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SINGULAR BINDING DEPENDENCE IN THE HADRONIC WIDTHS
OF 1^{++} AND 1^{+-} HEAVY QUARK ANTIQUARK BOUND STATES

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A B S T R A C T

The annihilation rates into hadrons of P wave heavy quark-antiquark bound states are calculated within $SU(3)$ colour gauge theory (in particular for the charm scheme). An interesting feature we find is a logarithmic divergence for small binding for the states 1^{++} and 1^{+-} . Implications for the asymptotic freedom approach to the decay rates of the new particles are discussed. An attempt to use quantitatively the obtained results for all the C-even P-waves gives $\Gamma_{\text{ann}}(0^{++}) : \Gamma_{\text{ann}}(2^{++}) : \Gamma_{\text{ann}}(1^{++}) \cong 15 : 4 : 1$.

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Asymptotic freedom¹⁾ of SU(3) colour-gauge theory²⁾, together with the heavy-quark picture³⁾, apparently provides for a suitable framework to discuss the rates of decay of the new particles. For the ratio of $\Gamma(\psi \rightarrow \text{hadrons})$, calculated in this scheme as a three-gluon annihilation, to $\Gamma(\psi \rightarrow e^+e^-)$ one has³⁾

$$\frac{\Gamma(\psi \rightarrow \text{hadrons})}{\Gamma(\psi \rightarrow e^+e^-)} \cong \frac{5(\tau^2 - 9)}{18\pi} \frac{\alpha_s^3}{\alpha^2} \quad (1)$$

where $\alpha_s = (g^2/4\pi)$, g being the running coupling constant of the colour-gauge theory, in the sense of renormalization group, at the ψ -mass value. Equation (1) gives $\alpha_s \cong 0.2$. For the predicted pseudoscalar state η_c one then expects³⁾

$$\frac{\Gamma(\eta_c \rightarrow \text{hadrons})}{\Gamma(\psi \rightarrow \text{hadrons})} \cong \frac{27\pi}{5(\tau^2 - 9)} \frac{1}{\alpha_s} \cong 100 \quad (2)$$

while, independently of the value of α_s ⁴⁾,

$$\frac{\Gamma(2^{++} \rightarrow \text{hadrons})}{\Gamma(0^{++} \rightarrow \text{hadrons})} = \frac{4}{15} \quad (3)$$

for the lowest 2^{++} , 0^{++} P-wave charmonium states, which are identified here as members of the multiplet lying between ψ and ψ' .

To understand the theoretical basis for such predictions, it is convenient first to imagine a situation where the constituent (heavy quark) has very large mass m [as it would perhaps better apply to still heavier quarks, like t and b ⁵⁾, if they should be required]. Then the short-distance Coulomb-like force of the gauge theory would presumably play the dominant role in describing the characteristics of the lowest bound states, giving, for instance, binding energies ϵ of order $m\alpha_s^2$. Assuming a smooth behaviour for vanishing light-quark masses, the total hadronic widths would then appear as a genuine short-distance effect, m being the *only* relevant mass scale ($\epsilon \sim m/\lg^2 m$ in this regime). The positronium-like predictions in Eqs. (1), (2), and (3) would then follow.

For the new particles, ψ , ψ' , etc., such a very massive limiting situation does not seem to apply. For instance, one would have a value of 8 for the ratio of the leptonic width of ψ to that of ψ' ³⁾, and other unsupported predictions like the ψ' degeneracy with P-states, etc. Clearly, retaining the general framework, the long-distance confinement mechanism comes into play to modify the short-distance predictions: technically, one can think of ϵ as bringing into the problem a small scale which prevents the naïve use of perturbative calculations.

A more sophisticated analysis⁶⁾ has now to be attempted to separate short-distance from long-distance effects to lead to the predictions such as in Eqs. (1), (2), and (3). The total rate for $\Gamma(\psi \rightarrow \text{hadrons})$, for instance, would, in general, be described as

$$\Gamma = \frac{1}{2} \int d^4p \text{Tr}[\psi(p)N(p)] d\phi \int d^4p' \text{Tr}[N^+(p)\psi^+(p)] \quad (4)$$

where $\psi(p)$ is the heavy quark-antiquark bound state wave function of relative momentum p , $N(p)$ symbolically denotes the annihilation amplitude, and $d\phi$ is the final-state phase space. [By definition $N(p)$ does not contain heavy quark-antiquark reducible diagrams in the annihilation channel, which are absorbed in the wave function⁶⁾.] If the low-momentum components of $\psi(p)$ are still the important ones, one has the factorization

