspondence ^{2), 7)} with experimental data.

ii) The $\lambda_0 > 2\alpha'\kappa$, i.e., $b > (2\alpha'\kappa)\xi$ region

In this case the saddle-point $k_0 = ib/2\alpha'\xi$ in (14) (at $\delta\alpha \rightarrow 0$) occurs just on the upper cut in the k plane in Fig. 4. Let us shift the integration contour $C \rightarrow C'$ as is shown in Fig. 4, and develop $e^{\xi\delta\alpha(-k^2)}$ in powers of $\delta\alpha$. It gives

$$v(\xi, b) = 2g^2 e^{\xi\Delta} \int_{\Delta C} e^{\varphi_b(\xi, k)} \sqrt{\frac{k}{2\pi i b}} (1 + \varepsilon(\alpha^2 + k^2)^\gamma \xi + \dots) dk + v',$$

where ΔC is the part of the C' contour which goes around the upper cut in Fig. 4, from $i\kappa$ to $k_0 = (ib/2\alpha'\xi)$; v' is the small contribution (as will be seen) to the integral (13) due to the rest part of the C' contour. The integral over ΔC is proportional to the discontinuity of the integrand over the upper cut. The first term gives no contribution to the discontinuity, and the second one leads to

$$v(\xi, b) = 2g^2 \varepsilon_1 \xi e^{\xi\Delta} \int_{k=i\kappa}^{k_0} e^{\varphi_b(\xi, k)} (-2i) \sqrt{\frac{k}{2\pi i b}} (k^2 + \alpha^2)^\gamma dk + v' \quad (15)$$

as the discontinuity of $\delta\alpha$ over the cut is just $-2i\varepsilon_1(\kappa^2 + k^2)^\gamma$, where $\varepsilon_1 = \varepsilon \sin \pi\gamma$.

Designating $k = i(\kappa + r)$ we obtain

$$\xi\Delta + \varphi_b(\xi, k) = \alpha(a\xi - b) - r(b - 2\alpha'\alpha\xi) + \alpha'\xi r^2$$

where the length

$$a = b/\alpha + \alpha'\alpha \quad (16)$$

will be very important for the following. Note that

$$a > 2\alpha'\alpha > a_0$$

at $\Delta > \alpha'\kappa^2$, so that the length

$$a_- = a - 2\alpha'\alpha = b/\alpha - \alpha'\alpha \quad (17)$$

is positive.

The dependence of the function $\varphi_b(\xi, k) - \varphi_b(\xi, i\kappa)$ for $k = i(\kappa + r)$ on r is plotted in Fig. 5. For $\delta = \lambda_0 - 2\alpha'\kappa > 0$ this function has a deep minimum at $k = k_0$, i.e., at $r = r_0 = (\delta/4\alpha')\xi$, where it equals $-(\delta^2/4\alpha')\xi$. It means that the exponent $\exp[\varphi_b(\xi, k) - \varphi_b(\xi, i\kappa)]$ decreases rapidly with the rising of r , and is negligibly small at $r = r_0$. This allows us to disregard the v' part of the integral (15) (which is proportional to this exponent at $r = r_0$), as well as the term $\alpha'\xi r^2$ in $\varphi_b(\xi, k)$, as $r \sim 1/\delta \cdot \xi$, and extend the integration over r to infinity. As a result we obtain

$$\begin{aligned} v(\xi, b) &= 2g^2\xi \int_0^\infty \frac{\sqrt{2\alpha'}}{\sqrt{\pi b}} e^{\alpha(a\xi - b)} \cdot \int_0^\infty e^{-(b - 2\alpha'\alpha\xi)r} (2\alpha r)^\gamma dr = \\ &= \frac{B \xi \exp[\alpha(a\xi - b)]}{\sqrt{b} (b - 2\alpha'\alpha\xi)^{\gamma+1}}, \end{aligned} \quad (18)$$

where

$$B = \frac{2g^2\xi}{\sqrt{\pi}} (2\alpha)^{\gamma+1/2} \Gamma(\gamma+1)$$

is some constant. Unitarity requires it to be positive, so $e_1 = e \sin \pi\gamma > 0$. Remember that this result holds only at $\Delta > \alpha'\kappa^2$ and means that there is, at large ξ , a large region

$$2\alpha'\alpha\xi < b < a\xi$$

where the Pomeron profile remains very large and is determined by the threshold singularity, Eq. (18). The function (12) is quite small in this region, as $2\alpha'\kappa > a_0$.

Figure 2 gives the value of the function $F_0 = 1 - e^{-v}$. It is close to the zero Froissaron profile $F \simeq F_0$ (see below), and will be very important in what follows. The substitution of the function (12) for v would lead to the profile shown by the broken line in the left-hand side of Fig. 2. However, just at the edge of this profile, which is most essential for the scattering

problem, Eq. (12) does not hold and the function (18) represents the actual value of $v(\xi, b)$. It means that the threshold singularity influences the Froissaron profile to extend up to values b of the order of $a\xi$.

Equation (18) shows that $v(\xi, b) = 1$ at $b = b_0$, where

$$b_0 = a\xi - \frac{1}{\alpha} \ln \frac{\sqrt{b_0} (b_0 - 2\alpha' a\xi)^{\gamma+1}}{B\xi}.$$

Putting approximately $b_0 \approx a\xi$ under the logarithm, we obtain

$$b_0 = a\xi - \beta \ln \eta \xi \quad (19)$$

where

$$\beta = \frac{\gamma+1/2}{\alpha}, \quad \eta = a \cdot (B/\sqrt{a})^{1/(\gamma+1)}. \quad (20)$$

For b close to b_0 , the function (18) can be approximately represented by

$$v(\xi, b) \approx e^{(b_0 - b)\beta_0} \quad (21)$$

where

$$\beta_0 = \left(\frac{-\partial v}{v \partial b} \right)_{b=b_0} = \alpha + \left(\frac{1}{2\alpha} + \frac{\gamma+1}{\alpha} \right) \frac{1}{\xi} \approx \alpha$$

for $\xi \rightarrow \infty$. It means that the width $\Delta b \sim 1/\beta_0$ of the distribution in the right-hand edge of Fig. 2 does not depend on the energy and equals $1/\alpha$.

Note that :

i) terms of high order in powers of $\xi(\kappa^2 + k^2)^\gamma$ in the integral (13) for $v(\xi, b)$ are small at $\gamma > 1$ and at $\xi \rightarrow \infty$, since [according to (18)] they are proportional to $\xi(2\kappa r)^\gamma$ and in the integral (18) r is of the order of $1/\delta$. Thus, they yield only a small correction to $v(\xi, b)$ of the order of $1/\xi^{\gamma-1}$.

ii) only the first threshold singularity at $t = \kappa_1^2$ which fulfils the condition $\Delta > \alpha' \kappa_1^2$ is essential. For all high order singularities at $t = (n\kappa)^2$, $n = 2, 3, \dots$ this condition is violated ($\Delta < (n\kappa)^2 \alpha'$) and they have no influence on the profile function (13). The same would happen with the first singularity should the condition $\Delta > \alpha' \kappa^2$ be violated. However, in

this case the s channel unitarity cannot be preserved [see Ref. 24], and we will not consider it. It has to be mentioned, however, that in the case $\Delta < \alpha' \kappa^2$ there would be no difference in the form of $v(\xi, b)$ in the two regions $b < 2\alpha' \kappa \xi$ and $b > 2\alpha' \kappa \xi$, because here the exponent $\exp(\varphi_b(\xi, n))$ in the integral line (12) would decrease with the increase of k along the upper cut of Fig. 4, even for $b > 2\alpha' \kappa \xi$.

