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EFFECTIVE POTENTIAL APPROACH
TO THE QUANTUM SCATTERING OF SOLITONS

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A B S T R A C T

Systems of solitons are approximately described in terms of a finite number of "effective degrees of freedom" interacting via "effective potentials". These are reconstructed, principally from knowledge of solutions to the classical field equations, by a procedure involving the isometric mapping of a sector of the field theoretical Hilbert space onto the Hilbert space of non-relativistic point particles. The quantum dynamics of solitons is then approximately formulated and solved in terms of ordinary Schrödinger-type equations. As an interesting exercise, our method is applied to an analysis of soliton-antisoliton binding and scattering in the sine-Gordon model. With the exclusion of exceptional values of the coupling constant, corresponding to solutions of an eigenvalue equation, backward scattering is found to occur near threshold and to decay exponentially with the centre-of-mass energy. One of the exceptional, reflectionless sine-Gordon models is, not surprisingly, found to correspond to vanishing coupling in the massive Thirring model.

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The main obstacle to the quantization of classical fields may be identified with the existence of an infinite number of degrees of freedom. In addition to being responsible for divergences which complicate the interpretation of the theory, their presence often frustrates attempts to calculate in instances where the usual perturbative machinery is not applicable. This may be the case even in absence of non-weak couplings if the phenomena of interest are intrinsically non-perturbative in character. We have in mind field theoretical bag models of hadrons and other soliton models ¹⁾. These models have recently attracted a great deal of interest because of properties of their classical solutions which are reminiscent of some aspects of a familiar picture of hadrons.

Semi-classical methods have been applied to the quantization of these solutions for the simplest two-dimensional models in the weak coupling or semi-classical limit ²⁾. In particular, an extensive study of the sine-Gordon equation has uncovered a rich and interesting spectrum of states ³⁾.

However, these methods have inherent limitations. Thus no satisfactory treatment of scattering has been given to date. This failure may be attributed to the nature of the approximations which were devised to make the methods cope with the presence of an infinite number of degrees of freedom.

In this note we would like to suggest a complementary approach that is based on an approximate description of systems of solitons in terms of "effective" degrees of freedom to be quantized non-perturbatively. Under certain conditions, such as weak coupling, these effective degrees of freedom interact through "effective potentials". Then the quantum dynamics may simply be formulated and solved in terms of ordinary Schrödinger-type equations.

The methods to be presented here may be applied to realistic soliton models of hadrons in four-dimensional space-time. They will be introduced without reference to a specific model and later applied to the soliton-antisoliton scattering problem in the quantum sine-Gordon equation. There, backward scattering is found to occur as a quantum mechanical effect, with a dependence on $\hbar$ of the form $\exp(-1/\hbar \dots)$. This phenomenon escapes detection in attempts based on semi-classical expansions in powers of $\hbar$ which incorrectly yield purely forward scattering. In fact, we find that :

- a) the soliton-antisoliton scattering is purely forward, independently of the energy, only for special values of the coupling constant or ratio of soliton to "elementary" meson mass M/m ; the occurrence of such "reflectionless" sine-Gordon models corresponds to solutions of an eigenvalue equation :

$$\sin(\pi^2 M/m) = 0, \quad \pi^2 M/m \geq 2\pi.$$

It should be noted that the occurrence of the first of these pseudo-free sine-Gordon models was already implied by the equivalence⁴⁾ with the massive Thirring model. From this equivalence and the identification⁴⁾ $\pi^2 M/m = 2\pi + 2g$, we may now infer absence of backward scattering in massive Thirring models corresponding to $g = n\pi/2$.

- b) With the exception of these special cases, the reflection coefficient equals one at threshold, yielding purely backward scattering, and decays exponentially with the energy on the scale set by the meson mass.

For a consistency check, we also calculate the spectrum of states in the soliton-antisoliton channel and recover, in good approximation, the meson spectrum of Dashen, Hasslacher and Neveu³⁾.

Let $\phi_s(x, \tau)$ denote a time (τ) -dependent extended solution to the classical field equations. We shall refer to $\phi_s(x, \tau)$ as a "classical degree of freedom" and to τ as its "co-ordinate". The solution $\phi_s(x, \tau)$ may correspond to a freely moving soliton or to a bound soliton-antisoliton pair ("breeder mode") or to a scattering solution involving an arbitrary number of solitons. Thus the co-ordinate τ specifies the state of an "effective" degree of freedom which generally describes a system of solitons.

