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CONSISTENT SUPERGRAVITY

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A B S T R A C T

A combined spin 2 - spin $\frac{3}{2}$ extension of general relativity is given which is both free of the usual higher spin inconsistencies and invariant under local supersymmetry transformations.

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The unification of the gravitational field with a spin $\frac{3}{2}$ system is a natural goal within the framework of supersymmetry ¹⁾. In constructing such a theory one faces obvious problems due to the highly non-linear nature of general relativity and to the well-known difficulties ²⁾ encountered in coupling higher spin fields in a consistent way. We shall show here that the simplest candidate, namely the sum of the Einstein action and that for a massless, minimally coupled, Rarita-Schwinger ³⁾ Majorana field fulfils the consistency criteria. As we shall also see, this fact is related to the invariance of the theory under local supersymmetry transformations ^{*}).

The key to our results lies in the use of the first order formalism for gravitation, in which vierbeins and connection coefficients are treated independently (a convenient description of the first order formalism can be found in a paper by Kibble ⁵⁾). Minimal coupling in this sense implies the existence of torsion, or of non-minimal contact interactions in second order language. The first order formulation with torsion is closely related to the description of supergravity in superspace ^{1),6)}.

The combined Lagrangian has the form ^{**)}

$$L = -\frac{1}{2} e R - \frac{i}{2} \epsilon^{\lambda\mu\nu\rho} \bar{\psi}_\lambda \gamma_5 \gamma_\mu D_\nu \psi_\rho \quad (1)$$

where

$$e = \det e_\mu^a, \quad R = e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}. \quad (2)$$

^{*}) Local supersymmetry transformations, with parameters depending arbitrarily upon the space-time co-ordinates were first discussed in Ref. 4), where their existence was stated for two space-time dimensions and conjectured for four dimensions. The commutator of two local supersymmetry transformations contains a general co-ordinate transformation.

^{**)} We choose units in which $\kappa = 1$. Our gamma matrices satisfy $(\gamma_0)^2 = (\gamma_5)^2 = -1$. If we use the Majorana representation the matrices γ_μ and γ_5 are real and the field ψ_μ Hermitean, with $\bar{\psi}_\mu = \psi_\mu \gamma^0$. We also take $\epsilon_{0123} = -\epsilon_{0123} = 1$.

The covariant derivative on ψ_μ is defined according to its spin $\frac{1}{2}$ content only

$$D_\mu = \partial_\mu - \frac{1}{2} \omega_{\mu,ab} \Sigma^{ab}, \quad \Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] \quad (3)$$

and satisfies

$$[D_\mu, D_\nu] = - \frac{1}{2} R_{\mu\nu,ab} \Sigma^{ab}. \quad (4)$$

This form for D_μ is strongly suggested by the Maxwell-like gauge invariance of the flat space action under $\delta\psi_\mu = 2\partial_\mu \alpha$ and preserves the simplest definition of the curl. This results in a particularly simple form of the torsion. The vierbeins $e_{\mu a}$, the connection coefficients $\omega_{\mu,ab}$ and the Majorana vector-spinor ψ_μ are to be varied independently. Note that we have not introduced any auxiliary fields. They will be useful, however, when coupling to matter is included. The equations of motion are

$$R^\lambda \equiv \varepsilon^{\lambda\mu\nu\rho} \left(\gamma_\mu D_\nu \psi_\rho - \frac{1}{4} \gamma_\tau C_{\mu\nu}{}^\tau \psi_\rho \right) = 0 \quad (5)$$

$$C_{\mu\nu}{}^\tau = \frac{i}{2} \bar{\psi}_\mu \gamma^\tau \psi_\nu \quad (6)$$

and

$$G^{\tau\mu} = \frac{i}{2} \varepsilon^{\lambda\mu\nu\rho} \bar{\psi}_\lambda \gamma_5 \gamma^\tau D_\nu \psi_\rho. \quad (7)$$

Here

$$G^\tau{}_a = R^\tau{}_a - \frac{1}{2} e_a{}^\tau R, \quad R_{\tau a} = R_{\tau\lambda, a}{}^\lambda \quad (8)$$

is the (non-symmetric) Einstein tensor and

$$\begin{aligned} C_{\mu\nu}{}^a &= D_\mu e_\nu{}^a - D_\nu e_\mu{}^a \\ &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_{\mu,}{}^a{}_b e_\nu{}^b + \omega_{\nu,}{}^a{}_b e_\mu{}^b \end{aligned} \quad (9)$$

is the torsion. In general the consistency problem arises from the fact that, if one applies D_λ to the left-hand side of the Rarita-Schwinger equation, one obtains a constraint which strongly restricts the manifold of solutions. In our case, however, one can verify that $D_\lambda R^\lambda$ vanishes as a consequence of the other equations of motion (6) and (7). The calculation makes use of (4) and of the cyclic identity

$$\varepsilon^{\lambda\mu\nu\rho} (R_{\lambda\mu,\nu}{}^a - D_\lambda C_{\mu\nu}{}^a) = 0. \quad (10)$$

One finds, identically,

$$D_\lambda R^\lambda \equiv \frac{1}{2} G^{\lambda\mu} \gamma_5 \gamma_\lambda \psi_\mu + \frac{1}{4} \varepsilon^{\lambda\mu\nu\rho} C_{\lambda\mu}{}^\tau \gamma_\tau D_\nu \psi_\rho. \quad (11)$$

Inserting (6) and (7) and making use of a simple Fierz rearrangement one verifies that the two terms cancel ^{*)}.

