The long-standing deficit of measured versus predicted solar neutrino fluxes, is re-examined in light of possible astrophysical solutions. In the last decade, solar neutrino flux and helioseismic measurements have greatly strengthened the case for non-astrophysical solutions. But some model-independent tests remain open.

The solar neutrino problem has nagged physicists for over 30 years and, for most of us, has a natural solution in neutrino oscillations, either in vacuo or matter-enhanced (MSW effect). This modification of neutrino properties (with $m\nu < 10^{-2}$ eV and mixing $\sim 0.001–1$) requires an extension of the Standard Model not detectable in accelerators and consistent with many models of unification. But confidence in such a solution is based on a prior exclusion of astrophysical or nuclear physics explanations; only in the past decade has such an outcome become strongly credible.

Predictions of the solar neutrino flux $\phi$ are outputs of a solar model, which in turn are special cases of hydrogen-burning ($4\cdot\text{H} \rightarrow 4\cdot\text{He}$) main sequence stellar models (almost all $\text{pp}$ chain, with small CNO contribution). These predictions are usually quoted from a particular model (here the Bahcall-Pinsonneault 1998 model), and detailed models agree if the same inputs are used. But a more generic approach is attractive if it can free us from a specific model with fixed parameters. Greater generality is all the more important if it reveals basic properties of the Sun and $\phi$ that depend only on simple properties. Here I outline the results of such an approach. As far as observations now go, it confirms the results of detailed solar models but also specifies crucial solar observations that remain to be filled in.

1 Properties of Solar and Stellar Models

Simple solar models orient us with gross structure: the core ($r/R_\odot \lesssim 0.3$, where nuclear fusion generates the luminosity $L_\odot$), the radiative zone ($0.3 \lesssim r/R_\odot < 0.71$) and the convective zone ($\text{CZ}, r/R_\odot > 0.71$). Less massive stars ($M < M_\odot$) have, according to stellar models, even deeper convective zones; while for $M > M_\odot$, the outer convective zone disappears, and a convective inner core appears as the central temperature gradient surpasses a critical value. The Sun might have had a convective inner core, drastically changing
the $\phi$ predictions, but there is now decisive evidence against core convection (see Section 3 below). A higher central temperature would also have led to CNO-dominance and higher neutrino fluxes.  

A solar model is standard (an SSM) if the model contains all the physics of matter, gravitation, and nuclear fusion needed to obtain a star, but nothing more. Conceptually, stellar structure and evolution divide into three levels by time scales. For the Sun, chemical evolution needs $\sim 10$ Gyr; thermal equilibrium, about 10 Myr (Kelvin-Helmholtz time); and hydrostatic equilibrium, about 5 minutes. The hydrodynamic time controls the helioseismic $p$- and $g$-modes. (Late in the Sun’s evolution, the chemical time scale will be shortened and the hierarchy blurred.) Structure means only the thermal and mechanical features of a star. 

The properties of matter needed are constitutive relations giving pressure $P$, opacity $\kappa$, and specific luminosity generation $\varepsilon$ as functions of density $\rho$ and temperature $T$. If the thermal structure is given, then the equation of state reduces to a barotropic $P = P(\rho)$, and the mechanical structure alone becomes a closed problem characterized by the stiffness profile $\Gamma = d\ln P/d\ln \rho$. The special case of $\Gamma = \gamma \equiv 1 + 1/n = \text{constant}$ is a polytrope of index $n$. All $n < 5$ polytropes have finite mass and radius. $n = 0$ is the constant-density case.

The initial conditions are fixed mass $M$ and the element mass abundances $X_i$ (the zero-age star assumed chemically homogeneous). The boundary conditions are zero mass and luminosity at the center and (nearly) zero pressure and density at the surface. The $X_i$ develop gradients by evolution, as nuclear fusion and heavy element diffusion act, and additional helium accumulates in the core.

Even within the SSM framework, variations are possible. The $\kappa$ and $\varepsilon$ functions and fusion cross sections must be calculated from atomic and nuclear physics and extrapolated into regimes not directly testable. For the dominant luminosity-producing $ppI$ reactions (terminating through $^3$He fusion), $\varepsilon$ and the reaction rates are almost fixed by $L_\odot$. But reactions not strongly connected with $L_\odot$ are not well-constrained: the $ppII$ and $ppIII$ chains (terminating from $^7$Be through $^7$Li and $^8$B, respectively) are sensitive functions of the core temperature and nuclear cross sections, as well as mildly dependent on the core density. $L_\odot$ and other global properties place only weak constraints on the $ppII$ and $ppIII$ rates.