Configurations of the field neighbouring $\phi_s(x, \tau)$ will be named "transverse degrees of freedom". They will be "excited" together with $\phi_s(x, \tau)$ as quantum fluctuations for $\hbar \neq 0$. However, in the weak coupling, semi-classical limit, as quantum fluctuations subside and quantum solitons acquire classical sharpness, transverse degrees of freedom will "freeze". In this limit, the state of a system may be represented by a Fock space coherent state⁵⁾ centred around the classical meson field :

$$|z\rangle = \exp\left\{-\frac{i}{\hbar} \int dx [\phi_s(x, \tau) \dot{\phi}(x) - \dot{\phi}_s(x, \tau) \phi(x)]\right\} |0\rangle, (1)$$

Here $\phi(x)$ and $\dot{\phi}(x)$ denote the $t=0$ free field and its momentum conjugate and $|0\rangle$ is the Fock space no-particle state.

We propose to study aspects of soliton quantum field theories by constructing non-perturbative quantum expansions which begin by retaining and quantizing only effective classical degrees of freedom. This is obtained in the following way.

We consider a class of effective classical degrees of freedom. This class may include, for instance, soliton-anti-soliton scattering solutions for all possible initial velocities of the pair. From these solutions we form a class $\{|\tau\rangle\}$ of coherent states. An element of the class is specified by its co-ordinates. The linear span of all elements of $\{|\tau\rangle\}$ yields a subspace of the original Hilbert space, associated with a projection operator P . We define a restriction of the original theory by replacing the Hamiltonian operator $:H:$ by $H' = P:H:P$. Such a restriction may be studied as a quantum theory in its own right. It will be a relativistic quantum theory provided that the class of effective classical degrees of freedom considered is Poincaré invariant. The restricted quantum theory may also be regarded as an extension of classical field theory, for which the only allowed states may be identified with pure coherent states (no superposition allowed) ⁶⁾.

The inner products $\langle \tau' | \tau \rangle$ may be calculated from the following identities :

$$\langle \tau' | \tau \rangle = \exp \left\{ -\frac{1}{4\pi} \left[\|\phi'_S - \phi_S\| + \|\dot{\phi}'_S - \dot{\phi}_S\| + 2i((\phi'_S, \dot{\phi}_S) - (\dot{\phi}'_S, \phi_S)) \right] \right\} \quad (2)$$

$$\|\phi_S\| = \int dk K_0 |\phi_S(k)|^2, \quad \|\dot{\phi}_S\| = \int dk \frac{1}{K_0} |\dot{\phi}_S(k)|^2 \quad (3)$$

$$(\phi_S, \dot{\phi}_S) = \int dx \phi_S(x) \dot{\phi}_S(x) \quad (4)$$

The Hamiltonian $\mathcal{H}$ is normal ordered with respect to $|0\rangle$ and the matrix elements are, therefore, easily calculated with the help of the identities :

$$\langle \tau' | : \phi^n(x) : | \tau \rangle = \left[\frac{1}{2} (\phi'_s(x) + \phi_s(x)) - \frac{i}{2} (\dot{\phi}'_s(x) - \dot{\phi}_s(x)) \right]^n \langle \tau' | \tau \rangle, \quad (5)$$

$$\langle \tau' | : \dot{\phi}^n(x) : | \tau \rangle = \left[\frac{1}{2} (\dot{\phi}'_s(x) + \dot{\phi}_s(x)) + \frac{i}{2} (\phi'_s(x) - \phi_s(x)) \right]^n \langle \tau' | \tau \rangle, \quad (6)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int dk k_0 e^{-ikx} \phi_s(k), \quad (7)$$

$$\dot{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int dk \frac{1}{k_0} e^{-ikx} \dot{\phi}_s(k). \quad (8)$$

Knowledge of $\langle \tau' | \tau \rangle$ and of $\langle \tau' | \mathcal{H} | \tau \rangle$ suffices to enable one to derive the exact spectrum of $\mathcal{H}$ and scattering properties of the restricted quantum theory under consideration. For instance, in the τ representation, the evolution equation for an arbitrary state

$$|\Psi(t)\rangle = \int d\tau \psi(\tau, t) |\tau\rangle \quad (9)$$

is the integral equation :

$$i\hbar \partial_t \psi(\tau, t) = \int d\tau' \langle \tau | \mathcal{H} | \tau' \rangle \psi(\tau', t). \quad (10)$$

And the time-independent counter-part of this equation may be used to construct Poincaré covariant approximations to soliton states. For this purpose, the class $\{|\tau\rangle\}$ could, in two-dimensional space-time, be formed from all freely moving soliton solutions and actually be parametrized by two continuous labels, the velocity v and the time τ .