The identity (11) suggests that the action $I = \int d^4x$ corresponding to (1) is invariant under the local supersymmetry transformation

$$\delta e_\mu{}^a = i \bar{\alpha} \gamma^a \psi_\mu \quad (12)$$

$$\delta \psi_\mu = 2 D_\mu \alpha \quad (13)$$

$$\delta \omega_{\mu,}{}^{ab} = B_\mu{}^{ab} - \frac{1}{2} e_\mu{}^b B_c{}^{ac} + \frac{1}{2} e_\mu{}^a B_c{}^{bc}, \quad (14)$$

^{*)} Note the importance of the fact that in (11) enters the Einstein tensor $G^{\lambda\mu}$ and not the full uncontracted Riemann tensor, for which the field equations do not provide sufficient information. For a charged spin $\frac{3}{2}$ particle in interaction with the Maxwell or Yang-Mills field one obtains a dependence on $F_{\mu\nu}$ in the equation analogous to (11) and the check of consistency fails. Presumably a similar situation arises in a spin 2 - spin 5/2 extension of general relativity. We mention that, in our case, the vanishing of the constraint (11) implies the absence of anomalous characteristic hypersurfaces.

where

$$B_a{}^{\lambda\mu} = i\bar{\alpha}\gamma_5\gamma_a D_\nu\psi_\rho \varepsilon^{\lambda\mu\nu\rho}. \quad (15)$$

Indeed (11) implies that the terms linear in ψ_μ (and its derivatives) vanish in δI [it is amusing that the calculation done directly on the action avoids use of the cyclic identity (10)]. A simple Fierz rearrangement shows that the remaining terms in δI , which are cubic in ψ_μ , also cancel identically. From the invariance of the action

$$\delta I = \int \left(\frac{\delta I}{\delta e_\mu{}^a} \delta e_\mu{}^a + \frac{\delta I}{\delta \omega_{\mu,ab}} \delta \omega_{\mu,ab} + \frac{\delta I}{\delta \psi_\mu} \delta \psi_\mu \right) d^4x = 0 \quad (16)$$

one concludes that the left-hand sides of the equations of motion are not independent. Using (13) and integrating by parts one recovers immediately the result that $D_\lambda R^\lambda = 0$ is satisfied as a consequence of the equations of motion obtained varying $e_\mu{}^a$ and $\omega_{\mu,ab}$. In this sense local supersymmetry implies consistency. Notice that we do not expect a Noether current here, because we are dealing with a local symmetry and hence with conservation identities. The conserved spinor current is $J^\mu \equiv R^\mu(e_\lambda{}^a, \omega) - R(e_\lambda{}^a = \delta_\lambda{}^a, \omega = 0)$ just as $G_{\mu\nu} - G_{\mu\nu}^{\text{LIN}}$ is the conserved energy momentum tensor in pure gravity.

Our results can be compared with some interesting recent work by Freedman, van Nieuwenhuizen and Ferrara ⁷⁾. These authors, having observed that the minimal second order coupling does not admit a local supersymmetry, are led to introduce an additional quartic contact term, whose form they simply guess. The correspondence is most easily established by using the equivalence formulas of Kibble's paper ⁵⁾. The contact term obtained by substituting (6) into (1) is

$$L_4 = \frac{1}{32} \left[\bar{\psi}^b \gamma^a \psi^c (\bar{\psi}_b \gamma_a \psi_c + 2 \bar{\psi}_a \gamma_b \psi_c) - 4 (\bar{\psi}_a \gamma \cdot \psi)^2 \right] \quad (17)$$

which turns out to be a simplified form of the expression adopted by the above authors. Furthermore, the form they suggest for $\delta\psi_\mu$ can be obtained from ours if one uses in (13) the expression

$$\omega_{\mu,ab} = \omega_{\mu,ab}(e) - \frac{i}{4} (\bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_\mu \gamma_a \psi_b - \bar{\psi}_\mu \gamma_b \psi_a) \quad (18)$$

obtained by solving (6) for $\omega_{\mu,ab}$. Here

$$\omega_{\mu,ab}(e) = -\frac{1}{2} (c_{\mu,ab} - c_{a,b\mu} - c_{b,\mu a}) \quad ,$$

$$c^a{}_{,\mu\nu} = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a \quad , \quad (19)$$

is the usual expression for the connection in terms of the vierbein in the second order formalism. The verification that the action is invariant is much more complicated in second order form and involves checking the vanishing of terms up to complicated quintic powers in ψ_μ , while, as mentioned above, we find at most relatively simple cubic terms.

We now speculate on some implications of supergravity. At the one-loop level general relativity is non-renormalizable when coupled to spin 0, $\frac{1}{2}$, or 1 fields ²⁾, but source free gravity is renormalizable because of an identity reducing the invariant $(R_{\mu\nu\alpha\beta})^2$ to $(R_{\mu\nu})^2$ and R^2 . Similar identities for invariants constructed from ψ_μ also exist. These, together with the strong constraints imposed by supersymmetry on the possible one-loop counterterms may well keep supergravity one-loop renormalizable as well.

As for observational consequences, our action does not alter any of the long-range properties of Einstein theory, since spinor exchange is always short range. Likewise, even if the mass of the spin $\frac{3}{2}$ particle vanishes, there will be no radiative catastrophes. The only possible problem could be the instability of the proton (similarly for the electron) if the spinor current to which ψ_μ couples universally with gravitational strength does not conserve baryon number, and the other particle forming the current together with the proton is lighter than it, for instance the pion. However, this form of the theory is not a natural one : from the point of view of supersymmetry it is more natural to assume the existence of bosons carrying baryon number and that the spinor current conserves baryon number. At any rate, whatever mechanism breaks supersymmetry for the observed matter supermultiplets it would also be expected to give the field ψ_μ a mass, perhaps quite large.

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