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SUPERSYMMETRIC STRINGS AND COLOUR CONFINEMENT

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A B S T R A C T

The (infinite-dimensional) supersymmetry algebra in $1+1$ space-time dimension is extended in order to incorporate, in a non-trivial way, an internal symmetry. It turns out that this requirement implies that the internal symmetry is realized as a local gauge symmetry. Moreover, it is possible to construct string-like models with this underlying symmetry, where colour confinement is exactly realized as a consequence of the gauge constraints.

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The interest in generalizing the two-dimensional supersymmetry algebra¹⁾ to include an internal symmetry²⁾ is twofold.

Firstly, it is a new example of a graded Lie algebra, quite different from its four-dimensional counterpart, because of the exceptional properties of the conformal group in two dimensions.

Secondly, its properties can be useful to construct and to exploit features of supersymmetric invariant field theories in two dimensions. In this respect, we have in mind possible generalizations of the supersymmetric string model (Neveu-Schwarz-Ramond) to the case where an internal quantum number (charge or isotopic-spin like) is distributed along the string. In at least two examples [internal symmetry $U(1)$ and $SU(2)$] we will show that one can indeed construct new supersymmetric string models where all physical excitations of the string have the property of being colour singlets because of the gauge conditions. In the present framework this gives an exact algebraic formulation of colour confinement. The explicit Lagrangian formulation of the string with the Abelian $[U(1)]$ colour symmetry will be given in this paper.

We start with the extension of the supersymmetry algebra in $1+1$ space-time dimension, which contains the infinite dimensional conformal algebra (the trivial c-number part has not been written for simplicity)

$$[L_n, L_m] = (n-m) L_{n+m} \quad (1)$$

and the supergauge charges with commutation and anticommutation relations

$$\begin{aligned} \{G_n, G_m\} &= 2 L_{n+m} \\ [G_n, L_m] &= \left(n - \frac{m}{2}\right) G_{n+m} \end{aligned} \quad (2)$$

to the case where the supergauge charges carry an internal symmetry index G_n^i .

The construction of the enlarged supersymmetry algebra is very simple. One can simply start from the composition law in the parameter space $(\sigma, \tau, \theta_\alpha^i)$ where σ, τ are space-time labels and θ_α^i are N Majorana spinors which are odd elements of a Grassmann algebra, i.e.

$$\theta_\alpha^i \theta_\beta^j = - \theta_\beta^j \theta_\alpha^i \quad (3)$$

which implies

$$\theta_\alpha^i \theta_\alpha^i = 0 \quad \forall \alpha, i \quad (4)$$

Because of the exceptional properties of conformal transformations in two dimensions, it is better to use light-cone parameters

$$\begin{aligned}\xi &= \sigma + \tau & \theta^i &= \frac{1}{2} (1 + \gamma_5)_{\alpha\beta} \theta_\beta^i \\ \zeta &= \sigma - \tau & \chi^i &= \frac{1}{2} (1 - \gamma_5)_{\alpha\beta} \theta_\beta^i\end{aligned}\quad (5)$$

and to confine ourselves to the ξ, θ^i part.

Under a restricted supersymmetry transformation ξ, θ^i undergo the following transformations

$$\begin{aligned}\delta\xi &= i \sum_i \alpha^i \theta^i \\ \delta\theta^i &= \alpha^i\end{aligned}\quad (6)$$

while under a conformal transformation of parameter $u(\xi)$:

$$\delta\xi = u \quad \delta\theta^i = \frac{\dot{u}}{2} \theta^i \quad (7)$$

The crucial point is that the most general transformation of the algebra can be generated combining the transformations (6) and (7).

The general formula one gets is the following

$$\begin{aligned}\delta\xi &= i (2-n) \alpha_{i_1 \dots i_n}(\xi) \theta^{i_1} \dots \theta^{i_n} \\ \delta\theta^i &= n \alpha_{i_1 \dots i_{n-1} i}(\xi) \theta^{i_1} \dots \theta^{i_{n-1}} + i \dot{\alpha}_{i_1 \dots i_n}(\xi) \theta^{i_1} \dots \theta^{i_n} \theta^i \\ n &= 1 \dots N\end{aligned}\quad (8)$$

where the $\alpha_{i_1 \dots i_n}$'s are commuting (anticommuting) parameters for n even (odd) and are completely antisymmetric in the indices, so that for a given n they have $\binom{N}{n}$ components. The total number of independent transformations is therefore $\sum_{n=0}^N \binom{N}{n} = 2^N$.