3. EIKONALIZATION AND t CHANNEL PROPERTIES OF THE ZERO-ORDER FROISSARON

a) Eikonalization at $\alpha(0) > 1$

At $\xi \Delta > 1$ each Pomeron contributes a factor $\exp(\xi \Delta) \gg 1$ [see (10), or (12), at small b]. Thus, the exchange, see Fig. 1a, of a large number of Pomerons is important. As the contribution of the graphs in Fig. 1a is simply

$$\frac{-(-v)^n}{n!} \cdot \frac{N_n^A N_n^B}{(g_A g_B)^n}$$

we obtain, by summing over all n ,

$$F_0(\xi, b) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} v^n(\xi, b) \cdot C_n \quad (22)$$

where

$$C_n = N_n^A N_n^B / (g_A g_B)^n$$

are some dimensionless coefficients, N_n^A, N_n^B are Gribov's n Pomeron vertices.

The following approximations were used to obtain the explicit values of the numbers C_n :

- i) the eikonal model (Fig. 6a) corresponds to the assumption that $C_n = 1$ and gives the function shown in Fig. 2:

$$F_0(\xi, b) = 1 - e^{-v(\xi, b)} \quad (23)$$

- ii) a better approximation is given by the model of resonance states (Fig. 6b). It takes into account the possibility of the excitation of resonance states of each of the scattered particles after each re-scattering, and leads to

$$F_0(\xi, b) = 1 - \sum_v \lambda_v e^{-\rho_v v(\xi, b)}$$

where λ_v and ρ_v are some coefficients for which

$$\sum_v \lambda_v = 1, \quad \sum_v \lambda_v \rho_v = 1, \quad \rho_v > 0;$$

v labels different resonance states of both particles.

For the amplitude of the diffraction production of one resonance state (Fig.6c), or for the double diffraction production (Fig.6d), the same mechanism yields

$$F_{01}(\xi, b) = \sum_v \lambda'_v e^{-\rho_v v(\xi, b)}$$

with λ'_v (or λ''_v in the case of double diffraction) satisfying the condition

$$\sum_v \lambda'_v = \sum_v \lambda''_v = 0.$$

The coefficients $\rho_v, \lambda_v, \lambda'_v, \lambda''_v$ can be expressed in terms of the eigenvalues of the vertices of the transition in a Pomeron from the i th to the k th excited state of each particle $(g_A)_{ik}, (g_B)_{ik}$. In the case of two states, the corresponding expressions are given in the Appendix. The zero-order Froissaron $F_0(\xi, b)$ is close to unity (and F_{01} to zero) for $v \gg 1$, and $F_0 \sim v \rightarrow 0$ in the $b > b_0$ region, where $v \rightarrow 0$. Thus, $F_0(\xi, b)$ has the b dependence shown in Fig. 2. The amplitude differs from zero only at the edge of the profile in Fig. 2. This yields a Froissart-type behaviour of $\sigma^{\text{tot}} \sim \xi^2$ and the linear rise of the diffraction dissociation cross-section $\sigma^{\text{diffr}} = (2\pi/\kappa) a \xi$.

- iii) In general, at $\xi \rightarrow \infty$, the profile $F_0(\xi, b)$ can be put in the form (3) $F_0 = \Theta(b_0 - b) C(0)$ under the assumptions that $C_n = C(n)$ are analytical in n and $C(n)/\Gamma(n) \rightarrow 0$ at $n \rightarrow \infty$. This was shown by Cardy⁶⁾, and all physical plausible models show that $C(0)$ has to be equal to unity. The principal result obtained below will not depend on the explicit form of the "eikonalization", i.e., of the dependence

$$F_0 = E(v)$$

provided only

$$E(v) = \begin{cases} 1, & v \gg 1 \\ v, & v < 1 \end{cases}$$

Therefore the simple eikonal form (22) $E(0) = 1 - e^{-v}$, which has these properties, will be used for the general analysis, although more accurate expressions have to be used in the numerical calculations of the observed quantities. For the Pomeron theory with $\alpha(0) > 1$, the "eikonalization" is very essential as it restores the s unitarity of the theory [at least in the form $|F| < 2$ which is clearly violated by the Pomeron contributions (12)-(21) only] and leads to the physical picture of the scattering in a black sphere with expanding radius (19).

b) t channel features of the Froissaron

At $F_0 = 1 - e^{-v}$, the scattering amplitude (5) can be put in the form :

$$M_0(\xi, \kappa) = \int_0^\infty M_b(\xi, \kappa) e^{-v} dv \quad (24)$$

where M_b is the diffraction scattering amplitude (4) and b has to be expressed in terms of v by means of Eq. (21)

$$b = b_0 - \frac{1}{\alpha} \ln v = a\xi - \frac{1}{\alpha} \ln(\eta \xi)^{\gamma + 1/2} v. \quad (25)$$

The last equation is valid because only the $v \sim 1$ region is essential in the integral (24), and there $v \approx e^{\kappa(b-b_0)}$.

Let us calculate the t channel partial wave (6) of this amplitude, writing it in the form :

$$f(\omega, \kappa) = \int_0^\infty e^{-v} f_v(\omega, \kappa) dv \quad (26)$$

with

$$f_v(\omega, \kappa) = \int_0^\infty e^{-\omega \xi} \mathcal{J}_m M_b(\xi, \kappa) d\xi, \quad (27)$$

where v and b are connected by Eq. (25).