2 Generalizing the Standard Solar Model

Generic properties of solar structure are restricted by boundary conditions. The SSM can then be generalized in two ways. One is to calibrate with the
specific model. This procedure defines a generalized SSM family, although not the most general.

Its most convenient implementation is through homology or power-law scaling, derivable analytically or made evident by numerical solutions. Exploration of model space by varying SSM inputs is actually a special case of homology, which amounts to “small” perturbations of the logarithms of inputs and outputs. Such “perturbative” analysis works over a surprisingly large range, so long as the power law-relations are stable. \(^4,7\)

The first signs of homological behavior in SSMs were found in the 1000 SSM Monte Carlo study of Bahcall and Ulrich. \(^8\) Subsequent work over a wider model range revealed a much broader validity for homology. The underlying analytic structure was derived by Bludman and Kennedy. \(^4\) Starting with structure alone, one keeps only dimensional and scaling behavior of macroscopic variables, dropping the differential nature of the equations. Assuming multifactor power laws for the equation of state, opacity, and luminosity generation, we found homological relations for the mechanical and thermal structure, assuming fixed powers. This requirement restricts the homology to the radiative and core regions, below the CZ. The dominant luminosity production is by \(ppI\), carried outwards entirely by radiative diffusion. The constitutive relations are

\[
P/\rho = RT/\mu , \quad \kappa(\rho, T) = \kappa_o(X_i)\rho^nT^{-s} , \quad \varepsilon(\rho, T) = \varepsilon_o(X)\rho^\lambda T^\nu . \tag{1}
\]

Expanded about the SSM, the exponents are: \(n = 0.43\), \(s = 2.5\), \(\lambda = 1.0\), \(\nu = 4.2\). \(\mu, \kappa_o,\) and \(\varepsilon_o\) are composition-dependent. The luminosity constraint reads: \(\phi(pp) + (0.977)\phi(Be) + (0.751)\phi(B) + (0.956)\phi(CNO) = 6.55 \times 10^{10} \text{ cm}^{-2} \text{ sec}^{-1} \). \(^4,2\)

The boundary conditions are imposed in a way appropriate to a single star: \(M_\odot, L_\odot,\) and \(R_\odot\) fixed. Homology then gives a family of possible non-convective interiors consistent with observed outer solar features and parametrized by \(\rho_c\) and \(T_c\): \(\rho_c \sim \varepsilon_o^{-0.34}\kappa_o^{-0.40}\mu_c^{0.52}L_\odot^{0.085}, \ T_c \sim \varepsilon_o^{-0.13}\kappa_o^{-0.034}\mu_c^{0.22}L_\odot^{0.17}\) The resulting \(\phi\) scale stably with \(\rho_c\) and \(T_c\) over a large range, giving the two-parameter homological mechanical/thermal variations of the SSM: \(\phi(i) \sim \rho_c^{\alpha_i} \cdot T_c^{\beta_i}\), with \((\alpha_i, \beta_i)\) for \(pp\), Be, and B \(\nu\)’s being \((-0.1, -0.7), (0.7, 9),\) and \((0.3, 21)\), respectively. The highest reactions in the \(pp\) chain have the famous extreme sensitivity to \(T_c\), while all sensitivities to \(\rho_c\) are mild and arise from the small luminosity contribution made by the \(ppII, ppIII,\) and CNO chains. It should be stressed that \(\rho_c\) and \(T_c\) are model outputs, like the \(\phi(i)\). These exponents reproduce the 1000-SSM Monte Carlo and clarify that the entire homological class of SSMs has the wrong pattern of fluxes to explain the observed energy dependence of \(\phi\): lower energies are
more suppressed. (Variation of nuclear cross sections also fail to explain the pattern.) This conclusion depends only on \( ppI \)-dominance in \( \varepsilon \), radiative diffusion in \( \kappa \), and the ideal gas law.