The algebra induced from Eqs. (8) contains as a particular case ($N = 0, 1$) the Virasoro algebra given by Eq. (1) and the Neveu-Schwarz algebra given by Eqs. (1) and (2). For $N = 2$, i.e. with the θ 's transforming as an $O(2)$ doublet, one gets the first graded Lie algebra which generalizes (1) and (2) to the case of an Abelian internal symmetry. The independent transformations obtained in this case from the general formula (8) are

$$\text{conformal transformations} \quad \delta\xi = u, \quad \delta\theta^i = \frac{\dot{u}}{2} \theta^i \quad (9a)$$

$$\text{supergauge transformations} \quad \delta\xi = i \alpha^i \theta^i, \quad \delta\theta^i = \alpha^i + i \dot{\alpha}^j \theta^j \theta^i \quad (9b)$$

$$\text{local } O(2) \text{ transformations } \delta \xi = 0, \delta \theta^i = T \varepsilon^{ij} \theta^j \quad (9c)$$

where $u(\xi)$, $T(\xi)$ and $\alpha(\xi)$ are arbitrary functions and $\alpha(\xi)$ is an anticommuting quantity [$\alpha^2(\xi) = 0$].

The abstract graded Lie algebra is (up to c-numbers which will be discussed in full detail elsewhere)

$$\begin{aligned} [L_n, L_m] &= (n-m) L_{n+m} \\ [L_n, G_m^i] &= \left(\frac{n}{2} - m\right) G_{n+m}^i \\ \{G_m^i, G_m^j\} &= 2 \delta^{ij} L_{n+m} + i \varepsilon^{ij} (n-m) T_{n+m} \\ [T_m, T_m] &= 0 \\ [T_m, G_m^i] &= i \varepsilon^{ij} G_{n+m}^j \\ [T_n, L_m] &= n T_{n+m} \end{aligned} \quad (10)$$

where the subscripts n, m run over the integers for the bosonic operators L_n, T_n and over the half integers for the supergauge operators G_n^i .

The interesting feature which emerges from this algebra is that (contrary to the four-dimensional case) the internal symmetry is realized as a local gauge symmetry in the string-space as is already evident from the general composition law given by Eqs. (8). The global transformation corresponds to the $n = 0$ mode T_0 (ξ -independent transformation), which is nothing but the generator of the $O(2)$ group. Note that the global group generator T_0 closes in a finite subalgebra (with L_0, L_1, L_{-1} and the supergauges $G_{\frac{1}{2}}, G_{-\frac{1}{2}}$) of the infinite system given by Eqs. (10). The supergauge charges G_n^i are $O(2)$ vectors with respect to both global and local $O(2)$ transformations. This will no longer be the case in the non-Abelian situation with $O(3)$ as an internal symmetry. In fact, for $N = 3$ the independent transformations derived from Eqs. (8) are

$$\text{conformal transformations } \delta \xi = u, \delta \theta^i = \frac{\dot{u}}{2} \theta^i \quad (11a)$$

$$\text{supergauge transformations (triplet)} \quad \delta \xi = i \alpha^i \theta^i, \delta \theta^i = \alpha^i + i \alpha^j \theta^j \theta^i \quad (11b)$$

$$\text{supergauge transformations (singlet)} \quad \delta \xi = \frac{1}{3!} \alpha \varepsilon^{ijk} \theta_i \theta_j \theta_k, \quad \delta \theta^i = \frac{i}{2} \alpha \varepsilon^{ijk} \theta_j \theta_k \quad (11c)$$

$$\text{local } O(3) \text{ transformations} \quad \delta \xi = 0, \quad \delta \theta^i = T^{ij} \theta^j + \frac{i}{2} T_{j\ell} \theta_\ell \theta^j \theta^i \quad (11d)$$