This partial amplitude will be shown to have singularities in the j plane at $\omega = j-1 = \pm i\kappa a$, and to have threshold singularities at $\kappa = \pm i\omega$, in the κ plane of Fig. 4. Consider it in different regions of variables ω and κa .

i) The region $\omega < \Delta$, but $\omega > |ak|$

determines the profile of the scattering amplitude (at $\xi > 1/\Delta$) in the main part, $b < a\xi$, of Fig. 2. Inserting (5) into (27) and putting there $b = \bar{a}\xi$, where

$$\bar{a} = a - \frac{1}{2\xi} \ln[(\eta\xi)^{\delta+1/2} v],$$

we obtain

$$f_v(\omega, k) = \int_0^\infty e^{-\omega\xi} \frac{(\bar{a}\xi)^2 J_1(\bar{a}k\xi)}{2\bar{a}k\xi} d\xi \approx \frac{a^2}{2\omega^3} \int_0^\infty e^{-x} \frac{J_1(\frac{k}{\omega}\bar{a}x)}{\frac{k}{\omega}\bar{a}x} x^2 dx, \quad (28)$$

where the factor $\bar{a}^2 \approx a^2$ was taken out of the integral at $\xi \sim 1/\omega \gg 1$. In the integral the values $x \sim 1$ are important and at $z = k\bar{a}/\omega \approx ka/\omega < 1$ the function $J_1(z)/z$ is close to $\frac{1}{2}$. This implies

$$f(\omega, k) \approx f_v(\omega, k) \approx \frac{a^2/2}{\omega^3}, \quad k \rightarrow 0,$$

in the limit $ka/\omega \rightarrow 0$. Terms of the order of $(ka/\omega)^n$ can be obtained by putting $\bar{a} = a$ in the last integral [this approximation holds only at small ω , when $\omega \ln 1/\omega \ll 2\Delta$, as $\bar{a} \approx a(1 - (\omega/\kappa a x)) \ln(\eta x/\omega)^{\delta+1/2} v$ and $x \sim v \sim 1$]. This leads to the Schwarz form (7)

$$f(\omega, k) \approx f_v(\omega, k) \approx \frac{a^2/2}{(\omega^2 + a^2 k^2)^{3/2}}$$

as

$$\int_0^\infty e^{-x} \frac{J_1(ux)}{ux} x^2 dx = (1+u^2)^{-3/2}$$

This result is only valid at $\omega \neq |ka|$ as the integrand in the integral over ξ includes factors of the form $\exp[-(\omega\xi \pm i\kappa b)]$, where

$$\omega\xi \pm i\kappa b = (\omega \pm i\kappa a) \pm \frac{i\kappa}{2} \ln(\eta\xi)^{\delta+1/2} v$$

and where the last term was omitted. The latter can be done only when ω is far from the singularities, i.e., at $\omega \pm i\kappa a \neq 0$.

ii) The case $|\omega \pm ika| \rightarrow 0$, or $|\omega \pm ika| < |ka| \sim \omega < 1$

requires a separate analysis. Here the large values of ξ of the order of $1/|\omega \pm ika|$ are important in the integral (28). In the case $\omega \rightarrow -ika$, considered below, we have in this integral

$$|ak|\xi \sim |ak| \cdot \frac{1}{|\omega + ika|} > 1$$

as $\xi \sim 1/|\omega + ika|$. This means that the asymptotic form of the Bessel function can be used

$$\text{Im } M_b = b^2 \frac{J_1(kb)}{2kb} \approx \frac{b^2}{\sqrt{\pi}} \left[\frac{e^{-ikb}}{(-2ikb)^{3/2}} + \frac{e^{ikb}}{(2ikb)^{3/2}} \right]$$

and (28) hence implies

$$f(\omega, k) = f^{(+)}(\omega, k) + f^{(-)}(\omega, k)$$

where

$$f^{(\pm)}(\omega, k) = \int_0^\infty f_v^{(\pm)}(\omega, k) e^{-v} dv$$

and

$$f_v^{(+)}(\omega, k) = \int_0^\infty \frac{b^2}{\sqrt{\pi}} \frac{e^{-ikb - \omega \xi}}{(-2ikb)^{3/2}} d\xi \approx \frac{a^2 v^{ik/\alpha}}{(-2ika)^{3/2} \sqrt{\pi}} \int_0^\infty e^{-\frac{(\omega + ika)\xi}{\xi^{1/2} \Gamma(\xi)}} d\xi, \quad (29)$$

where Eq. (25) for $b = b(v, \xi)$ was inserted in $\exp(-ikb)$ (and $b \approx a\xi$ was put otherwise), and where $\beta = \gamma + \frac{1}{2}/\alpha$. The integral over ξ yields a Γ function and implies

$$f_v^{(+)}(\omega, k) = \frac{a^2}{\sqrt{\pi}} \frac{\eta^{ik\beta} v^{ik/\alpha} \Gamma(ik\beta + \frac{3}{2})}{(-2ika)^{3/2} (\omega + ika)^{3/2 + ik\beta}}.$$

The part $f_v^{(-)}(\omega, k)$ of $f(\omega, k)$ remains regular at $\omega \rightarrow -ika$.

Note that the singularities of the function $\Gamma(ik\beta + \frac{3}{2})$ at the integer negative values of $ik\beta + \frac{3}{2}$ (which actually equals $\frac{3}{2} - \beta\sqrt{t}$ at the upper bound of the cut on the t plane) are not essential. They are due to the region of small values of ξ in the integral (29), which has actually to be excluded by a change of the lower limit $\xi = 0$ of the integral to $\xi = \xi_0$, where $\xi = \xi_0$ is some constant [since the Pomeron asymptotic value (10) used in (29) is valid only at $\xi > \xi_0$]. This yields only the substitution of

$$\Gamma(ik\beta + \frac{3}{2}, (\omega + ika)\xi_0)$$

for $\Gamma(ik\beta + \frac{3}{2})$ in the last equation, where

$$\Gamma(x, y) = \int_y^\infty e^{-u} u^{x-1} du$$

is the incomplete Γ function. The representation of $\Gamma(x, y)$ in the form of a series in y

$$\Gamma(x, y) = \Gamma(x) + \sum_n \frac{(-1)^{n+1} y^{n+x}}{n! (n+x)!}$$

where $y = \xi_0(\omega + ika) \rightarrow 0$ near the singular point, shows that this does not change the singular part of the function $f^{(+)}(\omega, k)$. Only a number of regular terms at $|\omega + ika| \rightarrow 0$ appear there.

The substitution of $f_v^{(+)}$ into (26) gives, at $ik = -\kappa$

$$f^{(+)}(\omega, k) = \frac{a^2}{\sqrt{\pi}} \frac{\eta^{ik\beta} \Gamma(1 + \frac{ik}{2}) \Gamma(ik\beta + \frac{3}{2}, (\omega + ika) \xi_0)}{(-2ika)^{3/2} \cdot (\omega + ika)^{3/2 + ik\beta}} \quad (30)$$

The branch points of the partial wave $f(\omega, k) = f^+ + f^-$ in the ω plane are shown in Figs. 7a and 7b. Due to the $\ln(\xi\eta)^{\gamma + \frac{1}{2}}$ term in (25) the corresponding singularities turn out to be "soft" at $\gamma > 1$. Indeed, at $\omega \rightarrow a\sqrt{t_+}$, where $\sqrt{t_+} = -ik > \kappa$, we have

$$f(\omega, k) = B_1 \cdot (\omega - a\sqrt{t})^{(\gamma + 1/2)\sqrt{t/a^2} - 3/2} + B_2$$

where B_1 and B_2 are some functions of t only, and the power $(\gamma + \frac{1}{2})\sqrt{t/a^2} - \frac{3}{2}$ is positive at $t > \kappa^2 = 4u^2$ under the condition $\gamma > 1$.