$$\Gamma \approx |\phi(0)|^2 \int d\phi |N(0)|^2 \equiv |\phi(0)|^2 M(\epsilon) \quad (5)$$

where $\phi(0)$ is the wave function in coordinate space at zero separation. $N(0)$ is the annihilation amplitude slightly below threshold ($s \approx 4m^2 - 4\epsilon m$) and for slightly off-shell heavy-quark momenta ($p_1^2 \approx m^2 - \epsilon m$). Now, provided the small binding limit, $\epsilon \rightarrow 0$, of $M(\epsilon)$ exists and is smooth enough, a separation of short-distance and long-distance effects is achieved, and one recovers the predictions in Eqs. (1), (2), and (3) also for realistic constituent mass values (but, of course, not for the ratio of the leptonic widths of ψ and ψ' , etc.). The same procedure applies in the case of the P-wave decays⁴⁾, the only modification being that the widths become proportional to the derivative of the spatial wave function and not to the wave function itself, which vanishes at the origin. However, *the limit $\epsilon \rightarrow 0$ of the corresponding $M(\epsilon)$ does not always exist.* We shall here, in fact, extend to the remaining P-states 1^{++} and 1^{+-} , the calculation already performed for 0^{++} and 2^{++} ⁴⁾, and find the result of a logarithmic divergence as $\epsilon \rightarrow 0$ in the Born approximation for $M(\epsilon)$. As a matter of fact, the same phenomenon will manifest itself also in computing higher-order contributions to the decays of the other P-waves as well as relativistic corrections to Eq. (5) for the S-wave decays.

To illustrate some of the technical features involved, let us present a corresponding QED calculation, exhibiting a logarithmic divergence in zero-binding limit. We consider a $\mu^+\mu^-$ system bound in $J^{PC} = 1^{++}$ and calculate its decay rate into a photon and a free e^+e^- pair, which represents the leading annihilation process since a vector particle cannot decay into two massless photons. The relevant diagrams are shown in Fig. 1. We introduce a fictitious width $\Gamma(a)$ for the decay of $(\mu^+\mu^-)$ into a virtual photon of mass squared a plus a real photon. We can write^{*}

$$\Gamma((\mu^+\mu^-) \rightarrow \gamma e^+e^-) = \frac{\alpha}{\pi^2} \int_{4m_e^2}^{\infty} \frac{da}{a} \Gamma(a) \text{Im} \Pi^{(2)}(a/m_e^2) \quad (6)$$

*) For the 1^{++} state, we are considering here, $\Gamma(0) = 0$, and then no divergence is exhibited by Eq. (6) in the limit $m_e \rightarrow 0$. For the other P-wave decays (0^{++} , 2^{++}), the absence of this kind of divergence is again ensured if one subtracts the photon propagator at a Euclidean point. In doing so one would, in fact, have an additional piece to the r.h.s. of Eq. (6) proportional to $\Gamma(0)$, which would cancel the divergence of the term shown. This point is relevant in connection with the smoothness of the behaviour for vanishing light-quark masses of the annihilation widths of the heavy-quark bound states.

where $\text{Im } \pi^{(2)}(a/m_e^2)$ is the electron-pair contribution to the vacuum-polarization spectral function and M is the bound-state mass. To calculate $\Gamma(a)$, from the P-wave bound state, the wave function $\psi(p)$ (see Fig. 2, where the momenta are indicated) must be known, including first-order relativistic corrections⁴⁾. The prescription giving $\psi(p)$ to this order is known from positronium theory⁷⁾

$$\psi_{JML1}(p) = \frac{1}{4m^2} (-i\mathbf{k}/2 - i\mathbf{p} + m) \frac{1}{2} (1 + \gamma_4) \sum_{e,m} \epsilon^{(e)} (i\mathbf{k}/2 - i\mathbf{p} + m) \cdot \frac{i}{12} \langle JM | 1L e m \rangle \tilde{\varphi}_{Lm}(\vec{p}) \delta(p_0) \quad (7)$$

where m is the constituent mass (muon mass here), $\tilde{\varphi}_{Lm}(\vec{p})$ is the L-wave Schrödinger wave function, and $\epsilon_{\mu}^{(\pm)} = -[1/\sqrt{2} (0, \pm 1, i, 0)]$, $\epsilon_{\mu}^{(0)} = (0, 0, 0, 1)$. The width $\Gamma(a)$, in Eq. (6), is given by

$$\Gamma(a) = \frac{M^2 - a}{M^2} \int \frac{d\Omega_k}{2^5 \pi^2} A_{\mu\nu} A^{\mu\nu} \quad (8)$$