Homology applied to the mechanical structure alone automatically leads to a polytrope. But the Bludman-Kennedy homology is more general than a polytrope, as it applies to both mechanical and thermal structure. It also scales correctly in the evolved core, where the molecular weight \( \mu \) changes substantially, reflecting the chemical evolution that makes the present Sun differ from its zero-age incarnation. No polytrope fits this behavior.\(^4,5\)

A more complete version of homology is possible if the \textit{differential} structure is retained and rewritten using scale-invariant \textit{homology variables}.\(^5\) For the entire mechanical, thermal, and chemical structural system, the dimensionless differential equations are not less complex than a full SSM. But if the structure is restricted to the mechanical alone and a barytrope \( P(\rho) \) assumed, simple dimensionless structure equations follow. The key to the mechanical structure turns out to be the \( \Gamma \) profile. Constant \( \Gamma \) gives a polytrope again; in fact, \( \Gamma_{\text{SSM}} \approx 4/3 \) outside the inner core, up to the CZ, where it rises to \( 5/3 \) (the adiabatic value). But within the core, \( \mu \) rises and \( \Gamma \) drops towards the center, where \( \Gamma_c \approx 3/4 \). Described in terms of a polytrope, the effective index \( n_{\text{eff}} = (\Gamma - 1)^{-1} \) rises in the core, diverging at \( \Gamma = 1 \). Further towards the center, \( n_{\text{eff}} \) rises from minus infinity to a finite negative value at the center. Such behavior in the inner core is not even approximately polytropic and explains why attempts to use polytropes to approximate the Sun only work (and only crudely) over the \textit{whole} Sun and fail badly in the core.

The other approach to generalizing the SSM is to work with a few reasonable assumptions, following these to simple, testable predictions. Some powerful results are available, although restricted to mechanical structure only,\(^6\) to which helioseismology is the key: sound waves are mechanical perturbations with information about sound speed, equation of state, and density and pressure profiles.\(^9\) Taking full advantage of these results requires both \( p \)- (pressure) and \( g \)- (gravity) modes. Their spectra can be inverted to yield sound speed \( c_{ad}(r) \) and buoyancy frequency \( N(r) \) profiles, from which follows the complete mechanical structure.

The thermal and chemical structures cannot be directly probed by helioseismology, as their associated time scales are so long. Simple homology implies \( L \sim \mu^4 M^3 \kappa^{-1} \varepsilon_o^{-0} \), but changes in \( L_\odot \) large enough to explain the solar \( \nu \) deficit are probably ruled out by paleoclimatology.\(^10\) Direct tests of these aspects of the SSM require comparison to other Sun-like stars. Such stars vary from the Sun somewhat in mass and chemical composition and could lie anywhere on their respective evolutionary tracks. Comparison properties
include luminosity, surface temperature, and photospheric radius. With accurate photometry and parallaxes, accurate luminosities and colors are achievable. The intermediary between these observations and stellar structure is stellar atmosphere models, which have advanced considerably in the last 30 years. Although still oversimplified, the models are good enough for solar-type stars to infer ranges for surface $T$, $g$, abundances, and turbulence. An exciting possibility will be opened by asteroseismology of Sun-like stars, as observation of stellar seismic modes (especially $g$-modes) would lead to direct characterization of stellar interiors.

### 3 Observational Issues

$M_\odot$, $L_\odot$, $R_\odot$, and surface $T$, as well as surface and proto-solar (meteoric) abundances, are well measured. Helioseismic observations have directly or indirectly captured millions of $p$-modes, allowing an accurate inversion of $c_{ad}(r)$ down to $r/R_\odot = 0.05$. A convective core ($\gamma = 5/3$) is ruled out, although circulation of heavy elements not affecting heat transport cannot be at present.

The inferred sound speed peaks off-center at $r/R_\odot \simeq 0.07$. Hydrostatic equilibrium requires $dc_{ad}/dr = 0$ at the center, but an off-center peak occurs where $\Gamma = 1$. This peak and $dc_{ad}/dr > 0$ for $r/R_\odot < 0.07$ indicate $\Gamma < 1$ there, a crucial confirmation of the core’s chemically evolved state. A complete profile down to the center becomes possible with the lowest $p$-modes. But complete inversion for model-independent mechanical structure would be possible only if the higher $g$-modes were observed; analogous inversion would yield $N(r)$, and together $N$ and $c_{ad}$ yield $\Gamma$ and other mechanical profiles, the first truly independent test of the SSM.

Comparing the Sun with other sun-like stars has been possible for many decades, albeit at poor precision. But the recent Hipparcos-Tycho star catalogs have revolutionized astrometry, raising the accuracy of nearby stellar parallaxes by up to a factor of 10 or better (1 m-arcsec). Luminosities accurate to < 5% for nearby stars (within 25 pc) can now be inferred. With good color measurements and best current stellar atmosphere models, surface $T$’s can be limited to 1%. Even more dramatic astrometric improvements could come from the proposed SIM and GAIA orbital systems, to be launched in 2006 and 2009, respectively: 4 µ-arcsec parallax errors and luminosity errors limited only by photometry.
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