In spite of (7), the partial amplitude in this form can preserve the t channel unitarity (8), even at $\omega \rightarrow a\sqrt{t}$, because here the singular term vanishes and the discontinuity of the regular one is not zero and is proportional to $(t - 4u^2)^{\omega + \frac{3}{2}}$ (see below).

iii) The $\sqrt{t} \rightarrow \kappa$ region, i.e., $k \rightarrow i\kappa$

Here the partial wave (30) reveals the singularity due to a divergence at $v \rightarrow 0$ of the integral (26), determining the function

$$\Gamma(1 + \frac{ik}{2}) = \int_0^\infty e^{-v} v^{ik/2} dv.$$

Near this singularity, at $|ik| < |\omega + i\kappa|$, Eq. (30) is not valid and has to be rewritten more accurately.

To separate the singular part of $f^{(+)}(\omega, k)$ at $|k+ik| \rightarrow 0$, let us introduce the fixed small value $v=v_1 < 1$ and define

$$f(\omega, k) = f_{\text{sing}}(\omega, k) + f_{\text{reg}}(\omega, k).$$

Here

$$f_{\text{reg}}(\omega, k) = \int_{v_1}^{\infty} f_v(\omega, k) e^{-v} dv$$

does not contain divergences at $|k+ik| \rightarrow 0$ and

$$f_{\text{sing}}(\omega, k) = \int_0^{v_1} f_v(\omega, k) e^{-v} dv = \int_0^{\infty} e^{-\omega \xi} \text{Im } M_{\text{sing}}(\xi, k) d\xi \quad (31)$$

is the contribution of the region of small $v < v_1$, i.e., of large $b > b_1$ where

$$b_1 = a \xi - \frac{1}{2} \ln(\eta \xi)^{\delta+1/2} v_1$$

is just Eq. (24) at $v=v_1$. By M_{sing} the contribution of this region $b > b_1$ to the scattering amplitude (5) is defined, i.e., $\text{Im } M_{\text{sing}} = \mathcal{A}_{\text{sing}}^{(+)} + \mathcal{A}_{\text{sing}}^{(-)}$ where

$$\mathcal{A}_{\text{sing}}^{(+)}(\xi, k) = \frac{1}{2} \int_{b_1}^{\infty} v(\xi, b) \frac{e^{-ikb} b db}{(-2\pi i k b)^{1/2}}. \quad (32)$$

The Bessel function $I_0(kb)$ in (5) was put in the form $\frac{1}{2}(H_0^{(1)} + H_0^{(2)})$ and the asymptotic form

$$\frac{1}{2} H_0(kb) \approx e^{-ikb} / \sqrt{-2\pi i k b}$$

for the Hankel function was used, since only large values of $|kb| > 1$ are important (at $b > b_1$). The singular part of the amplitude M_{sing} at $k \rightarrow i\kappa$ is defined by the first term in the expansion of the profile $F(\xi, b) = 1 - e^{-v} \approx v$ in powers of v [it corresponds to the substitution $e^{-v} \rightarrow 1$ in (26)]; other powers $\sim v^n/n!$ leads in (31) to the singularities at $k \rightarrow i\kappa n$.

Now, let us put the Pomeron contribution (18) into (31). It gives

$$\mathcal{A}_{\text{sing}}^{(+)}(\xi, k) = \frac{B \xi (\alpha + i k)^{\delta}}{\sqrt{2\pi(-ik)}} e^{a(\alpha + i k)\xi - i a k \xi} \int_{x_0}^{\infty} e^{-x} \frac{dx}{x^{\delta+1}}$$

where $x_0 = (\kappa + i\kappa)(b_1 - 2\alpha'\kappa\xi)$ and a_- is the length (17). Insertion into Eq. (31) gives, after integration over ξ :

$$f^{(+)}(\omega, k) = \frac{B \cdot \Gamma(-\gamma) (\alpha + i\kappa)^\gamma}{2\sqrt{2\pi\alpha} (\omega + i\alpha\kappa)^2} + f_{\text{Reg}}^{(+)}(\omega, k) \quad (33)$$

where $\Gamma(-\gamma)$ was substituted for the non-complete Γ function $\Gamma(-\gamma, (b_1 - 2\alpha'\xi)(\kappa - i\kappa))$ at $|\kappa + i\kappa| \rightarrow 0$ provided (by definition) γ is not integer.

In conclusion, let us sum the obtained forms of the partial amplitude $f(\omega, k)$ in the different regions of ω and $\alpha\kappa$. They are as follows :

$$f(\omega, k) = \begin{cases} \frac{a^{3/2}}{(\omega^2 + a^2\kappa^2)^{3/2}}, & \text{for } \omega > |\alpha\kappa|, \quad \omega < \Delta < 1 \\ \frac{a^2}{\sqrt{\pi}} \frac{\eta^{ik\beta} \Gamma(1 + \frac{ik}{\alpha})}{(2\omega)^{3/2}} \frac{\Gamma(ik\beta + \frac{3}{2}, y)}{(\omega + i\alpha\kappa)^{3/2 + ik\beta}}, & \text{for } |\omega + i\alpha\kappa| < |\alpha\kappa| < 0 \\ B_1 \frac{(\alpha + i\kappa)^\gamma}{(\omega + i\alpha\kappa)^2} + B_2, & \text{for } |\kappa + i\kappa| < |\omega + i\alpha\kappa| \rightarrow 0 \end{cases} \quad (34)$$

where $y = \xi_0(\omega + i\alpha\kappa)$, $\beta = (\gamma + \frac{1}{2})/\kappa$.

At $\gamma > 1$ the singularities of $f(\omega, t)$ at $\omega = \pm a\sqrt{t}$ become more and more "soft" as $\sqrt{t} = -i\alpha\kappa$ increases over the value of $\kappa = 2u_{\text{cr}}$.

We note that :

- α) the plateau shown in Fig. 2 will develop only at very large $\xi > \xi_0 \sim 1/\Delta$ (to which refers all what was considered above). The value ξ_0 can be a little more accurately obtained from the condition $b_0 > 0$, where b_0 is given by Eq. (19). It implies

$$\xi > \xi_0, \quad \xi_0 = \frac{\beta}{\alpha} \ln \eta \xi_0 \approx \frac{\gamma + 1/2}{2\Delta} \ln \frac{\gamma + 1/2}{2\Delta}, \quad (35)$$

where $\kappa a = \Delta + \alpha'\kappa^2$ has a value slightly smaller than 2Δ .

- β) The zero-order Froissaron (34) at $\Delta > \alpha'\kappa^2$, essentially differs from that at $\Delta < \alpha'\kappa^2$. In the $\Delta < \alpha'\kappa^2$ case, the form (12) for $v(\xi, b)$ holds for all values of the impact parameter b . Its use in (24) and (27) leads, as can be shown [see Ref. 6] to the expression $f(\omega, k)$, which develops all

Regge singularities with the increase of $\sqrt{t}$: the Pomeron pole for $\sqrt{\Delta/\alpha'} < \sqrt{t} < 2\sqrt{\Delta/\alpha'}$, the two Pomeron cut for $2\sqrt{\Delta/\alpha'} < \sqrt{t} < 3\sqrt{\Delta/\alpha'}$, etc. However, at $\Delta > \alpha' k^2$ all these singularities have disappeared from the second Riemann sheet connected with the Schwarz singularities at $\omega = \pm ika$. Actually, at $t < 0$, $f(\omega, k)$ is determined as the sum of their contributions.