The corresponding abstract graded Lie algebra is

$$\begin{aligned} [L_n, L_m] &= (n-m) L_{n+m} & [L_n, G_m^i] &= \left(\frac{n}{2} - m\right) G_{n+m}^i \\ [L_n, \Gamma_m] &= \left(-\frac{n}{2} - m\right) \Gamma_{n+m} & [\Gamma_m, T_m^i] &= 0 \\ \{\Gamma_n, \Gamma_m\} &= 0 & \{\Gamma_n, G_m^i\} &= T_{n+m}^i \\ [T_m^i, G_m^j] &= i \varepsilon^{ijk} G_{n+m}^k + n \Gamma_{n+m} \delta^{ij} & [T_m^i, T_m^j] &= i \varepsilon^{ijk} T_{n+m}^k \\ \{G_m^i, G_m^j\} &= 2 \delta^{ij} L_{n+m} + i \varepsilon^{ijk} (n-m) T_{n+m}^k & [T_m^i, L_m] &= n T_{n+m}^i \end{aligned} \quad (12)$$

which show that a fourth supergauge charge Γ_n is generated from a local $O(3)$ rotation of the supergauges G_n^i . Γ_n is a local singlet, while the G_n^i 's are only global triplets.

For a general $O(N)$ symmetry the algebra (12) generalizes as follows

$$\begin{aligned} [G_m^{i_1 \dots i_R}, G_m^{j_1 \dots j_S}]_{(-1)^{RS+1}} &= i^{-RS} \{ [m(2-S) - n(2-R)] G_{n+m}^{i_1 \dots i_R j_1 \dots j_S} \\ &\quad - i \sum_{h=1}^R \sum_{k=1}^S (-1)^{h+k+S} \delta^{i_h j_k} G_{n+m}^{i_1 \dots \hat{i}_h \dots i_R j_1 \dots \hat{j}_k \dots j_S} \} \end{aligned} \quad (13)$$

where one has a commutator (anticommutator) if RS is even (odd). $G_n^{i_1 \dots i_R}$ is a completely antisymmetric tensor of the global $O(N)$ group and it is a bosonic (fermionic) quantity if R is even (odd). The hat on the indices i_h and j_k means that these indices are omitted.

The algebra (13) contains just 2^N independent (local) generators of which $2^{N/2}$ are bosons and $2^{N/2}$ are fermions. This implies that, in addition to the $[N(N+1)/2] + 1$ charges that one would expect from the conformal, supergauge and internal symmetry generators, there are $2^N - 1 - [N(N+1)/2]$ new charges.

An interesting new structure can be extracted from the algebra (13) in the case where $N = 4$, due to the peculiar property $O(4) = O(3) \otimes O(3)$. For $N = 4$ the algebra (13) contains 16 generators:

$$L_n, G_n^i, T_n^{ij}, \Gamma_n^{ijk} = \varepsilon^{ijkl} \Gamma_n^l, \Delta_n^{ijkl} = \varepsilon^{ijkl} \Delta_n$$

respectively the conformal, supergauge and internal symmetry generators plus the additional new charges. If we make now the following combinations:

$$\mathcal{L}_n = L_n + \frac{n^2}{2} \Delta_n, \mathcal{G}_n^i = G_n^i + i n \Gamma_n^i, \mathcal{T}_n^{ij} = \frac{1}{2} \left(T_n^{ij} + \frac{1}{2} \varepsilon^{ijkl} T_n^{kl} \right)$$

these new charges close under the following algebra

$$\begin{aligned} [\mathcal{L}_n, \mathcal{L}_m] &= (n-m) \mathcal{L}_{n+m} \\ \{\mathcal{G}_n^i, \mathcal{G}_m^j\} &= 2i(n-m) \mathcal{T}_{n+m}^{ij} + 2\delta^{ij} \mathcal{L}_{n+m} \\ [\mathcal{L}_n, \mathcal{G}_m^i] &= \left(\frac{n}{2} - m\right) \mathcal{G}_{n+m}^i \\ [\mathcal{L}_n, \mathcal{T}_m^{ij}] &= -m \mathcal{T}_{n+m}^{ij} \\ [\mathcal{G}_n^k, \mathcal{T}_m^{ij}] &= \frac{i}{2} [\delta^{ki} \mathcal{G}_{n+m}^j - \delta^{kj} \mathcal{G}_{n+m}^i] - \frac{1}{2} \varepsilon^{kijl} \mathcal{G}_{n+m}^l \\ [\mathcal{T}_n^{ij}, \mathcal{T}_m^{kl}] &= i [\delta^{ik} \mathcal{T}_{n+m}^{jl} + \delta^{jl} \mathcal{T}_{n+m}^{ik} - \delta^{il} \mathcal{T}_{n+m}^{jk} - \delta^{jk} \mathcal{T}_{n+m}^{il}] \end{aligned} \quad (14)$$