$$A_{\mu\nu} = ie^2 \int d^4p \text{Tr} [\psi(p) N_{\mu\nu}(p)] \quad (9)$$

$$N_{\mu\nu} = \gamma_{\mu} [-i\mathbf{k}/2 - i\mathbf{p} + i\mathbf{k} - m]^{-1} \gamma_{\nu} + \gamma_{\nu} [i\mathbf{k}/2 - i\mathbf{p} - i\mathbf{k} - m]^{-1} \gamma_{\mu}, \quad (10)$$

where k is the momentum of the massless photon [in c.m. $|\vec{k}| = (M^2 - a)/2M$]. After performing the indicated traces and integrations, one finds, for $a \rightarrow 4m^2 \approx M^2$,

$$\Gamma(a) \approx \frac{32\alpha^2}{m^2} \frac{|\varphi_1'(0)|^2}{4m^2 - a} \quad (11)$$

where $\phi_1(r)$ is the radial wave function $\phi_{1m} = \phi_1 Y_{1m}$. The singular behaviour of $\Gamma(a)$ can be traced back to the divergence in $N_{\mu\nu}$ in the zero binding limit as $k \rightarrow 0$. Inserting Eq. (11) into Eq. (6) one has finally

$$\begin{aligned} \Gamma((\mu^+ \mu^-)_{1++} \rightarrow \gamma e^+ e^-) &\approx \frac{8}{3\pi} \frac{\alpha^3}{m^4} |\varphi_1'(0)|^2 \lg \frac{4m^2}{4m^2 - \mu^2} \approx \\ &\approx \frac{16}{3\pi} \frac{\alpha^3}{m^4} |\varphi_1'(0)|^2 \lg \left(\frac{1}{\alpha} \right) \end{aligned} \quad (12)$$

where the terms α^3 and $O[(m_e/m)^2]$ are neglected. Equation (12) is the QED prediction for the leading contribution to the total annihilation rate of a $J^{PC} = 1^{++} \mu^+ \mu^-$ bound state^{*)}. The origin of the logarithmic divergence in the zero binding limit, $\lg [1 - (M^2/4m^2)] \approx \lg(\epsilon/m)$ is in the zero photon mass and the consequent non-analytic behaviour of Green's function for annihilation of off-shell free muon-antimuon in the on-shell limit. Although such a behaviour does not manifest

*) This QED result does not exist in the literature; its experimental verification is impossible at present.

itself as an infrared singularity of the on-shell free-particle amplitude^{*}), still it has the effect of introducing a singular dependence on the binding energy in the expression for the physical width in Eq. (12).

A calculation of annihilation of a 1^{++} charmonium state into hadrons in asymptotic free-gauge theory differs in many respects from the simpler QED calculation we have shown. However, the feature of the logarithmic divergence in the zero-binding limit is again found. In the following, we shall present the main aspects of the calculation of the total annihilation state of 1^{++} and 1^{+-} charmonium states. Annihilation via three gluons goes through a process illustrated by diagrams like those shown in Fig. 3a (these diagrams have QED analogues) and in Fig. 3b (typical of non-Abelian theory). The diagram in Fig. 3a is as in QED, except for the colour factor, proportional to $\text{Tr}[\lambda^a \lambda^b \lambda^c]$, where a, b, c are the colour indices of the emitted gluons in a definite order. C-odd states decay through d-couplings, C-even states through f-couplings. The diagram in Fig. 3b is relevant only for C-even states. To the same order (α_s^3) in the chromodynamic fine-structure constant α_s one has for C-even states the diagrams in Fig. 1, where now the decay is into a gluon and a light-quark pair. For the bound-state wave functions $\psi_{JMLS}(p)$ we have Eq. (7) for $S = 1$, while for $S = 0$:

$$\psi_{JMLS}(p) = \frac{1}{4m^2} (-i\mathbf{k}/2 - i\not{p} + m) \frac{1}{2} (1 + \gamma_4) \gamma_5 (i\mathbf{k}/2 - i\not{p} + m) \frac{\delta(\mathbf{p}_0)}{\sqrt{2}} \delta_{LJ} \tilde{\varphi}_{LM}(\vec{p}) \quad (13)$$

Equations (7) and (13) apply strictly in the electromagnetic case⁷⁾. They apply here as well for large constituent mass.

Starting from Eq. (4), with the appropriate expressions for $N(p)$, the calculation of the decay amplitudes proceeds in the same way as in the QED example we have already discussed. The results^{**)} of this lengthy, but essentially straightforward analysis can be summarized as follows:

- i) The three-gluon decay width of the P-wave 1^{++} state has no zero-binding divergence, although divergences appear in individual diagrams. The leading contribution to the 1^{++} annihilation width thus comes from one-gluon plus light-quark-pair annihilation giving, with a minor modification of Eq. (12) due to colour factors,

$$\Gamma_{ann}(1^{++}) \cong \frac{n}{3} \frac{128}{3\pi} \frac{\alpha_s^3}{M^4} |\varphi_1'(0)|^2 \lg \frac{4m^2}{4m^2 - M^2} \quad (14)$$

where n is the number of light-quark flavours.