3. ENHANCED GRAPHS AND s UNITARITY

Let us use the results obtained above for the analysis of the Froissaron enhanced graphs shown in the right-hand side of Fig. 1.

It was shown earlier ^{7),8)}, taking into account all these graphs, that the amplitude profile is determined by the integral equation

$$F(\xi, b) = E(v) \approx 1 - e^{-V(\xi, b)} \quad (36)$$

where

$$V(\xi, b) = v(\xi, b) + C(\xi, b) + D(\xi, b) \quad (37)$$

contains only contributions of the graphs which are not reducible against "eikonalization", i.e., do not contain graphs of the "eikonal" set. For instance, it includes the Pomeron contribution, Fig. 3, and the graphs shown in Figs. 1b, 1d, but does not include graphs in Fig. 1a (with $n > 2$) and in Fig. 1c.

The $C(\xi, b)$ term includes the graphs shown in Fig. 8a, which have the form of a chain along the t channel. They consist of a number of blocks $Z(\xi, b)$ of Froissaron graphs connected only by vertices g_{00} of Froissaron interactions. The term $D(\xi, b)$ represents the contribution of more complicated graphs, not having the form of a chain along the t channel. Examples of them are given in Fig. 8b. To each Froissaron line in Fig. 8 corresponds the exact Froissaron value $F(\xi, b)$.

Therefore, $C(\xi, b)$ and $D(\xi, b)$ can be expressed by means of Gribov rules, through $F(\xi, b)$ and Eqs. (36) and (37) can be considered as an integral equation system for the exact Froissaron contribution $F(\xi, b)$.

Let us show that the solution of this system can be obtained by a simple iteration method with zero approximation $C = D = 0$, $F \approx F_0 = 1 - e^{-v}$, where v is the Pomeron contribution (12)-(18). Substituting F by F_0 we can then obtain the first approximation values of C and D .

Remember 7), 8) that the t channel partial amplitude $c(\omega, k)$ of the chain contribution Eq. (40) is represented (from Fig. 8a) by

$$c(\omega, k) = \frac{g_{00} f^2(\omega, k)}{1 + g_{00} f(\omega, k)} = \frac{1}{f^{-1}(\omega, k)} - \frac{1}{f^{-1}(\omega, k) + g_{00}}$$

where $f(\omega, k)$ is the Froissaron t channel amplitude (6). The chain profile $C(\xi, b)$ is determined by an equation of the form of (1) or (9) :

$$C(\xi, b) = \int_{-\infty}^{\infty} \frac{2}{\xi} \tilde{C}(\xi, k) J_0^{(1)}(kb) k dk \approx \int_{-\infty}^{\infty} e^{ikb} \frac{2}{\xi} \tilde{C}(\xi, k) \frac{k dk}{\sqrt{-2\pi i k b}}, \quad (38)$$

where the right-hand side equality holds only at $|k_0 b| > 1$ (k_0 is the essential value of k), and where $\tilde{C}(\xi, k)$ is connected to $c(\omega, k)$ by the transformation (6).

Both the convergence of the iteration solution and the s unitarity of the theory will be preserved, provided not too large negative values of $C(\xi, b)$ arise after the substitution of Eq. (35), for the Froissaron contribution, into (38).

It is, at least, necessary to have

$$|C(\xi, b)| < v(\xi, b), \quad |D(\xi, b)| < v(\xi, b) \quad (39)$$

at each step of the iteration. However, the substitution of Eq. (35) into (38) leads to the Regge poles in (38), which can yield the exponentially rising value of $C(\xi, b)$ at $\xi \rightarrow \infty$. In principle, it can violate inequalities (39).

Note that the graphs of class $D(\xi, b)$ are not dangerous in this connection. The sum of their contributions can be put in the form of the eikonal expansion of type (22) and (23), where the chain contribution $|C(\xi, b)|$ is substituted for $v(\xi, b)$. This means that $D(\xi, b)$ is of the order of unity and is small at $\xi \Delta \gg 1$ as compared to the exponentially large value of $v(\xi, b)$; we will not consider it here (see Section 5).

Let us estimate the quantity $C_1(\xi, b)$ having a t channel amplitude in the form of the second term in $c(\omega, k)$

$$c(\omega, k) = \frac{1}{f^{-1}(\omega, k) - |g_{00}|} \quad (40)$$

It represents the most dangerous part of $C(\xi, b)$ and at $\xi\Delta > 1$ the inequality holds $|C(\xi, b)| < C_1(\xi, b)$. Let $\omega = \omega_0(k)$ be that value of ω for which the denominator in $C_1(\omega, k)$ equals zero. Then $\tilde{C}_1(\xi, k) \sim \exp(\omega_0(k)\xi)$ and it follows from (38) that

$$C_1(\xi, b) = \int_{-\infty}^{\infty} e^{ikb + \omega_0(k)\xi} dk$$

up to the non-exponential factor which is not essential (and can easily be calculated).

Consider separately the region $b < a\xi$ inside the black disk, Fig. 2, (putting $b = \lambda_0 \xi$, $\lambda_0 < a$) and the region $(a\xi - b) < a\xi$ at the edge of the disk. The central part $b = \text{const}$ at $\xi \rightarrow \infty$ gives no new results.

a) The region $b = \lambda_0 \xi < a\xi$ inside the disk

As will be seen here, the values ω and k are essential, for which $\omega > |ka|$. Therefore $f(\omega, k)$ can be put in (39) in the Schwarz form (7). It yields

$$\omega_0(k) = (g^{2/3} - a^2 k^2)^{1/2} \quad (41)$$

where $g = (a^2/2)|g_{00}|$ is a dimensionless Froissaron coupling constant.

The integral (38) [in the form of the right-hand part of (38), because inequality $|kb| > 1$ holds readily] can be calculated by the saddle-point method

$$C_1(\xi, b) = \int_{-\infty}^{\infty} e^{\varphi(k)} dk \approx \sqrt{\frac{\pi}{\gamma}} e^{\varphi(k_0)},$$

where the function in the exponent

$$\varphi(k) = ikb + \omega_0(k) \approx \varphi(k_0) - \frac{\gamma}{2} (k - k_0)^2 \quad (42)$$

($\gamma = -\varphi''(0)$) depends on both ξ and b , and $\varphi'(k_0) = 0$.