where the last commutator is just a realization of one of the $O(3)$ symmetries of $O(4)$. In particular, we observe that the two complex doublets $(G_3 \pm iG_4, G_1 \pm iG_2)$ just transform according to the spinor representation (and its complex conjugate) of $SU(2)$. The interesting fact is that no additional charges, besides conformal, supergauge, and internal symmetry, are present in this algebra and that it can be easily realized in terms of canonical fields, leading to the possibility of a new dual model with a non-Abelian $SU(2)$ colour symmetry^{*)}. (We call the local internal symmetry a colour symmetry because, as we shall see in detail for the $U(1)$ case, the gauge conditions imply that the physical states are singlet with respect to the local gauge group.) It is remarkable that, besides the local $O(3)$ algebra contained in Eqs. (14), one can build, in terms of the same canonical fields, another $O(3)$ algebra. Even if these charges are not gauge operators they are a symmetry of the model, because their commutators with the gauges is a combination of the gauges themselves, so that the physical states must belong to multiplets of this "flavour" global $O(3)$ symmetry. A detailed analysis of this model will be given in a following publication.

Let us now consider in more detail the $O(2)$ case (Abelian colour symmetry). We will show that, as in the Neveu-Schwarz model, one can construct a Lagrangian which is invariant under the algebra given by Eqs. (10) and which gives rise to a new dual model.

*) We remark that the Fock space representations of the $U(1)$ and $SU(2)$ strings are respectively obtained by replacing the (real) bosonic and fermionic oscillators of the Neveu-Schwarz model with complex and quaternionic oscillators.

The most general superfield³⁾ one can write is^{*}

$$X(\sigma, \tau, \theta^i) = \phi + i \bar{\theta}^i \psi^i + i \bar{\theta}^i \theta^j F^{ij} + \frac{i}{2} \varepsilon^{ij} \bar{\theta}^i \gamma^5 \theta^j D \\ + \frac{i}{2} \varepsilon^{ij} \bar{\theta}^i \gamma^\mu \theta^j B_\mu + i \bar{\theta}^i \theta^i \bar{\theta}^i \chi^i + (\bar{\theta}^i \theta^i)^2 G \quad (15)$$

This field can be reduced by imposing on it the supersymmetric condition

$$(\bar{D}^i D^j - \frac{1}{2} \bar{D}^h D_h \delta^{ij}) X = 0 \quad (16)$$

D^i being the covariant derivative²⁾. The reduced superfield becomes

$$X = (1 - \frac{1}{8} (\bar{\theta}^i \theta^i)^2 \square) \phi + i \bar{\theta}^i (1 + \frac{i}{2} \bar{\theta}^i \theta^i \not{\partial}) \psi^i + \\ + \frac{i}{2} \bar{\theta}^i \theta^i F + \frac{i}{2} \varepsilon^{ij} \bar{\theta}^i \gamma^5 (\not{D} - \not{\partial} C) \theta^j \quad (17)$$

The invariant action is

$$S = \int d\sigma d\tau d^4\theta X^2(\sigma, \tau, \theta_\alpha^i) = \\ = \int d\sigma d\tau \left\{ \frac{1}{2} \phi \square \phi - \frac{1}{2} \gamma^\mu C \partial_\mu C - \frac{i}{2} \bar{\psi}^i \not{\partial} \psi^i + \frac{1}{2} (F^2 + D^2) \right\} \quad (18)$$

which gives the following equations of motion

$$\square \phi = 0 \quad \square C = 0 \quad \not{\partial} \psi^i = 0 \quad F = D = 0 \quad (19)$$