But, for the moment, let us concentrate on approximations to our theories which are suggested by the weak coupling or $\hbar \rightarrow 0$ limit. The inner product $\langle \tau' | \tau \rangle$ of Eq. (2) undergoes fast exponential fall-off with increasing separation $\epsilon = \tau' - \tau$ in that limit. This suggests an expansion of $\langle \tau' | \tau \rangle$ and retention only of the leading terms in the exponent in Eq. (2). We obtain

$$\langle \tau' | \tau \rangle \underset{\hbar \rightarrow 0}{\simeq} \exp \left\{ -\frac{1}{4\hbar} [A(\tau)(\tau' - \tau)^2 + 2i B(\tau)(\tau' - \tau)] \right\}, \quad (11)$$

where

$$A(\tau) = \int dk [K_0 |\dot{\phi}(k, \tau)|^2 + K_0^{-1} |\ddot{\phi}(k, \tau)|^2], \quad (12)$$

$$B(\tau) = \int dk [|\dot{\phi}(k, \tau)|^2 - \ddot{\phi}(k, \tau) \phi(k, \tau)]. \quad (13)$$

We now remark that the inner product given by Eq. (11) is reminiscent of the inner product between certain coherent states of a hypothetical system with one rather than an infinite number of degrees of freedom. Let us denote by q and p the co-ordinate and momentum of such a hypothetical particle and by $\alpha = \alpha_1 + i\alpha_2 = \sqrt{\hbar} q + i p/\sqrt{\hbar}$ the complex co-ordinate of a point in its phase plane. A one-particle coherent state $|\alpha\rangle$ is defined by :

$$|\alpha\rangle = e^{\frac{1}{\sqrt{2\hbar}}(\alpha a^\dagger - \alpha^* a)} |0\rangle = e^{-\frac{1}{4\hbar}|\alpha|^2} e^{\frac{\alpha}{\sqrt{2\hbar}} a^\dagger} |0\rangle, \quad (14)$$

where $[a, a^\dagger] = 1$ and $a|0\rangle = 0$. The inner product between two such states is

$$\langle \alpha' | \alpha \rangle = \exp \left\{ -\frac{1}{4\hbar} [|\alpha' - \alpha|^2 + (\alpha'^* \alpha - \alpha' \alpha^*)] \right\}. \quad (15)$$

For states corresponding to points on a phase plane path defined by $\alpha = \alpha(\tau)$ and for $\hbar \rightarrow 0$, Eq. (15) reduces to :

$$\langle \alpha(\tau') | \alpha(\tau) \rangle \underset{\hbar \rightarrow 0}{\approx} \exp \left\{ -\frac{1}{4\hbar} [|\dot{\alpha}|^2 (\tau' - \tau)^2 + 2i \operatorname{Im} (\dot{\alpha}^* \alpha) (\tau' - \tau)] \right\}. \quad (16)$$

Comparing Eq. (16) with Eq. (11), we then discover that the isomorphism induced by

$$|\tau\rangle \longleftrightarrow |\alpha(\tau)\rangle \quad (17)$$

is an isometry

$$\langle \tau' | \tau \rangle = \langle \alpha(\tau') | \alpha(\tau) \rangle \quad (18)$$

provided that

$$|\dot{\alpha}(\tau)|^2 = A(\tau), \quad (19')$$

$$\operatorname{Im} \dot{\alpha}^*(\tau) \alpha(\tau) = B(\tau). \quad (19'')$$

These equations, supplemented by the specification of (compatible) initial conditions ^{*)} ($\alpha(0)$), uniquely define the one-particle phase plane path $\alpha = \alpha(\tau)$ ^{**) .} As implied by analyticity properties of coherent states (or, equivalently, by the completeness of families of one-particle coherent states parametrized by continuous labels), the condition :

$$\langle \tau' | \mathcal{H} | \tau \rangle = \langle \alpha(\tau') | \mathcal{H} | \alpha(\tau) \rangle \quad (20)$$

^{*)} These are often determined by symmetry requirements.

^{**) In fact, Eq. (19') specifies for any τ the absolute magnitude $|\underline{W}|$ of the phase plane velocity $\underline{W} = \dot{\alpha}$, $\alpha \equiv (\alpha_1, \alpha_2)$, while Eq. (19'') determines its orientation by specifying the curl $\underline{W} \times \underline{\alpha}$.}

also uniquely defines a Hamiltonian operator $\mathcal{H}$ in the one-particle Hilbert space.

We have therefore shown that, in the weak coupling limit, it is possible to reformulate "restricted quantum field dynamics" as quantum dynamics of point particles. Intuitively, this equivalence is a consequence of the high degree of coherence of the soliton field in the weak coupling limit : while the soliton is extended and built from an infinite number of degrees of freedom, their coherent excitation may be described in terms of one effective co-ordinate. In fact, if $\phi_s(x, \tau)$ corresponds to a classical system of solitons such a co-ordinate will describe the entire system. Needless to say, if the restricted field theoretical quantum dynamics under consideration was obtained by simultaneously retaining N different classical degrees of freedom, $\phi_s^i(x, \tau)$, the above construction will yield, as an equivalent problem, the quantum dynamics of a system of N interacting point particles.