The explicit form (41) of $\omega_0(k)$ yields

$$k_0 = \frac{i b g^{1/3}}{a \sqrt{a^2 \xi^2 - b^2}} \approx \frac{i b g^{1/3}}{a^2 \xi},$$

$$\varphi(k_0) = \frac{g^{1/3}}{a} \sqrt{a^2 \xi^2 - b^2}, \quad \gamma = -\varphi''(k_0) = \frac{(a^2 \xi^2 - b^2)^{3/2}}{g^{1/3} a \xi^2},$$

which implies

$$C_1(\xi, b) \sim e^{g^{1/3} \sqrt{\xi^2 - b^2/a^2}} \approx e^{g^{1/3} \xi}, \quad b < a\xi. \quad (42)$$

This value has to be compared with (12)

$$v(\xi, b) \sim e^{\xi \Delta}, \quad b < a\xi.$$

It is smaller at

$$g^{1/3} < \Delta$$

i.e., at $g < \Delta^3$.

We note that :

- i) inequality $\omega > |ak|$ actually holds, i.e., $\omega_0 > |ak_0|$, because at $k=k_0$ the quantity

$$\omega_0(k_0) = \frac{g^{1/3} a \xi}{\sqrt{a^2 \xi^2 - b^2}} \approx g^{1/3}$$

differs from $|ak_0| \approx g^{1/3} (b/a\xi)$ by the large factor $(a\xi/b) > 1$.

- ii) the mean number $\langle N \rangle$ of links in the chain $C_1(\xi, b)$ in Fig. 8a has a value of the order of $g^{1/3} \xi$ [corresponding to Eq. (42)], i.e., the mean length in the rapidity space of each link is

$$\langle y \rangle = \xi / \langle N \rangle \approx g^{-1/3}$$

At $g^{1/2} < \Delta$ this length is larger than the value (35), which is needed to develop the plateau of the form shown in Fig. 2 for the Froissaron in each link of the chain. [The inequality $\langle y \rangle > \xi_0$ leads, according to (35), to the restriction $g^{1/3} < B_1 \Delta$, where $B_1 = 2/(\gamma + \frac{1}{2}) \ln(\gamma + \frac{1}{2}/\Delta)$ has at $\gamma > 1$ a value a little smaller than unity.]

- b) The region of the edge of the disk $|\xi a - b| \ll a\xi$

Here the small values of $|\omega + ika| \ll |ka|$ and of $v = \frac{3}{2} + i k \beta$ are essential, i.e., the imaginary values of $k \approx 3i/2\beta$ are important. The calculations can be done with the form (30) of the Froissaron amplitude, where

$$\Gamma(v, y) \approx \frac{1}{v} (1 - y^v), \quad v < 1, \quad y^v < 1,$$

and $y = \xi_0(\omega + ika)$. Thus, we put in (40)

$$f(\omega, k) \simeq f_{\text{sing}}^{(+)}(\omega, k) \simeq \frac{a^2}{2} A_0 \frac{\Gamma(\nu, y)}{(\omega + ika)^\nu} \simeq \frac{a^2 A_0}{2} \frac{1 - (\omega + ika)^\nu \xi_0^\nu}{\nu (\omega + ika)^\nu}$$

where

$$A_0 = \frac{2}{\sqrt{\pi}} \left(\frac{2\gamma+1}{3a\alpha} \right)^{3/2} \Gamma\left(\frac{2\gamma-1}{2\gamma+1}\right)$$

is a constant. The condition of the pole in (40) gives $\omega_0(k) = -ika + [G(\nu)]^{-1/\nu}$ or

$$\varphi(k) = ikb + \omega_0(k)\xi = ik(b - a\xi) + \xi/[G(\nu)]^{1/\nu} \quad (43)$$

with

$$G(\nu) = \xi_0^\nu + \nu/gA_0$$

and $\nu = \frac{3}{2} + i k \beta$, $\beta = (\gamma + \frac{1}{2})/\kappa$. The equation $\varphi'(k_0) = 0$ leads to

$$\frac{a\xi - b}{\beta\xi} = \frac{G \ln G - \nu G'}{\nu^2 G^{1+1/\nu}} \quad (44)$$

and $G' = dG/d\nu$. The right-hand side of this equation is equal to zero at small $\nu = \nu_0 = GA_1 g \sim \Delta^3 \ll 1$, where $G(\nu_0)$ has a value of the order of several units [the estimations can be done by developing $G \ln G - \nu G'$ in (44) in the series in powers of ν , or by putting $G \simeq \nu/gA_0$. In the first case we obtain $\nu_0 = g$, $G(\nu_0) = 2$, in the second $\nu_0 = eg$, $G \simeq e/A_0$.]

Putting in (43) $\nu = \nu_0$ and inserting, from (44),

$$[G(\nu_0)]^{-1/\nu} = \frac{a\xi - b}{2\beta\xi} \Phi_0$$

where

$$\Phi_0 = \nu_0^2 / (\ln G - \nu G/G)_{\nu=\nu_0}$$

is a quantity of order of unity, we obtain

$$\varphi(k_0) = (ik_0 - \frac{\Phi_0}{2\beta})(b - a\xi) = - \left(\frac{3 - 2gA_0 + \Phi_0}{2\gamma + 1} \right) \alpha (b - a\xi)$$

and

$$C_1(\xi, b) \sim e^{\varphi(k_0)}$$

as at vB_0g we have $-ik = (3 - 2A_0g)/2\beta$.

A comparison with the Pomeron contribution (18) $v(\xi, b) \sim \exp[\kappa(a\xi - b)]$ shows that at $a\xi - b > 0$ the inequality $C_1(\xi, b) \gg v(\xi, b)$ holds at $3 + \Phi_0 2A_0'g < 2\gamma + 1$, i.e., at

$$\gamma > 1 + \frac{\Phi_0 - 3A_0g}{2}$$

This s unitarity condition is close to the condition $\gamma > 1$ of the t unitarity obtained in Section 3.

It is now easy to check that the condition $|\omega + ika| \ll |ka|$ is really fulfilled for $\omega = \omega_0(k_0)$, $k = k_0$. Indeed

$$|\omega_0(k_0) + ika| \approx [G(v_0)]^{1/v_0} = \exp \left[-\frac{1}{gA_0} \ln G(gA_0) \right]$$

is very small, as compared to unity, because $\ln G(A_0g)$ is of the order of unity, and $1/g \sim 1/\Delta^3$. On the other hand, $|ak_0| \approx 3a/2\beta = 3\alpha\kappa/(2\gamma + 1)$ is only of the order of Δ . Thus the ratio $|k_0a/(\omega_0 + k_0a)| \sim \Delta \exp(C/\Delta^3)$ is very large at $\Delta \ll 1$.