together with the boundary conditions

$$(\not{D}_1^i - \not{D}_2^i) X = (\not{D}_1^2 + \not{D}_2^2) X = 0 \quad \text{at } \sigma=0 \quad \text{and } \theta_1^i + \theta_2^i = \theta_1^2 - \theta_2^2 = 0 \\ (\not{D}_1^i + \not{D}_2^i) X = (\not{D}_1^2 - \not{D}_2^2) X = 0 \quad \text{at } \sigma=\pi \quad \text{and } \theta_1^i - \theta_2^i = \theta_1^2 + \theta_2^2 = 0 \quad (20)$$

which are due to the fact that the σ integration is in the finite region $0, \pi$.

For simplicity, we limit our considerations to the meson sector of the model.

The invariance of the action under conformal, supergauge, and local $O(2)$ transformations implies via the Noether theorem the existence of a conserved supercurrent⁴⁾, which in our case is given by

$$V^\mu(\xi^\mu, \theta_\alpha^i) = \varepsilon^{ij} \bar{D}^i X \gamma^\mu D^j X \quad (21)$$

Since we started from the linearized form of the superstring Lagrangian we have lost the constraint equations which follow from the requirement of invariance

*) X stands for a Lorentz vector field X_μ in a D-dimensional space-time and a Lorentz index μ has been omitted. When X appears in a bilinear form, summation over this index is understood.

under general reparametrization of the superstring coordinates $(\sigma, \tau, \theta_\alpha^i)$. Therefore, in analogy with the case of the conventional string (as well as in the Neveu-Schwarz string) in order to impose general reparametrization invariance we require at the classical level the vanishing of the supercurrent

$$V^\mu(\xi^\mu, \theta_\alpha^i) = 0 \quad (22)$$

Constructing from the supercurrent the corresponding charges one recognizes that the coefficients of the θ expansion of the "supercharge" are just given by the operators T_n , G_n^i , and L_n , which satisfy the algebra (10).

As a consequence of Eq. (22) in the quantum theory one finds that the physical states are annihilated not only by the conformal and supergauge charges (as already happens in the Neveu-Schwarz model) but also by the internal symmetry generator so one gets the colour confinement condition

$$T_n |\psi_{\text{phys}}\rangle = 0 \quad n \geq 0 \quad (23)$$

i.e. that the physical states must be singlets with respect to the local (colour) $O(2)$ group.

At this stage one is very tempted to consider $\psi = \psi_1 + i\psi_2$ as a quark field carrying, in this oversimplified case, an Abelian colour quantum number $[U(1)$ charge colour group]. The detailed structure of this model and the explicit construction of the dual amplitudes, as well as the supersymmetric string formulation of the conventional Neveu-Schwarz model, will be the subject of a forthcoming publication.

As a final point, we would like to point out the difficulties and perhaps the interesting features which emerge in the extension of the present model to higher non-Abelian groups.

In order to understand this more general case, it is better to come back to the original Neveu-Schwarz model whose linearized Lagrangian is

$$\mathcal{L}(\sigma, \tau, \theta_\alpha) = \bar{X} X D X \quad (24)$$

while for the $O(2)$ symmetric string it is

$$\mathcal{L}(\sigma, \tau, \theta_\alpha^i) = X^2 \quad (25)$$

The reason for this is just due to conformal invariance. In fact, one should observe that the invariant measure is in general

$$\int d\sigma dz d^{2N}\theta \quad (26)$$

for an $O(N)$ symmetry, so it carries a dimension $d_v = N - 2$. Therefore the Lagrangian density must have a dimension $d_{\mathcal{L}} = 2 - N$, in order to have a conformal invariant action. For $N \geq 3$ this last condition generally implies that subcanonical fields must be present in the superfield expansion^{*)}. These subcanonical fields are just the sources of the new charges which are present in the algebra (12). The presence of these subcanonical fields greatly complicates the quantization of a supersymmetric string with a non-Abelian colour symmetry; nevertheless we think that these difficulties have some analogies with the case of supersymmetric QED in two and four dimensions⁵⁾. We hope to come back to these problems in a future publication.

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*) An important exception is however the $SU(2)$ case previously discussed, where its realization does not need any subcanonical field.

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