What are the effective interactions of these fictitious particles ? As implied by Eqs. (5), (6) and (20), we know that

$$\langle \alpha(\tau) | \mathcal{H} | \alpha(\tau) \rangle = \mathcal{H}(\phi_s) = \mathcal{E} \quad (21)$$

where $\mathcal{E}$ is the energy of the classical solution $\phi_s(x, \tau)$. Therefore, $\alpha(\tau)$ defines a phase plane path of constant classical energy. Thus, we may infer an effective classical potential $V(q)$, for our fictitious point particle, from the first integral of motion :

$$V(q) - \mathcal{E} = - \frac{1}{2M} p^2(q, \omega) = - \frac{\omega}{2M} |\alpha_2(\alpha_1)|^2. \quad (22)$$

Here $p(q, \omega)$ is the Cartesian representation in terms of co-ordinates and momenta of the path $\alpha = \sqrt{\omega} q + i p / \sqrt{\omega} = \alpha(\tau)$.

Notice that a non-relativistic energy momentum relation has been adopted in Eq. (22). This is in line with our use of a harmonic oscillator Fock space for the mapping (17). Such a choice is also motivated by the observation that a relativistic field is constructed from a sequence of non-relativistic oscillators: their coherent (collective) excitations are naturally described in terms of a similar one-degree-of-freedom system.

Also notice that the coherent state "width parameter" ω and the fictitious particle mass M , which appear in the definition of the potential $V(q)$, are still undetermined, as they are not constrained by the isometry condition (18) and by the diagonal condition (21).

The potential $V(q)$, obtained up to these ambiguities from Eq. (22), defines a Schrödinger Hamiltonian operator :

$$\mathcal{H}_S = - \frac{\hbar^2}{2M} \nabla^2 + V(q) \quad (23)$$

Differences between this operator and the exact Hamiltonian, defined by Eq. (20), will manifest themselves through terms which are of non-leading order in $\hbar$ in Eq. (20). Exploiting knowledge of the off-diagonal matrix elements in the left-hand side of this equation, such differences may be partially corrected by properly tuning the leftover parameters ω and M . Thus an optimal Schrödinger-type Hamiltonian, Eq. (23), and a quantum evolution equation for our duplicate world are established.

Analysis of this Schrödinger equation will generally yield a spectrum of bound states and scattering solution. The bound state spectrum may be directly identified with the spectrum of the restricted field theory from which we started, while from the scattering solutions a reconstruction of the S matrix will be possible.

As an interesting exercise, we have applied the method described above to an analysis of the quantum dynamics in the soliton-anti-soliton channel of the sine-Gordon model. We report here the results of this analysis.

We consider as a classical degree of freedom a soliton-anti-soliton scattering solution or a breeder mode ³⁾, $\phi_s(x, \tau)$, taken in the centre-of-mass and near "threshold". Following the method previously discussed, we then proceed to : construct coherent states and a Hilbert space ; calculate, with the help of Eqs. (2)-(8), inner products $\langle \tau' | \tau \rangle$, and Hamiltonian matrix elements $\langle \tau' | \mathcal{H} | \tau \rangle$; derive from Eqs. (19) a phase plane path and from Eqs. (20)-(22) an effective potential for the fictitious particle. This is approximately given by :

$$V(q) \approx 2M \tanh^2(mq), \quad (24)$$

where M and m denote the soliton and "elementary" meson mass, respectively. The effective mass of the fictitious particle is $\simeq 2M$ which is also the depth of the effective potential. As expected, this potential is attractive * and has a range of the order of the inverse meson mass ⁷⁾.

We then pick up a book of elementary quantum mechanics ⁸⁾ and read off all that we need to know about the Schrödinger theory of such a potential. The spectrum of bound states is ($\hbar=1$) :

$$M_n \simeq n \cdot m - n^2 \cdot \frac{m^2}{\pi M}, \quad (25)$$

where

$$n = 1, 2, \dots \lesssim \pi M/m \quad (26)$$

The reflection and transmission coefficients are

$$|R|^2 = \frac{1}{1+p^2}, \quad |T|^2 = \frac{p^2}{1+p^2}, \quad (27)$$

where

$$p \simeq \frac{\sinh(\pi k/m)}{\sin(\pi^2 M/m)}. \quad (28)$$

Here k is the initial momentum of the fictitious particle. In the duplicate, soliton world, k should be identified with twice the initial soliton (antisoliton) momentum. Noteworthy properties of Eqs. (25)-(28) have already been remarked in the introduction ^{**)}.

*) An analogous calculation carried out for soliton-soliton scattering yields a repulsive potential. This is reflected in the absence of bound states in the soliton-soliton channel.

**) We find it amusing to observe that purely forward scattering occurs for values of the ratio M/m corresponding to which a bound state appears (disappears).

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