5. CONCLUSIONS. THE REGION $\xi > 1$, $\xi\Delta < 1$ OF THE ACCESSIBLE ENERGIES

The main results of the analysis made above for a range of extremely large energies, where $\xi > 1/\Delta \gg 1$, are as follows :

a) The threshold singularity contribution was found to be very important at the edge of the Pomeron profile at $\Delta > a'\kappa^2$. In the $\Delta < a'\kappa^2$ case, they are of no importance. However, there are strong difficulties with the s channel unitarity condition. The singularity of the type $(t - \kappa^2)$ in Pomeron trajectory, where $\kappa^2 = 4u_\pi^2$, leads to the dependence $\sim e^{-b\kappa}$ of the Pomeron profile on the impact parameter b for large $b > 2\alpha'\kappa\xi$.

b) The partial t channel amplitude, corresponding to the multi-Pomeron exchange, Fig. 1a, (to the Froissaron) was calculated - taking into account the Pomeron profile threshold behaviour. It was found to have "soft" singularities of the type $(\omega \pm i\kappa)^{-\nu}$, where $(-\nu)$ is positive for $\sqrt{t} > \kappa$ at the condition $\gamma > 1$. It preserves the t channel two-particle unitarity condition.

c) The analysis of the integral equation for the total contribution of all enhanced Froissaron graphs has shown that, at a sufficiently small Froissaron coupling constant, at $g_{00} a^2/2 < \Delta^3$ and at $\gamma > 1$ (more exactly at $\gamma > \gamma_0$, where γ_0 exceeds a lot unity), it has a solution in the form of an iteration expansion in powers of $g = g_{00} a^2/2$. Actually, the behaviour of the sum of contributions of the t channel chains of Froissaron graphs $C(\xi, b)$, Fig. 8a, was analyzed in this paper. The set of more complicated graphs of the $D(\xi, b)$ class, Fig. 8b, was not considered. They can be represented by the eikonal set, Fig. 9, of the t channel chains, Fig. 8a, connected to each other by a large number of Froissaron lines. Rejection of these lines leads to a considerable enhancement of the contribution of these graphs, because some of these lines lead to extremely small factors of the type $1 - F(y_1, b_1)$ (the so-called Cardy cancellation). After this rejection $D(\xi, b)$ is represented (Fig. 10b) by the eikonal expansion of type (22)

$$D(\xi, b) \simeq \sum_n \frac{(-1)^{n-1}}{n} |C(\xi, b)|^n C'_n,$$

where C'_n are some coefficients, such as C_n in (22). At $C'_n \lesssim \Gamma(n)$ this series reduces to the form (23)

$$D(\xi, b) \simeq 1 - e^{-|C(\xi, b)|} < |C(\xi, b)|$$

and, hence, is smaller than $v(\xi, b)$ if $|C(\xi, b)| < v$.

We cannot give a strong proof of this result. However, we could not find any set of graphs belonging to the class D and rising at $\xi \rightarrow \infty$ steeper than $v(\xi, b)$.

d) Results a)-c) mean that the Froissaron theory can be a good candidate for the real self-consistent scheme of strong interactions at extremely high energy $\xi > 1/\Delta \gg 1$ and at small $p_\perp \sim m$. The zero order approximation of this theory is the zero order F or the well-known (since the AFS paper!) eikonalized Pomeron. It can be an extremely good approximation in the available energy region $E_{LAB} \sim 10^2 - 10^4$ GeV, $\xi \sim 6-8$, where the enhanced graphs are not at all essential. Here the scattering amplitude

$$M \simeq M_0(\xi, k) = g^2 e^{\xi_1 \Delta} \sum_n \frac{(-z)^{n-1}}{n \cdot n!} e^{-\frac{\Delta k^2}{n}} C_n, \quad (45)$$

depends, in general, on the form of the "eikonalization" (i.e., on the choice of C_n). The rescattering parameter

$$z = g^2 e^{\xi_1 \Delta} / \lambda$$

with $\lambda = R^2 + \alpha'(0)\xi_1$, $\xi_1 = \xi - (i\pi/2)$, determines (at $C_n = 1$) the relative values of each successive rescattering and R is the Pomeron residue radius: $g^2(k^2) = g^2 \exp(-R^2 k^2)$. The parameter z is almost real, and $-$ as a function of $\xi -$ has a minimum, shown in Fig. 10, at $\xi = \xi^0$, where

$$\frac{z'}{z} = \Delta - \frac{\alpha'}{\lambda(\xi = \xi^0)} = 0 \quad (46)$$

In the region of this minimum, it is almost constant. At $z \simeq \text{const}$, the amplitude (45) reveals the properties of the geometrical scaling and yields, in particular, the equality

$$\frac{\sigma^{\text{tot}'} }{\sigma^{\text{tot}}} = \frac{B'(\xi)}{B(\xi)} = \Delta, \quad (47)$$

where the prime means derivative over ξ ($\sigma^{\text{tot}'} = d\sigma^{\text{tot}}(\xi)/d\xi$) and B is the diffractive cone slope $B = 2\text{Re}[dM/dk^2 M(k^2)]_{k^2=0}$. From (45), the value of B is found to be close to 2λ (at $C_n \equiv 1$). The four Pomeron parameters

$$\Delta, \quad \alpha', \quad R^2, \quad g^2$$

can be chosen so as to fit Eq. (46) and to provide a geometrical scaling in the Batavia-ISR region, where $\xi = \xi^0 = 6$, and where experiment strongly supports Eq. (47), giving for the ratio $\sigma^{\text{tot}'} / \sigma^{\text{tot}}$ the value $\Delta = 0.07 \pm 0.01$. Simultaneously, it gives in the pp scattering case $B/2 \simeq \lambda(\xi = \xi^0) \simeq 5(\text{GeV}/c)^{-2}$, which implies, from (46), $\alpha' \simeq \Delta \cdot B/2 = 0.35(\text{GeV}/c)^{-2}$.

Note that the condition $\Delta > \alpha' \kappa^2$ is fulfilled strongly as $\alpha' \kappa^2 = \alpha' 4\mu^2 \simeq 0.35(2/7)^2 \simeq 0.03$ and $\Delta = 0.07$. The value of R^2 can be obtained from the equation $\lambda = R^2 + \alpha' \xi^0 = B/2 = 5(\text{GeV}/c)^2$, giving $R^2 = 3(\text{GeV}/c)^2$, and g^2 can be readily found from the experimental data for the ratio $\sigma^{\text{tot}}/B(\Delta)$ [which, because of (45), turns out to be the function of z only].

A more careful numerical analysis confirms these values of the parameters and gives - taking account of other poles f, w, ρ, A contributions ²⁾, a good description of the experimental data for $\sigma^{\text{tot}}, d\sigma/dt, B(\Delta), \text{Re} M(5,0)$ etc. The details are beyond the scope of this paper. We mention only that the use of values of the coefficients C_n in (22) and (45) corresponding to the resonance state model, Fig. 5, leads to a better fit than the rough eikonal model, where $C_n \equiv 1$. With the same set of parameters the theory gives a good fit of some of the features of the particle production processes. The cross-sections of different types of such processes can be associated through the AGK ²³⁾ rules with different terms in the imaginary part of the amplitude (46). In particular, the theory gives a good description of the effect of a broadening of the distribution over multiplicities with increasing energy ^{3),4)} the mean multiplicity, (45), it gives a value rising as s^Δ at large energies ^{4),6)}.