^{*}) For $\vec{p} = 0$ there is no divergence: in the infrared limit the numerator in $N_{\mu\nu}$ becomes $[\gamma_i(1 + \gamma_4)\gamma_j + \gamma_j(1 - \gamma_4)\gamma_i]$ ($\mu = i, \nu = j$ for real photons); ψ has the spinor factor $(1 + \gamma_4)\vec{E} \times \vec{Y}$, giving $\text{Tr}(N_{\mu\nu}\psi) = 0$.

^{**)} The actual calculation was made using SCHOONSCHIP, written by M. Veltman⁸⁾.

- ii) The three-gluon decay width of the P-wave 1^{+-} state has the logarithmic zero-binding divergence. We find

$$\Gamma_{\text{ann}}(1^{+-}) \cong \frac{5}{18} \frac{2}{\pi} \frac{\alpha_s^3}{M^4} |\varphi'_1(0)|^2 \lg \frac{4m^2}{4m^2 - M^2} \quad (15)$$

Equation (15) is what one would obtain for P-state 1^{+-} positronium 3γ -annihilation, except for the colour factor $5/18$, which would be unity for positronium and for the substitution $\alpha_s \rightarrow \alpha$.

Recalling also what we said in the Introduction, the emerging picture is then the following: Eqs. (14) and (15) have to be considered as genuine quantitative predictions of the SU(3) colour-gauge theory, with $\lg [4m^2/(4m^2 - M^2)] \approx -2 \lg \alpha_s$, in the limiting situation of very high constituent mass. This statement applies as well to Eqs. (1), (2), and (3) to the extent that logarithmic terms of the type we have encountered here are going to be present also when calculating non-leading contributions to the other P-wave total annihilation rates [for example, in the contribution from $0^{++}(2^{++}) \rightarrow q\bar{q} + \text{gluon}$] or relativistic corrections to the S-wave (1^{--} and 0^{-+}) annihilations. One may still attempt to use the results for an approximate description of the actual situation^{*)}. From Eqs. (14) and (15), taken together with the results of Ref. 4, one gets, provided $\varepsilon \geq \langle p^2 \rangle / m$, where $\langle p^2 \rangle$ is the average squared relative momentum in the bound state^{**) ,}

$$\frac{\Gamma_{\text{ann}}(1^{++})}{\Gamma_{\text{ann}}(0^{++})} \cong \frac{4}{9} \frac{\alpha_s}{\pi} \lg \frac{m}{\varepsilon}, \quad \frac{\Gamma_{\text{ann}}(1^{+-})}{\Gamma_{\text{ann}}(0^{++})} = \frac{10}{27} \frac{\alpha_s}{\pi} \lg \frac{m}{\varepsilon} \quad (16)$$

For $m/\varepsilon \sim 10$, $\alpha_s \sim 0.2$ one finds, for instance, $\Gamma_{\text{ann}}(1^{++})/\Gamma_{\text{ann}}(0^{++}) \approx 0.06$, and, with the absolute normalization coming from a previous phenomenological analysis⁴⁾ one has: $\Gamma_{\text{ann}}(0^{++}) \approx 2.4 \text{ MeV}$, $\Gamma_{\text{ann}}(2^{++}) \approx 640 \text{ keV}$, $\Gamma_{\text{ann}}(1^{++}) \approx 140 \text{ keV}$.

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*) Qualitatively one may also want to predict an approximate two-jet structure in 1^{++} annihilation. This is so since most of the energy goes to the light-quark pair because of the dominant logarithmic divergence for small binding, coming from the soft-gluon emission.

**) For $\varepsilon < \langle p^2 \rangle / m$ the long-distance force plays the important role. In this case the log factor in Eq. (16) will be replaced by $\lg(m^2/\langle p^2 \rangle)$. For a linear confining potential $V = \lambda r$ such log factors would be $\sim \lg(m^2/\lambda)$.

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Figure captions

Fig. 1 : Diagrams for $(\mu^+\mu^-)_{\text{bound}} \rightarrow \gamma + e^+ + e^-$.

Fig. 2 : Bound-state wave function $\psi(p)$.

Fig. 3 : Representative diagrams for the three-gluon annihilation of charmonium states.

FIG. 1

FIG. 2

FIG. 3