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APPENDIX

RESCATTERINGS IN THE RESONANCE STATE MODEL (FIG. 6)

The formulae for the scattering amplitude $F_0(\xi, b)$ (Fig. 6b) for the amplitude of the diffraction production of one-resonance state $F_{01}(\xi, b)$ (Fig. 6c) and for the double diffraction production (Fig. 6d) was written in Section 3a. They include coefficients $\rho_v, \lambda_v, \lambda'_v, \lambda''_v$ which we give here for the case of the collision of two identical particles, with the Pomeron vertex of the form of the matrix $g = \{g_{ik}\}$,

$$g_{00} = g_0(1+\alpha), \quad g_{01} = g_{10} = g_0\beta, \quad g_{11} = g_0(1-\alpha).$$

Here g_{00} is the Pomeron vertex of the elastic scattering of particle A. g_{01} corresponds to the $A \rightarrow A^*$ transition on the Pomeron, and g_{11} is the vertex of the elastic scattering of the resonance state $A^* \rightarrow A^*$.

The coefficients $\rho_v, \lambda_v, \lambda'_v, \lambda''_v$ for $v=1,2,3$ have the following explicit values

$$\rho_1 = (1+\gamma)^2, \quad \rho_2 = (1-\gamma)^2, \quad \rho_3 = 1-\gamma^2$$

$$\lambda_1 = \left(\frac{1+\alpha\gamma}{2}\right)^2, \quad \lambda_2 = \left(\frac{1-\alpha\gamma}{2}\right)^2, \quad \lambda_3 = \beta^2/2\gamma^2$$

$$\lambda'_1 = \frac{\beta}{2\gamma}(1+\alpha\gamma), \quad \lambda'_2 = \frac{\beta}{2\gamma}(1-\alpha\gamma), \quad \lambda'_3 = -\alpha\beta/2\gamma^2$$

$$\lambda''_1 = \lambda''_2 = (\beta/2\gamma)^2, \quad \lambda''_3 = -\beta^2/2\gamma^2$$

and $\gamma = \sqrt{\alpha^2 + \beta^2}$. They fulfil the equations $\Sigma \lambda'_v = \Sigma \lambda''_v = 0$, $\Sigma \lambda_v = \Sigma \rho_v \lambda_v = 1$.

REFERENCES

- 1) A. Capella and J. Kaplan, Phys. Letters 52B, 448 (1974);
A. Capella, J. Tran Thanh Van and J. Kaplan, Preprint LPTHE 75/72.
- 2) A.M. Lapidus, V.I. Lisin, K.A. Ter-Martirosyan and P.E. Volkovitsky,
Preprint ITEP-115 (1976).
- 3) K.A. Ter-Martirosyan, Phys. Letters 44B, 337 (1973).
- 4) K.A. Ter-Martirosyan and Y. Shabelsky, JEPT (1976), in print.
- 5) V.N. Gribov, JEPT 53, 654 (1967).
- 6) J. Cardy, Nuclear Phys. B75, 413 (1974).
- 7) M.S. Dubovikov and K.A. Ter-Martirosyan, Proceedings of Coral Gables
Conference 1976, Preprint ITEP-37 (1976).
- 8) M.S. Dubovikov, L.I. Lapidus, B.Z. Kopeliovich and K.A. Ter-Martirosyan,
Dubna Preprint D2-9789 (1976).
- 9) V.N. Gribov and I.Ya. Pomeranchuk, JEPT 43, 308 (1962).
- 10) M. Cheng and T.T. Wu, Phys. Rev. Letters 24, 1465 (1970); Phys. Rev.
D1, 2775 (1970).
- 11) S.J. Chang and T.M. Yan, Phys. Rev. Letters 25, 1586 (1970);
Phys. Rev. D4, 537 (1971).
- 12) S.J. Chang, J.K. Wolker and T.T. Wu, Phys. Letters B44, 97 (1974);
Phys. Letters B44, 283 (1974).
- 13) S. Auerbach, R. Avro, R. Blankenbeckler and R. Sugar, Phys. Rev.
Letters 29, 522 (1972); Phys. Rev. D6, 2266 (1972).
- 14) L.D. Soloviev, JEPT Letters 18, 455 (1973); 19, 185 (1974);
L.D. Soloviev and A.V. Schelkachev, Sov. J. of Particles and Nuclei,
Vol. 6 (1975), p. 3.
- 15) V.F. Edneral, S.M. Troshin and N.E. Tyurin, IHEP Preprint OTQ 76-54
(1976).
- 16) V.F. Edneral, O.A. Khrustalev, S.M. Troshin and N. Tyurin, CERN
Preprint TH.2126 (1976).
- 17) J.B. Bronzan, Phys. Rev. D9, 2347 (1974).
- 18) J.L. Cardy, Nuclear Phys. B75, 413 (1974); SLAC-PUB 1784 (1976).
- 19) H.D.I. Abarbanel, J.B. Bronzan, R.L. Sugar and A. Schwimmer, to be
published in Phys. Rev. D (1976).
- 20) D. Amati, M. Ciafaloni, M. Le Bellac and G. Marchesini,
Nuclear Phys. B112, 107 (1976).
- 21) D. Amati, G. Marchesini, M. Ciafaloni and G. Parisi,
Nuclear Phys. B114, 483 (1976).
- 22) A.R. White, CERN Preprint TH.2259 (1976).
- 23) V.A. Abramovsky, V.N. Gribov and Q.V. Kanchely, Yad. Fiz. 18, 895 (1973).
- 24) M.S. Dubovikov and K.A. Ter-Martirosyan, Yad. Fiz. (in press).

FIGURE CAPTIONS

- Fig. 1 : a) The multi-Pomeron exchange forming the zero-order Froissaron.
 b) The enhanced graphs consisting of two multi-Pomeron blocks:
 $y_1, \vec{b}_1$ are the rapidity and impact parameter of the vertex
 g_{00} on the lab. system, where the lower particle is at
 rest.
 c) and d) Different types of the Froissaron enhanced graphs
 consisting of three multi-Pomeron blocks.

Fig. 2 : The Froissaron profile.

Fig. 3 : The Pomeron exchange graph.

Fig. 4 : The integration contour on the k plane in (13).

Fig. 5 : The dependence of the exponent on (15) on $r = |k - ix|$ for
 the case $b > 2\alpha' x\xi$, $0 = \alpha' x^2$.

Fig. 6 : The eikonal model (a) and the model of resonance intermediate
 states (b) for the elastic scattering, (c) and (d) for the
 processes of the diffraction production of one- and two-
 resonance states.

Fig. 7 : The singular points on the $w = j - 1$ plane of the t channel
 partial wave (30).

Fig. 8 : The Froissaron graphs of class (a) $C(\xi, b)$ and (b) $D(\xi, b)$.

Fig. 9 : The reduction of graphs of the class D to the eikonal set of
 chains $C(\xi, b)$.

Fig. 10 : The dependence on ξ of the parameter $Z = g^2 e^{\xi\Delta} / \alpha' \xi + R^2$.

FIG.1

FIG.5

FIG.6

FIG.7

FIG.8

FIG.9

FIG.10